Wess-Zumino Anomaly and the Structure of the Decay $\tau^- \to K^+ K^- \pi^- \nu_{\tau}$

Abstract
We present the first study of the Wess-Zumino anomaly in the $\tau^- \to K^+ K^- \pi^- \nu_{\tau}$ decay, and determine its quantitative contribution $(55.7 \pm 8.4 \pm 4.9)\%$ to the decay width. The contributions from the axial vector current and the intermediate states are also given. The structure of the decay is also investigated simultaneously.

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1 Introduction

The $\tau$ decays involving $\eta$ violate $G$ parity in the standard vector current processes, and are expected to be quite small. But according to Wess and Zumino [1], one can construct an effective chiral Lagrangian involving three meson currents, the so-called chiral anomaly term which flips the parity of the three meson final state, allowing decays with an $\eta$ and at least two other mesons in the final state without isospin suppression. The anomaly violates the rule that the weak axial-vector and vector currents produce an odd and an even number of pseudoscalars, respectively, and it applies to $\tau$ decays to at least three mesons. Actually, the anomaly derived from the anomalous Ward identities and governed by the consistency conditions is universal and model independent [1] which plays an important role in the hadron dynamics [2], while the overall coupling constants in all vertices are model dependent and/or need to be determined by experiments. The triangle anomaly [3] describing the decay $\pi^0 \to \gamma\gamma$ which provides the evidence of colored quarks with $N_C = 3$ and the box anomaly [4] can be derived from the Wess-Zumino Lagrangian.

The $\tau$ hadronic decays provide a comprehensive test ground for the anomaly mechanism. The golden plated mode to study the anomaly is the $\tau^- \to \eta \pi^- \pi^0 \nu_\tau$ decay without the contribution from the axial-vector current, which was first observed by the CLEO Collaboration with sizeable branching fraction [5] as predicted. While for the dominant $\tau$ 3-pion decays, $\tau \to (3\pi)^-\nu_\tau$, $G$ parity forbids the anomaly to contribute. The second alternative decay mode to study the anomaly is $\tau^- \to K^+ K^- \pi^- \nu_\tau$, but it has contributions from both the axial-vector and vector currents [6] which need to be isolated by experiments. For this purpose, the $\pi/K$ separation capability is crucial to help suppress the backgrounds. Because of the historical difficulty in separating kaons from pions in tau decays, the study of the anomaly in the decay $\tau^- \to K^+ K^- \pi^- \nu_\tau$ was difficult. Good pion and kaon separation capability provided by the RICH detector at CLEO III makes the study possible. Taking advantage of the good $\pi/K$ separation capability with the RICH, we have improved the measurements of the branching fraction for $\tau$ decays to three charged hadrons with 0, 1, 2 and 3 kaons [7]. Moreover, the structure of the decay $\tau^- \to K^+ K^- \pi^- \nu_\tau$ is also sensitive to the contribution from the anomaly. ALEPH and OPAL attempted to compare the structure of $\tau^- \to K^+ K^- \pi^- \nu_\tau$ [8] with the default model KORALB [9] based on their limited data samples and limited $\pi/K$ separation provided by $dE/dx$ which results in large backgrounds from the $\tau$ cross feed decays. They observed a $K^*$ peak and obvious $K\bar{K}\pi$ mass shift compared to the default model KORALB. In Fig. 2, the comparison based on 3.26 fb$^{-1}$ CLEO III datasets shows clear discrepancies between the data and MC generated using the default KORALB, it means the decay is not well modeled in KORALB. So far there is no experimental measurements of the model parameters for the decay, therefore, the study of its structure will help improve modeling of the decay.

In this note, based on 7.77 fb$^{-1}$ datasets collected with the CLEO III detector on or near the $Y(4S)$, we use the unbinned maximum likelihood method which will be discussed in Sect. 3 to determine two model parameters, and therefore the quantitative contribution from the anomaly to the decay width of $\tau^- \to K^+ K^- \pi^- \nu_\tau$ and its structure. The event selection, background estimates and momentum dependent particle identification efficiency corrections are described in CBX-02-29 and CBX-03-2 [7] in details. The dataset dependent particle identification efficiency corrections using the whole datasets are also taken into account throughout this note as done in CBX-02-29 and CBX-03-2.
2 Theoretical Aspects of \( \tau^- \to K^+ K^- \pi^- \nu_\tau \) Decays

2.1 Model Independent Part

The matrix element \( \mathcal{M} \) for the semileptonic \( \tau \) decays into three pseudoscalar mesons \( h_1, h_2, h_3 \)

\[
\tau(\ell, s) \to \nu(\ell', s') + h_1(q_1, m_1) + h_2(q_2, m_2) + h_3(q_3, m_3)
\]

can be expressed in terms of a leptonic \( (M_\mu) \) and a hadronic current \( (J^\mu) \) as

\[
\mathcal{M} = \frac{G}{\sqrt{2}} \left( \frac{\cos \theta_C}{\sin \theta_C} \right) M_\mu J^\mu. \tag{1}
\]

where \( G \) is the Fermi coupling constant. The cosine and the sine of the Cabibbo angle \( (\theta_C) \) in Eq. 1 have to be used for Cabibbo-allowed \( |\Delta S| = 0 \) and Cabibbo-suppressed \( |\Delta S| = 1 \) decays, respectively. The leptonic \( (M_\mu) \) and hadronic \( (J^\mu) \) currents are given by

\[
M_\mu = a(\ell', s')\gamma_\mu(g_V - g_A\gamma_5)u(\ell, s)
\]

with \( g_V = g_A = 1 \) in the Standard Model, and

\[
J^\mu(q_1, q_2, q_3) = \langle h_1(q_1), h_2(q_2), h_3(q_3)\rangle|V^\mu(0) - A^\mu(0)|0\rangle.
\]

where \( V^\mu \) and \( A^\mu \) are the vector and axial-vector quark currents, respectively. The most general ansatz for the hadronic current is characterized by four form factors

\[
J^\mu(q_1, q_2, q_3) = V_1^\mu F_1 + V_2^\mu F_2 + iV_3^\mu F_3 + V_4^\mu F_4,
\]

with

\[
V_1^\mu = q_1^\mu - q_5^\mu - Q^\mu \frac{Q(q_1 - q_3)}{Q^2},
\]

\[
V_2^\mu = q_2^\mu - q_5^\mu - Q^\mu \frac{Q(q_2 - q_3)}{Q^2},
\]

\[
V_3^\mu = \epsilon^{\mu\alpha\beta\gamma} q_{1\alpha} q_{2\beta} q_{3\gamma},
\]

\[
V_4^\mu = q_1^\mu + q_2^\mu + q_3^\mu = Q^\mu.
\]

\( q_i \) are 4-momenta of the observed particle \( i \). The vector current \( (J^P = 1^-) \) originating from the Wess-Zumino anomaly gives rise to the term proportional to \( F_3 \). The terms proportional to \( F_1 \) and \( F_2 \) originate from the axial-vector current \( (J^P = 1^+) \). The term proportional to \( F_4 \) is due to the spin zero scalar \( (J^P = 0^+) \) part of the axial-vector current is expected to be small \([6, 9]\) and will be set to be zero throughout this note, but we will consider its contribution to systematics. All form factors are in general functions of \( Q^2 = (q_1 + q_2 + q_3)^2 \), \( s_1 = (q_2 + q_3)^2 \), \( s_2 = (q_1 + q_3)^2 \) or \( s_3 = (q_1 + q_2)^2 \).

The differential decay rate is obtained from \([11]\)

\[
d\Gamma(\tau \to 3h \nu_\tau) = \frac{1}{2m_\tau} \frac{G^2}{2} \left( \frac{\cos^2 \theta_C}{\sin^2 \theta_C} \right) \{L_{\mu\nu} H^{\mu\nu}\} d\mathcal{P} S^{(4)}.
\]
where $L_{\mu\nu} = M_{\mu}(M_{\nu})^\dagger$ and $H^{\mu\nu} = J^\mu(J^\nu)^\dagger$. The $\tau \rightarrow (3h)\nu_\tau$ phase space element is

$$dPS^{(4)} = (2\pi)^4\delta^4(\ell - \ell' - q_1 - q_2 - q_3) \times \frac{d^3\ell' \, d^3q_1 \, d^3q_2 \, d^3q_3}{2E_{\ell'} \, 2E_1 \, 2E_2 \, 2E_3 \, (2\pi)^3}.$$

For $\tau$ decays to three mesons, it is easily analyzed in the hadronic rest frame $q_1 + q_2 + q_3 = 0$. The orientation of the hadronic system is characterized by three Euler angles ($\alpha, \beta$ and $\gamma$), their definition is shown in Fig. 1, details can be found in Ref. [11].

\[
\begin{align*}
\cos \beta &= \hat{n}_L \cdot \hat{n}_L, \\
\cos \gamma &= -\frac{\hat{n}_L \cdot \hat{q}_3}{|\hat{n}_L \times \hat{n}_L|}.
\end{align*}
\]

where $\hat{n}_L = -\hat{n}_Q$, with $\hat{n}_Q$ the direction of the hadrons in the laboratory frame, $\hat{n}_L = \hat{q}_1 \times \hat{q}_2$, the normal to the plane defined by the momenta of particles 1 and 2. Note the angle $\gamma$ defines a rotation around $\hat{n}_L$ and determines the orientation of the three hadrons within their production plane.

![Figure 1: Definition of the polar angle $\beta$ ($0 \leq \beta < \pi$) and the azimuthal angle $\gamma$ ($0 \leq \gamma < 2\pi$). $\beta$ denotes the angle between $\hat{n}_L$ and $\hat{n}_L$. $\gamma$ denotes the angle between the $(\hat{n}_L, \hat{n}_L)$ plane and the $(\hat{n}_L, \hat{q}_3)$ plane. The unobservable angle $\alpha$ ($0 \leq \alpha < 2\pi$, not shown) denotes the angle between two planes $(\hat{n}_L, \hat{n}_\gamma)$ and $(\hat{n}_L, \hat{n}_\gamma)$.](image)

After integrated over the unobservable neutrino, the phase space element reads

$$dPS^{(4)} = \frac{1}{(2\pi)^5} \frac{1}{\delta^4} \frac{(m_\tau^2 - Q^2)}{m_\tau^2} \frac{dQ^2}{Q^2} \frac{ds_1 ds_2}{2\pi} \frac{d\alpha}{2\pi} \frac{d\gamma}{2\pi} \frac{d\cos \beta}{2} \frac{d\cos \theta}{2}.$$

and $L_{\mu\nu} H^{\mu\nu}$ is given by

$$L_{\mu\nu} H^{\mu\nu} = \sum_x L_X W_X = (g_V^2 + g_A^2)(m_\tau^2 - Q^2) \sum_x T_X(\alpha, \beta, \gamma, \theta, Q^2) W_X(Q^2, s_1, s_2).$$
The density matrix elements $\bar{T}_X$ stand for the averages of $L_X$ over the $\tau$ neutrino. Since the Euler angle $\alpha$ is unobservable, we have to integrate over it to obtain the differential decay width at low energies for a single $\tau$.

$$d\Gamma(\tau \to 3\nu_\tau) = \frac{G^2}{4m_\tau} \left( g_Y^2 + g_A^2 \right) \left( \frac{\cos^2 \theta_C}{\sin^2 \theta_C} \right) \left\{ \sum_X \bar{T}_X(\beta, \gamma, \theta, Q^2)W_X(Q^2, s_1, s_2) \right\}$$

$$\times \frac{1}{(2\pi)^5} \frac{1}{64} \frac{1}{m^2_\tau} \frac{1}{(m^2_\tau - Q^2)^2} \frac{dQ^2}{2\pi} \frac{ds_1 ds_2}{2\pi} \frac{d\gamma}{2} \frac{d\cos \beta}{2} \frac{d\cos \theta}{2}. \quad (3)$$

The angle $\theta$ in Eq. 3 denotes the angle between the flight direction of the $\tau$ in the laboratory frame and the direction of the hadronic system as seen in the $\tau$ rest frame. It is related to the energy $E_h$ of the hadronic system in the laboratory frame by

$$\cos \theta = \frac{2x m_\tau^2 - m_\tau^2 - Q^2}{(m_\tau^2 - Q^2)^2 \sqrt{1 - 4m_\tau^2/s}}. \quad (4)$$

with $x = 2E_h/\sqrt{s}$ and $s$ is the center of mass energy. The angle $(\psi)$ between the flight direction of the $\tau$ and that of the laboratory ($-Q$) as seen from the hadronic rest frame is given by

$$\cos \psi = \frac{x(m_\tau^2 + Q^2) - 2Q^2}{(m_\tau^2 - Q^2)^2 \sqrt{x^2 - 4Q^2/s}}. \quad (5)$$

which depends on $E_h$. $\sum_X \bar{T}_XW_X$ is a sum of nine density matrix elements $\bar{T}_XW_X$ of the hadronic system in a spin-1 state. The explicit expressions of $\bar{T}_X$ are given in [11].

The structure functions $W_X$ contain the dynamics of the hadronic decays and depend in general on $s_1, s_2$ and $Q^2$, which are given by nine structure functions expressed in terms of the form factors $F_i$. The hadronic tensor $H^{\mu\nu}$ is conveniently calculated in the hadronic rest frame with the $z$ and $x$ axes aligned along $\hat{n}_1$ and $q_3/|q_3|$. The momenta of the hadrons in the rest frame of the hadronic system are given as follows

$$q_3^\mu = (E_3, q_3^x, 0, 0), \quad q_2^\mu = (E_2, q_2^x, q_2^y, 0), \quad q_1^\mu = (E_1, q_1^x, q_1^y, 0).$$

with

$$s_1 + s_2 + s_3 = Q^2 + m_1^2 + m_2^2 + m_3^2, \quad E_i = \frac{Q^2 - s_i + m_i^2}{2\sqrt{Q^2}}, \quad q_3^x = \sqrt{E_3^2 - m_3^2}, \quad q_2^x = \frac{(2E_2E_3 - s_1 + m_2^2 + m_3^2)/2q_3^x,}{(2E_1E_3 - s_2 + m_1^2 + m_3^2)/2q_3^x,} \quad q_1^x = \frac{-\sqrt{E_2^2 - (q_2^x)^2 - m_2^2},}{\sqrt{E_1^2 - (q_1^x)^2 - m_1^2} = -q_1^y.}$$

where $q_i^x(q_i^y)$ denotes the $x$($y$) component of the momentum of the meson $i$ in the hadronic rest frame. The following variables $x_i$ are useful to express the general results for the hadronic structure
functions \( W_X \)

\[
\begin{align*}
x_1 &= V_1^x = q_1^x - q_3^x, \\
x_2 &= V_2^x = q_2^x - q_3^x, \\
x_3 &= V_1^y = q_1^y = -q_2^y, \\
x_4 &= V_3^z = \sqrt{Q^2 x_3 q_3^x}.
\end{align*}
\]

With these variables, the structure functions can be in general expressed in terms of the form factors as follows:

\[
\begin{align*}
W_A &= (x_1^2 + x_3^2)|F_1|^2 + (x_2^2 + x_3^2)|F_2|^2 + 2(x_1 x_2 - x_3^2)\text{Re}(F_1 F_2^*), \\
W_B &= x_4^2 |F_3|^2, \\
W_C &= (x_1^2 - x_3^2)|F_1|^2 + (x_2^2 - x_3^2)|F_2|^2 + 2(x_1 x_2 + x_3^2)\text{Re}(F_1 F_2^*), \\
W_D &= 2\{x_1 x_3|F_1|^2 - x_2 x_3|F_2|^2 + x_3(x_2 - x_1)\text{Re}(F_1 F_2^*)\}, \\
W_E &= -2x_3(x_1 + x_2)\text{Im}(F_1 F_2^*), \\
W_F &= 2x_4\{x_1 \text{Im}(F_1 F_3^*) + x_2 \text{Im}(F_2 F_3^*)\}, \\
W_G &= -2x_4\{x_1 \text{Re}(F_1 F_3^*) + x_2 \text{Re}(F_2 F_3^*)\}, \\
W_H &= 2x_3x_4\{\text{Im}(F_1 F_3^*) - \text{Im}(F_2 F_3^*)\}, \\
W_I &= -2x_3x_4\{\text{Re}(F_1 F_3^*) - \text{Re}(F_2 F_3^*)\}, \\
W_{SA} &= Q^2 |F_1|^2, \\
W_{SB} &= 2\sqrt{Q^2}\{x_1 \text{Re}(F_1 F_3^*) + x_2 \text{Re}(F_2 F_3^*)\}, \\
W_{SC} &= -2\sqrt{Q^2}\{x_1 \text{Im}(F_1 F_3^*) + x_2 \text{Im}(F_2 F_3^*)\}, \\
W_{SD} &= 2\sqrt{Q^2}\{\text{Re}(F_1 F_4^*) - \text{Re}(F_2 F_4^*)\}, \\
W_{SE} &= -2\sqrt{Q^2}\{\text{Im}(F_1 F_4^*) - \text{Im}(F_2 F_4^*)\}, \\
W_{SF} &= -2\sqrt{Q^2}\{x_4 \text{Im}(F_3 F_4^*)\}, \\
W_{SG} &= -2\sqrt{Q^2}\{x_4 \text{Re}(F_3 F_4^*)\}. \\
\end{align*}
\]

with

\[
\begin{align*}
q_3^x &= \sqrt{\frac{(s_1 + s_2 - m_1^2 - m_2^2)^2}{4Q^2} - m_3^2}, \\
x_1^2 + x_3^2 &= \frac{(s_1 - s_3 - m_1^2 + m_3^2)^2}{4Q^2} + s_2 - 2(m_1^2 + m_3^2), \\
x_2^2 + x_3^2 &= \frac{(s_2 - s_3 - m_2^2 + m_3^2)^2}{4Q^2} + s_1 - 2(m_2^2 + m_3^2), \\
2(x_1 x_2 - x_3^2) &= \frac{(s_1 - s_3 + m_3^2 - m_1^2)(s_2 - s_3 + m_3^2 - m_2^2)}{2Q^2} + Q^2 - 2s_3 + m_1^2 + m_2^2 - 3m_3^2, \\
(x_1 + x_2)^2 &= \frac{9(s_1 + s_2 - m_1^2 - m_2^2)^2}{4Q^2} - 9m_3^2 = 9(q_3^x)^2, \\
q_1^x &= \frac{1}{4Q^2 q_3^x} [Q^2(s_1 - s_2 + m_1^2 - m_2^2 + 2m_3^2) - s_1(s_1 + s_2) + s_1(2m_1^2 + m_2^2)]
\end{align*}
\]
\[(q_1^2)^2 = \frac{1}{4(q_3^2)} \left\{ m_1^2m_2^2 + m_1^2m_3^2 + m_2^2m_3^2 - m_4^4 + \frac{s_1s_2s_3}{Q^2} + s_2(m_3^2 - m_2^2) + s_1(m_2^2 - m_1^2) - m_3^2Q^2 \right. \\
+ \frac{1}{Q^2} \left[ m_1^2s_2(m_3^2 - m_2^2) + m_2^2s_1(m_3^2 - m_1^2) + m_1^2m_2^2(m_1^2 + m_2^2 - m_3^2) \right] \right\},
\]
\[x_4^2 = Q^2(q_1^2)^2(q_3^2)^2.\]

The structure functions \(W_{B,E,G,H,I,LSF,SG}\) are related to the anomaly form factor \(F_3\). Here we derived Eq. 7 in general case of \(m_1 \neq m_2 \neq m_3\). If \(m_1 = m_2 = m_3\), many terms with \(m_i^2 - m_j^2\) will be zero, it simplifies as Equation 49 in Zeit. Phys. C56, 661 (1992) [11]. The equation is also tested by comparing the mass spectra from the integrations and from KORALB with the same model parameters, see Fig. 3.

Equation 3 together with Eq. 6 provides the full description of the angular distributions of the decay products from a \(\tau\) decay. They reveal that the measurements of the structure functions \(W_i\), and therefore result in the measurements of the anomaly form factor \(F_3\). For \(\tau\) decays to three mesons, the decay width can be calculated from [11]

\[\Gamma(\tau \rightarrow 3h\nu_r) = \frac{G^2}{24m_r} \left( g_V^2 + g_A^2 \right) \left( \frac{\cos^2 \theta_C}{\sin^2 \theta_C} \right) \frac{1}{(4\pi)^5} \int Q^2 \left( \frac{m_c^2 - Q^2}{Q^4} \right) \left( W_A + W_B \right) + 3W_{SA} \}
\]

where \(W_A\) and \(W_B\) are given in Eq. 6 and 7 (only the coefficients for \(W_A\) and \(W_B\) are derived). It shows that \(W_A\) (\(W_{SA}\)) and \(W_B\) govern the rate and the distributions in the Dalitz plot, the remaining functions determine the angular distributions; and there is no interference between the axial vector current contributions (\(F_1, F_2\)) and the vector current contribution (\(F_3\)) in the decay width (their \(J^P\) are totally different). It will help us to determine the anomaly contribution to the total decay width without extracting 16 structure functions.

The model predicted contribution from the Wess-Zumino anomaly varies from 10% to 100% which is strongly dependent on the model parameters, the model favored one is about 40%, the contribution from the scalar part is of \(10^{-1}\) which is negligible [9, 10].

### 2.2 Integration Intervals

For a given value of \(s_1\), the range of \(s_2\) is determined by its values when \(q_1\) is parallel or antiparallel to \(q_8\) (see “Kinematics” section in the PDG [12]):

\[(s_2)_{\text{max}} = (E_1 + E_3)^2 - \left( \sqrt{E_1^2 - m_1^2} - \sqrt{E_3^2 - m_3^2} \right)^2,\]
\[(s_2)_{\text{min}} = (E_1 + E_3)^2 - \left( \sqrt{E_1^2 - m_1^2} + \sqrt{E_3^2 - m_3^2} \right)^2.\]

Here \(E_1 = (Q^2 - s_1 - m_1^2)/(2\sqrt{s_1})\), \(E_3 = (s_1 - m_2^2 + m_3^2)/(2\sqrt{s_1})\) are the energies of particle 1 and 3. The ranges of \(s_1\) and \(Q^2\) are determined by

\[(m_2 + m_3)^2 \leq s_1 \leq \left( \sqrt{Q^2 - m_1^2} \right)^2,\]
\[(m_1 + m_2 + m_3)^2 \leq Q^2 \leq m_\tau^2. \tag{10}\]

Similarly, for a given value of \(s_2\), the range of \(s_3\) is determined by its values when \(\mathbf{q}_1\) is parallel or antiparallel to \(\mathbf{q}_2\)

\[
(s_3)_{\text{max}} = (E_1 + E_2)^2 - \left(\sqrt{E_1^2 - m_1^2} - \sqrt{E_2^2 - m_2^2}\right)^2,
\]
\[
(s_3)_{\text{min}} = (E_1 + E_2)^2 - \left(\sqrt{E_1^2 - m_1^2} + \sqrt{E_2^2 - m_2^2}\right)^2. \tag{11}\]

Here \(E_1 = (s_2 - m_2^2 + m_1^2)/(2\sqrt{s_2})\), \(E_2 = (Q^2 - s_2 - m_2^2)/(2\sqrt{s_2})\) are the energies of particle 1 and 2 and

\[
Q^2 + m_1^2 + m_2^2 + m_3^2 - s_2 - (s_3)_{\text{max}} \leq s_1 \leq Q^2 + m_1^2 + m_2^2 + m_3^2 - s_2 - (s_3)_{\text{min}},
\]
\[(m_1 + m_3)^2 \leq s_2 \leq (\sqrt{Q^2 - m_2^2})^2,
\]
\[(m_1 + m_2 + m_3)^2 \leq Q^2 \leq m_\tau^2. \tag{12}\]

### 2.3 Model Dependent Part

The parameterization of the form factors for the decay \(\tau^- \to K^- \pi^- K^+ \nu_\tau\) is given by [9]

\[
F_1(Q^2, s_2, s_3) = -\frac{\sqrt{2}}{3f_\pi} \text{BW}_{a_1}(Q^2)T_\rho^{(2m)}(s_2),
\]
\[
F_2(Q^2, s_1, s_3) = -\frac{\sqrt{2}}{3f_\pi} \text{BW}_{a_1}(Q^2)T_\rho^{(2m)}(s_1),
\]
\[
F_3(Q^2, s_1, s_2) = -\frac{1}{2\sqrt{2\pi^2 f_\pi^3}} T_\rho^{(3m)}(Q^2) T_{\omega K^*}(s_2, s_1),
\]
\[
F_4(Q^2, s_1, s_2) = R_{\pi K} \text{BW}_{\pi^*}(Q^2) \left\{ s_2(s_1 - s_3)T_\rho^{(2m)}(s_2) + s_1(s_2 - s_3)T_\rho^{(2m)}(s_1) \right\}. \tag{13}\]

Here \(R_{\pi K}\) are coupling constants for 3\(\pi\) and \(K K \pi\) channels with \(R_\pi = 2R_K = \frac{f_{\pi^*} g_{\pi^* \pi \pi} g_{\pi \pi \rho}}{m_\pi^2 m_\rho^4}\) and \(f_{\pi^*} = 0.02\text{ GeV}, g_{\pi^* \pi \rho} = 5.8\) and \(g_{\pi \pi \rho} = 6.08\) [9]. \(\text{BW}_X(s)\) is the two particle Breit-Wigner propagator with an energy dependent width \(\Gamma_X(s)\):

\[
\text{BW}_X(s) \equiv \frac{M_X^2}{M_X^2 - s - i\sqrt{s}\Gamma_X(s)},
\]

where \(X\) stands for the various resonances of the two mesons channels. For a two body decay, the energy dependent width is

\[
\Gamma_X(s) = \Gamma_X \frac{M_X^2}{s} \left(\frac{p}{p_X}\right)^{2n+1},
\]
\[p = \frac{1}{2\sqrt{s}} \sqrt{(s - (M_1 + M_2)^2)(s - (M_1 - M_2)^2)},\]
\[p_X = \frac{1}{2M_X} \sqrt{(M_X^2 - (M_1 + M_2)^2)(M_X^2 - (M_1 - M_2)^2)},\]

\]
where \( n \) is the power of \(|\rho|\) in the matrix element and \( n = 1 \) here. For the \( a_1 \) decays, its Breit-Wigner function reads

\[
\text{BW}_{a_1}(Q^2) = \frac{m_{a_1}^2}{m_{a_1}^2 - Q^2 - i m_{a_1} \Gamma_{a_1} g(Q^2)/g(m_{a_1})}. \tag{14}
\]

The CLEO results on the \( a_1 \) parameterization [13] which have been implemented in KORALB [9] will be used. The \( K^* \), the \( \rho \) and the \( \omega \) resonances are parameterized as [9]:

\[
T^{(2m)}_{K^*}(s_1) = \frac{\text{BW}_{K^*}(s_1) + \beta_{K^*} \cdot \text{BW}_{K^*}(s_1)}{1 + \beta_{K^*}},
\]

\[
T^{(2m)}_{\rho}(s_2) = \frac{\text{BW}_{\rho}(s_2) + \beta_{\rho} \cdot \text{BW}_{\rho}(s_2)}{1 + \beta_{\rho}},
\]

\[
T^{(2m)}_{\omega}(s_2) = \frac{\text{BW}_{\omega}(s_2) + \epsilon \cdot \text{BW}_{\phi}(s_2)}{1 + \epsilon}. \tag{15}
\]

with \( \beta_{K^*} = -0.135 \) and \( \beta_{\rho} = -0.145 \), which are obtained from the measurements of the branching fraction for \( \tau^- \to K^{*-} \nu_{\tau} \) decay and of the spectral function for \( e^+e^- \to \pi^+\pi^- \) [9]. The \( \omega \to \phi \) mixing gives rise to \( \epsilon = 0.05 \), but we did not see any \( \phi \) signal in the \( K^+K^- \) mass spectrum (see, Fig. 15), so \( \epsilon \) will be set to be zero.\(^1\) While for the decay \( \tau^- \to K^+K^-\pi^-\nu_{\tau} \), the intermediate state \( K^{*-}/K^- \) is seriously phase space suppressed, moreover the branching fraction for \( K^{*-} \to K^+K^- \) is quite small with \( B(K^{*-} \to K^+K^-) = (6.6 \pm 1.3)\% \) [12] (the dominant \( K^{*-} \) decay is \( K^*\pi^0 \)), it implies \( \beta_{K^*} \sim 0 \), the experimental results from \( \tau^- \to (K\pi)^-\nu_{\tau} \) and \( \tau^- \to K^+K^-\pi^-\nu_{\tau} \) also favor \( \beta_{K^*} = 0 \) [7, 14], otherwise there should be a \( K^+\pi^- \) bump near the \( K^{*-} \), while we hardly observed any event with \( M_{K^+\pi^-} > 1.2\text{GeV}/c^2 \), see Fig. 15.

The remaining Breit-Wigner functions are given by

\[
T^{(3m)}_{\rho}(Q^2) = \frac{\text{BW}_{\rho}(Q^2) + \lambda \text{BW}_{\rho}(Q^2) + \delta \text{BW}_{\rho'}(Q^2)}{1 + \lambda + \delta},
\]

\[
T^{(2m)}_{\omega K^*}(s_1, s_2) = \frac{T^{(2m)}_{\omega}(s_2) + \alpha T^{(2m)}_{K^*}(s_1)}{1 + \alpha}. \tag{16}
\]

with [12]

\[
m_{\rho} = 0.770 \text{ GeV}, \quad \Gamma_{\rho} = 0.151 \text{ GeV}, \]
\[
m_{\rho'} = 1.465 \text{ GeV}, \quad \Gamma_{\rho'} = 0.310 \text{ GeV}, \]
\[
m_{\rho''} = 1.700 \text{ GeV}, \quad \Gamma_{\rho''} = 0.240 \text{ GeV}.
\]

The model default parameters \( \lambda = -0.25 \), \( \delta = -0.0385 \) and \( \alpha = -0.2 \) do not describe the data well, see Fig. 8 in CBX-02-29, and we will determine these parameters from fits to the data.

Taken together, Eq. 13 reads

\[
F_1(Q^2, s_2, s_3) = -\frac{\sqrt{2}}{3f_{\pi}} \text{BW}_{a_1}(Q^2)T^{(2m)}_{\rho}(s_2) = -\frac{\sqrt{2}}{3f_{\pi}} \text{BW}_{a_1}(Q^2) \frac{\text{BW}_{\rho}(s_2) + \beta_{\rho} \text{BW}_{\rho'}(s_2)}{1 + \beta_{\rho}}.
\]

\(^1\)Thanks Jon Urheim to point out the error made in Z. Phys. C56, 445 (1993) about the isospin violation for the presence of the intermediate state \( \rho^{(i)}\pi \) in the vector current processes where it is forbidden although its contribution is small, only the \( \omega\pi \) qualifies in the vector current process for this decay unlike the axial vector current processes where it is allowed, see also Z. Phys. C69, 243 (1996).
that the Wess-Zumino anomaly contribution increase. In this note, we use unbinned maximum likelihood method to fit the data and determine these two parameters.

For later convenience, the explicit expressions of $W_A$ and $W_B$ are given as functions of $Q^2, s_1$ and $s_2$:

\[
W_A = \left( \frac{2s_1 + s_2 - Q^2 - 2m_1^2 - m_2^2}{4Q^2} + s_2 - 2(m_1^2 + m_3^2) \right) \\
\times \left| \frac{\sqrt{2}}{3f_\pi} \text{BW}_{a_1}(Q^2) \frac{\text{BW}_{\rho}(s_2) + \beta_\rho \text{BW}_{\rho'}(s_2)}{1 + \beta_\rho} \right|^2 \\
+ \left( \frac{2s_2 + s_1 - Q^2 - 2m_2^2 - m_1^2}{4Q^2} + s_1 - 2(m_2^2 + m_3^2) \right) \left| \frac{\sqrt{2}}{3f_\pi} \text{BW}_{a_1}(Q^2) \cdot \text{BW}_{K^*}(s_1) \right|^2 \\
+ \frac{1}{2} \left( s_1 + s_2 + m_1^2 + m_2^2 - Q^2 \right) - 5m_3^2 \\
\times \text{Re} \left\{ \frac{2}{9f_\pi^2} \left| \text{BW}_{a_1}(Q^2) \right|^2 \cdot \frac{\text{BW}_{\rho}(s_2) + \beta_\rho \text{BW}_{\rho'}(s_2)}{1 + \beta_\rho} \cdot \text{BW}_{K^*}(s_1) \right\} \\
\]

\[
W_B = \frac{Q^2}{4} \left\{ \frac{m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2 - m_3^4 + s_1 s_2 (Q^2 + m_1^2 + m_2^2 + m_3^2) - s_1 - s_2}{Q^2} \\
+ s_2 (m_3^2 - m_2^2) + s_1 (m_3^2 - m_1^2) - m_3^2 Q^2 \\
- \frac{1}{Q^2} \left[ m_1^2 s_2 (m_3^2 - m_2^2) + m_2 s_1 (m_3^2 - m_2^2) + m_1^2 m_2 (m_1^2 - m_2^2) + m_1^2 m_2 (m_3^2 - m_2^2) \right] \right\} \frac{1}{8\pi^4} \\
\times \left| \frac{\text{BW}_{\rho}(Q^2) + \lambda \text{BW}_{\rho'}(Q^2) + \delta \text{BW}_{\rho''}(Q^2)}{1 + \lambda + \delta} \cdot \frac{\text{BW}_{\omega}(s_2) + \alpha \text{BW}_{K^*}(s_1)}{1 + \alpha} \right|^2 \\
\]

\[
W_{SA} = Q^2 |F_2| F_1^2. \\
\]

In our case of $m_1 = m_3 = m_K$ and $m_2 = m_\pi$, $W_A$ and $W_B$ (Eq. 18) becomes a little simpler, but still complicated. The parameterizations of the resonances as those in KORALB [9] have been discussed above, we need to fit the data to extract parameters $\alpha, \lambda$ and $\delta$, and Alan suggests to introduce two more parameters defining the relative strengths of $F_2/F_1$ and of the anomaly.
contribution $F_{WZ}$ and let the data determine their strengths instead of the model predicted ones, i.e., we have five parameters to be determined from the fits. To extract these five parameters, the method of unbinned extended maximum likelihood will be used.

3 Unbinned Maximum Likelihood Fit

The standard likelihood function, $\mathcal{L}$, has the form (see "Statistics" section in the PDG [12])

$$\mathcal{L} = \prod_{i=1}^{n} \mathcal{P}_i,$$

where $\mathcal{P}_i$ is the probability density function (PDF) evaluated with the experimental observables associated with event $i$, $n$ is the total number of events in the data sample. Here there is no binning of the data. The probability density functions are designed as functions of a set of parameters $\theta = (\theta_1, \ldots, \theta_k)$, the method of maximum likelihood takes the estimators $\hat{\theta}$ to maximize the likelihood function by solving the likelihood equations,

$$\frac{\partial \ln \mathcal{L}}{\partial \theta_i} = 0.$$

In evaluating the likelihood function, it is important that any normalization factor in the PDFs that involved be included.

In the case where the size $n$ of the sample $x_1, \ldots, x_n$ is small, the unbinned maximum likelihood is preferred, since binning can only result in a loss of information. The sample size $n$ can be treated as a Poisson-distributed variable, the so-called "extended maximum likelihood" reads:

$$\mathcal{L} = \frac{e^{-m} m^n}{n!} \prod_{i=1}^{n} \mathcal{P}_i,$$

where $m$ is the number of events predicted by the model.

In this note, the method of unbinned extended maximum likelihood will be used to extract the above five parameters $\alpha$, $\lambda$ and $\delta$, $F_2/F_1$ and $F_{WZ}$ in Eq. 18. Since any observed event can be either a signal or a background with the probability $\mathcal{P}_i$

$$\mathcal{P}_i = \frac{n^S \cdot \mathcal{P}^S_i + n^B \cdot \mathcal{P}^B_i}{n^S + n^B}.$$

Here, $n^S$ and $n^B$ are the numbers of signal and background events with $n = n^S + n^B$. $\mathcal{P}^S_i$ and $\mathcal{P}^B_i$ are the probabilities for event $i$ to be a signal or a background event. These probabilities are functions of these parameters and $Q^2$, $s_1$ and $s_2$, that is:

$$\mathcal{L} = \frac{e^{-m} m^n}{n!} \prod_{i=1}^{n} \frac{n^S \cdot \mathcal{P}^S_i(\tilde{\theta}, Q^2, s_1, s_2) + n^B \cdot \mathcal{P}^B_i(\tilde{\theta}, Q^2, s_1, s_2)}{n^S + n^B}.$$

Here $\tilde{\theta} = (\alpha, \lambda, \delta, F_2/F_1, F_{WZ})$. The probabilities are normalized while being fitted:

$$\int \mathcal{P}^S_i(\tilde{\theta}, Q^2, s_1, s_2) dQ^2 ds_1 ds_2 = 1,$n^B \cdot \mathcal{P}^B_i(\tilde{\theta}, Q^2, s_1, s_2) dQ^2 ds_1 ds_2 = 1.$$
3.1 Signal PDFs

We rewrite Eq. 8 with the scalar part neglected as

$$\frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{\Gamma_{tot} dQ^2 ds_1 ds_2} = \frac{1}{\Gamma_{tot}} \frac{G^2}{12 m_{\tau}} \left( \frac{\cos^2 \theta_C}{\sin^2 \theta_C} \right) \left( \frac{1}{4\pi} \frac{(m_{\tau}^2 - Q^2)^2}{Q^4} \left(1 + \frac{2Q^2}{m_{\tau}^2}\right) \right) (W_A + W_B) \tag{21}$$

where $W_A$ and $W_B$ have been defined in Eq. 18. If we extract the five parameters $\alpha, \lambda$ and $\delta$, $F_2/F_1$ and $F_{WZ}$ in $W_A$ and $W_B$ of Eq. 18 via unbinned maximum likelihood fitting, we can integrate $W_A$ and $W_B$ separately to obtain the contribution from the Wess-Zumino anomaly. The right hand side of Eq. 21 means the probability density of a signal event found to be within $(Q^2, Q^2 + dQ^2)$, $(s_1, s_1 + ds_1)$ and $(s_2, s_2 + ds_2)$. If we integrate over any two of the three variables, we can obtain the expected distribution $T$ for the third one.

$$T(s_1) = \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{\Gamma_{tot} ds_1} = \frac{1}{\Gamma_{tot}} \int_{Q^2} \int_{s_2} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} dQ^2 ds_2, \tag{22}$$

$$T(s_2) = \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{\Gamma_{tot} ds_2} = \frac{1}{\Gamma_{tot}} \int_{Q^2} \int_{s_1} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} dQ^2 ds_1, \tag{23}$$

$$T(Q^2) = \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{\Gamma_{tot} dQ^2} = \frac{1}{\Gamma_{tot}} \int_{s_1} \int_{s_2} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} ds_1 ds_2. \tag{24}$$

$\Gamma_{tot}$ is the total decay width which can be obtained from Eq. 8, the integration intervals are given in Eq. 9-12. We obtain the total width by integrating over $Q^2$, $s_1$ and $s_2$ in different order (see Eq. 22-24) and we get the consistent results. The comparison of the structure obtained from MC using KORALB [9] and integrations with the same model parameters are shown in Fig. 3, the agreement is perfect. The signal PDFs obtained via integrations without the detector simulation shown in Fig. 4. It is clear that the vector and axial vector contributions are well separated.

Due to the finite detection efficiency $\epsilon$, the signal PDFs of the measured mass squared $s_1 = m_{K^+\pi^-}^2$, $s_2 = m_{K^0}\pi^0$ and $Q^2 = m_{K^+K^-\pi^0}^2$ can be designed as follows:

$$P(s_1) = \frac{1}{N_{tot}} \epsilon(s_1) \int_{Q^2} \int_{s_2} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} dQ^2 ds_2,$$

$$P(s_2) = \frac{1}{N_{tot}} \epsilon(s_2) \int_{Q^2} \int_{s_1} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} dQ^2 ds_1,$$

$$P(Q^2) = \frac{1}{N_{tot}} \epsilon(Q^2) \int_{s_1} \int_{s_2} \frac{d\Gamma(\tau \rightarrow 3\nu_{\tau})}{dQ^2 ds_1 ds_2} ds_1 ds_2. \tag{25}$$

Here $N_{tot}$’s are the normalization factors, obtained from the triple integrations over the right sides incorporated with the mass dependent efficiencies with $1/N_{tot}$ omitted, respectively.

3.2 Background PDFs

For the decay $\tau^- \rightarrow K^+K^-\pi^-\nu_\tau$, there are several sources of backgrounds: $\tau$ cross feeds and continuum backgrounds. The dominant backgrounds come from the $\tau$ cross feed decays $\tau^- \rightarrow K^- + \pi^+ - \nu_\tau, \pi^- + \pi^- - \nu_\tau, K^+K^- \pi^0\nu_\tau$ due to the pion mis-identification or missing $\pi^0$ and $K^+K^-\pi^-\nu_\tau$ due to mistags (see Table 2 in CBX-02-29 for detail), each contributes to about 20% of the total backgrounds. These decays have different structures. In Figs. 5 and 6, we use the generated $\tau^- \rightarrow \pi^- + \pi^- + \nu_\tau, K^- + \pi^- - \nu_\tau, K^+K^- \pi^0\nu_\tau, K^+K^-\pi^-\nu_\tau$ and continuum events
and present the mass spectra (normalized to the observed numbers of background events) for these
backgrounds reconstructed as the signal mode $\tau^- \rightarrow K^+K^\pi^-\nu$. It is clear that these different
backgrounds have different structures. For the decay $\tau^- \rightarrow K^+K^\pi^-\nu$, due to mistags, we take
it as our singal which results in 2.24% increase of the overall efficiency. In Fig. 7- 8, we present
the fits to the mass spectra. The fitted functions will be used to model the background PDFs after
normalization. The background PDFs with and without $\tau^- \rightarrow K^+K^\pi^-\nu$ backgrounds due to
mistags will server as a cross check for the fit results.

3.3 Checks of Detector Smearing

The finite detector resolutions result in a broadening of the steeply falling portions of the generated
spectra. The effect can be studied through Monte Carlo simulation with the differences
between the generated and reconstructed masses, which is shown in Fig. 9, it gives us some idea
of the mass resolutions. In Fig. 10, we present the standard deviations of the generated and recon-
structed mass differences which are expected to be Gaussians centered at 0 with resolutions of 1.0
as observed which will be compared with those from the data for studying systematics due to the
detector smearing. In Fig. 11, the mass resolutions in the data and MC are shown, the comparison
shows a good agreement between the data and MC.

3.4 Detection Efficiencies

The detection efficiencies are obtained from signal MC samples with both momentum dependent
particle identification efficiency corrections and dataset dependent PID efficiency corrections taken
into account as done in CBX notes [7]. The kaon and pion identification efficiency using $D^{*+} \rightarrow
D^0\pi^+$ with $D^0 \rightarrow K^-\pi^+$ are shown in Fig. 12. Please note we don’t use RICH to identify pion due
to the dearth of the three-kaon events which in turn helps improve about 10% efficiency. In Fig. 13,
we present the overall efficiencies integrated over all tags as functions of $K^+K^-\pi^-$, $K^+\pi^-$ and
$K^+K^-$ masses which are fitted with $1st$ polynomials. The figures show that the efficiencies are
slightly dependent on the masses. The mass dependent efficiencies will be incorporated to obtain
the signal PDFs in Eq. 25.

4 Fitting and Results

4.1 Fitting Procedure

Minuit [15] is used for the final fit to minimize

$$-2 \ln \mathcal{L} = -2 \left\{ m - n \ln m + \sum_{i=1}^{n} \ln i + \frac{n_s \cdot \mathcal{P}_i^S(\tilde{\theta}, Q^2, s_1, s_2) + n_B \cdot \mathcal{P}_i^B(\tilde{\theta}, Q^2, s_1, s_2)}{n_s + n_B} \right\}.$$

(26)

The parameters in fitting are: $\alpha$, $\lambda$ and $\delta$, $F_2/F_1$ and $F_{WZ}$. 

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4.2 Tests of the Fitting Procedure

The MC sample is used to test the fitting procedure. Fig. 14 shows the fits to the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses of the prior MC sample generated with $\delta = 0$, $F_2/F_1 = 1$ and $F_{WZ} = 1$. The input and fitted parameters of the MC sample are given in Table 1, they agree well. We also use the parameters after fits from the data to generate MC sample to test the fitter, the fitted parameters are in good agreement with the input ones.

Table 1: The input and fitted parameters of MC sample with $\delta = 0$, $F_2/F_1 = 1$ and $F_{WZ} = 1$ fixed.

<table>
<thead>
<tr>
<th>parameters</th>
<th>input</th>
<th>fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.400</td>
<td>0.416±0.014</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-0.365</td>
<td>-0.361±0.005</td>
</tr>
</tbody>
</table>

4.3 Check of $B(\tau^- \rightarrow K^+K^-\pi^-\nu_{\tau})$

Based on more data ($7.77$ fb$^{-1}$, compared to that of $3.26$ fb$^{-1}$ used in CBX notes [7]) collected on or near the $\Upsilon (4S)$ with the CLEO III detector which correspond to $7.09 \times 10^6$ produced tau pairs, we observed $2255$ events that satisfy the event selection criteria. Out of which, $256\pm16\pm45$ events comes from the backgrounds, the errors are statistical and systematic, respectively. The details about the event selection and background estimates can be found in CBX notes [7]. The key point of the event selection is that RICH does help suppress $\tau$ cross feed backgrounds. After the momentum dependent and dataset dependent efficiency corrections, the overall efficiency integrated over tags is $(12.36\pm0.11)\%$. And we obtain $B(\tau^- \rightarrow K^+K^-\pi^-\nu_{\tau}) = (1.58 \pm 0.04(stat.,)) \times 10^{-3}$ which is in good agreement with our published result $B = (1.55 \pm 0.06 \pm 0.09) \times 10^{-3}$. (Out of $7.77$ fb$^{-1}$ data, $4.51$ fb$^{-1}$ is newly available datasets (data10,12-14), in Fig. 12, we also present the PID efficiencies for it. We observe $1323$ events with $151 \pm 14$ background events, the overall efficiency is $(12.28 \pm 0.11)\%$, and we obtain $B(\tau^- \rightarrow K^+K^-\pi^-\nu_{\tau}) = (1.60 \pm 0.05) \times 10^{-3}$, which is consistent with our published results.)

4.4 Fit Results

The fits to the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses for the data are shown in Fig. 15 with different contributions overlaid. The fitted results are $\alpha = 0.471 \pm 0.060$, $\lambda = -0.314 \pm 0.073$, $\delta = 0.101 \pm 0.020$, $F_{WZ} = 3.225 \pm 0.257$ and $F_2/F_1 = 0.976 \pm 0.153$ which yields the vector (W-Z anomaly) and axial vector currents contribute to $R_{WZ} = \frac{\Gamma_{WZ}}{\Gamma_{tot}} = (55.7 \pm 8.4)\%$ and $R_{AV} = \frac{\Gamma_{AV}}{\Gamma_{tot}} = (44.3 \pm 8.4)\%$ of the total decay width of $\tau^- \rightarrow K^+K^-\pi^-\nu_{\tau}$, respectively. Out of which, the intermediate states via $\rho^0\pi^- \rightarrow (K^+K^-)\pi^-$ and $\omega\pi^- \rightarrow (K^+K^-)\pi^-$ contribute $R_{\rho^0\pi^-} = (2.5 \pm 0.8)\%$ and $R_{\omega\pi^-} = (3.4 \pm 0.9)\%$ for the axial-vector and vector processess, respectively, and $K^*\pi^- \rightarrow (K^+\pi^-)K^-$ contributes $R_{K^*\pi^-} = (46.8 \pm 8.4)\%$ and $R_{K^*\pi^-} = (60.8 \pm 8.5)\%$.

\footnote{We can reproduce exactly the same results as those given at July CLEO meeting with the same parameterization of $\rho$, $\rho'$ and $\rho''$ and three free parameters as before, while we have five free parameters here.}

\footnote{The fits with $\tau^- \rightarrow K^+K^-\pi^-\nu_{\tau}$ due to mistags taken as signal or background (see Figs. 7-8) give consistent results.}
they do not sum up to the total contributions from the axial-vector and vector currents due to the interference between the two intermediate states.

5 Systematics

There are some sources which may affect the fit results: the efficiencies, detector smearing, the scalar contribution, background level and PDFs, parameterizations of $\rho$, $\rho'$ and $\rho''$ which will be discussed below. We determine the systematics by varying these quantities by 10% or by $\pm\sigma$. Here are the details:

- Efficiencies: the slopes of the fitted efficiency functions in Fig. 13 are varied by 10% instead of the bin-by-bin mass dependent efficiencies since the overall efficiencies will not affect the fitting results, only the shapes of the efficiencies will;

- detector smearing: since we obtain the signal PDFs from integrations, it means we do not consider the detector smearing effects on the signal PDFs. The effects can be studied by comparing the signal PDFs obtained from the generated signal MC before and after reconstruction. Unfortunately, our MC does not model the mass resolutions well. To study the effects, we select samples of $D^0 \rightarrow K^-\pi^+$, $D^0 \rightarrow K^+K^-$ from $D^{*+} \rightarrow D^0\pi^+$ decays and $D_s^- \rightarrow K^0K^-$ with $K^0 \rightarrow K^+\pi^-$ and obtain the standard deviations of $D^0$ and $D_s^-$ masses. $dE/dx$ and RICH are combined to provide pion and kaon separation. For $D^0 \rightarrow K^-\pi^+$ and $D_s^- \rightarrow K^0K^-$ decays, their momenta are required to be greater than 2.0 GeV/c and 3.0 GeV/c to suppress the combinatorial backgrounds, respectively; while such a requirement is not applied for $D^0 \rightarrow K^+K^-$ decay to increase the statistics. The $\Delta m = M_{D^{*+}} - M_{D^0}$ and $K^0$ sidebands are used to subtract the remaining combinatorial backgrounds. In Figs. 16 - 18, we present the fits to the $D^0$ and $D_s^-$ masses and the standard deviations of the masses from the fitted values after the backgrounds have been subtracted. The fits show the measured masses are systematically 0.5 MeV/$c^2$ lower than the PDG values [12]. We throw three random standard deviations for the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses according to the distributions of the standard deviations from the $D^0$ and $D_s^-$ samples. After the mass errors ($\delta\sigma$) scaled by these three random numbers ($R$), together with the generated masses ($G$), we obtain the reweighted mass ($G + R\delta\sigma$) spectra with the mass resolutions obtained from the data. In Fig. 19, we present the comparison among the generated, reweighted and reconstructed masses, the agreement is very good. (The mass errors show good agreement between the data and MC, see Fig. 11) The differences between the fitted parameters obtained from the reweighted and generated masses are taken as the systematic uncertainties due to the detector smearing.

- scalar contribution: the scalar component is turn on. We tried to let the relative contribution from the scalar part float, i.e., we have one more free parameter and six in total, but the fit failed to converge, so the relative contribution is fixed at the model predicted value to obtain the systematics from it, while the scalar contribution is only of $10^{-4}$ of the total decay width.

- background error: it is varied by one standard deviation $\pm\sigma$;

- background PDFs: the correlations between the parameters of the fitted background PDFs in Fig. 7 are considered using their error matrix from the fits, the background PDFs are varied by one standard deviation $\pm\sigma$ are shown in Fig. 20;
parameterizations of \( \rho, \rho' \) and \( \rho'' \) which directly affect the results, their masses and widths are varied by \( \pm \sigma \) to evaluate their contributions to the systematics. Since \( \rho' \) and \( \rho'' \) are not well established from the PDG [12], it will contribute the dominant part of the systematics.

Their contributions to the systematics are listed in Table 2, the dominant systematics comes from the parameterizations of the \( \rho' \) and \( \rho'' \). There is a large uncertainty in the contribution of the \( \rho'' \) resonance to the vector (W-Z) current. This has little effect on the total vector (W-Z) contribution because it is strongly suppressed by phase space, and its contribution is easily absorbed into the contribution from the \( \rho' \) resonance. We obtain \( \alpha = 0.471 \pm 0.060 \pm 0.034, \lambda = -0.314 \pm 0.073 \pm 0.080, \delta = 0.101 \pm 0.020 \pm 0.156, F_{WZ} = 3.225 \pm 0.257 \pm 1.90, F_2/F_1 = 0.976 \pm 0.153 \pm 0.051 \) and the contributions from the vector (W-Z anomaly) and axial vector currents to the decay width of \( \tau^- \to K^+K^0\pi^-\nu_\tau \) are \( R_{WZ} = \frac{\Gamma_{WZ}}{\Gamma_{tot}} = (55.7 \pm 8.4 \pm 4.9)\% \) and \( R_{AV} = \frac{\Gamma_{AV}}{\Gamma_{tot}} = (44.3 \pm 8.4 \pm 4.9)\% \), respectively. The intermediate states \( \rho^0\pi^- \to (K^+K^-)\pi^- \) and \( K^{*0}K^- \to (K^+\pi^-)K^- \) contribute \( R_{AV}^{\rho^0} = (2.5 \pm 0.8 \pm 0.4)\% \) and \( R_{AV}^{K^*} = (46.8 \pm 8.4 \pm 5.2)\% \) for the axial vector current processes, and \( \omega\pi^- \to (K^+K^-)\pi^- \) and \( K^{*0}K^- \to (K^+\pi^-)K^- \) contribute \( R_{WZ}^{\omega} = (3.4 \pm 0.9 \pm 1.0)\% \) and \( R_{WZ}^{K^*} = (60.8 \pm 8.5 \pm 6.0)\% \) for the vector current ones (W-Z anomaly) which presents the first evidence for \( \omega \to K^+K^- \) decays.

One thing worth mentioning is that when we study the systematics, we find that no matter what varies:

- the parameter \( \alpha \) does not change much;
- the ratio \( F_2/F_1 \) does not change much;
- the anomaly contribution \( R_{WZ} \) does not change much (< 5\%);

### 6 Discussions

The direct and model independent study of the Wess-Zumino anomaly in the decay can be obtained through the analysis of the structure functions, similar to that done for \( \tau^- \to \pi^-\pi^0\pi^0\nu_\tau \) [13], but it demands much larger statistics and it only gives the qualitative evidence of the presence of the anomaly. Its quantitative contribution can be determined via the study of the mass spectra \((Q^2, s_1 \text{ and } s_2)\) as done above, while it gives indirect evidence of its presence and model dependent results since the parameterizations of the form factors are model dependent. We are trying to search for direct evidence of the presence of the the Wess-Zumino anomaly in the decay. In Fig. 21a-c, we present the comparison of the angular distributions of \( \cos \beta, \cos \theta \) and \( \cos \psi \) in Eqs. 2, 4, 5 between the data and the MC samples generated using the parameters from the fits to the data with both two components (axial vector and vector contributions) and axial vector contribution only. The comparison shows that the MC samples with and without the Wess-Zumino contribution both describes the data well within current statistics. It means these angular distributions are insensitive to the anomaly contribution.

The above analysis shows that the decay proceeds through both the hadronic axial vector \((J^P = 1^+)\) and vector \((J^P = 1^-)\) currents, the difference in spin-parity assignments for each component results in different angular distributions of the \( K^+ (\theta_K) \) in the \( K^{*0} \to K^+\pi^- \) rest frame evaluated in the hadronic rest frame. The expected angular distributions are uniform and \( 1 + 3 \cos^2 \theta \) for the axial vector current with s-wave \((L = 0)\) and d-wave \((L = 2)\) [16], respectively, and \( \sin^2 \theta \) for the
Table 2: Contributions to systematics, see details in the text, the larger errors are taken as systematics in case of +σ or −σ variations. Please note $R_{WZ} + R_{AV} = 1.0$, so we have $ΔR_{WZ} = −ΔR_{AV}$ and the systematics for $ΔR_{AV}$ is not shown in the table. The total systematic error is quadratic sum over individuals with the larger one used if two are available.

<table>
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<th>sources</th>
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<tr>
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<td>±0.002</td>
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<td>±0.017</td>
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vector current as shown in Fig. 22. Although the overall angular distribution for the axial vector current is totally unknown (since s- and d- wave contributions are unknown), the observed angular distribution favors that of the vector contributions over that of the axial vector contributions if the vector current overwhelmingly dominates the decay process. The vector current tends to contribute at higher $Q = M_{KK\pi}$, see Fig. 15, and the axial vector current at lower $Q$, it is expected that the angular distribution at higher $Q$ should favor that of the vector current contribution (convex) if it appears in the decay. In Fig. 21d, the distributions of the decay angle of the $K^+$ are shown for the data with $M_{KK\pi} > 1.5$ GeV/$c^2$ and $M_{KK\pi} < 1.5$ GeV/$c^2$, it shows the different shapes at higher and lower $M_{KK\pi}$. In Fig. 23, we present the fits to the $\cos \theta_K$ with a function of $p_1(1 + p_2 \cos^2 \theta_K)$ for $M_{KK\pi} > 1.5$ GeV/$c^2$ and $M_{KK\pi} < 1.5$ GeV/$c^2$. The fits show the different trends at lower and higher $KK\pi$ masses which presents the direct evidence of the presence of the Wess-Zumino anomaly in the decay as discussed above.

7 Summary

In summary, the direct evidence of the presence of the Wess-Zumino anomaly in the $\tau^- \rightarrow K^+K^-\pi^-\nu_\tau$ decay is given via the study of the angular distribution of the $K^+$ ($\theta_K$) in the $K^{*0} \rightarrow K^+\pi^-$ rest frame evaluated in the hadronic rest frame. We present the first study of the Wess-Zumino anomaly in the decay and determine its quantitative contribution $(55.7 \pm 8.4 \pm 4.9)\%$ to the decay width, and the axial vector current processes $R_{AV} = \frac{F_{AV}}{F_{tot}} = (44.3 \pm 8.4 \pm 4.9)\%$. For the axial vector processes, the contributions from two intermediate states $\rho^0\pi^- \rightarrow (K^+K^-)\pi^-$ and $K^{*0}K^- \rightarrow (K^+\pi^-)K^-$ are $R^0_{AV} = (2.5 \pm 0.8 \pm 0.4)\%$ and $R^{K^*K^-}_{AV} = (46.8 \pm 8.4 \pm 5.2)\%$, respectively, and $\omega\pi^- \rightarrow (K^+K^-)\pi^-$ and $K^{*0}K^- \rightarrow (K^+\pi^-)K^-$ are $R^{\omega\pi^-}_{WZ} = (3.4 \pm 0.9 \pm 1.0)\%$ and $R^{K^*K^-}_{WZ} = (60.8 \pm 8.5 \pm 6.0)\%$ for the vector current ones (W-Z anomaly). These four fractions do not sum up to 1 since there exists the interference between the two intermediate states. We present the first evidence for $\omega \rightarrow K^+K^-$ decay. The structure of the decay with the parameters $\alpha = 0.471 \pm 0.060 \pm 0.034$, $\lambda = -0.314 \pm 0.073 \pm 0.080$, $\delta = 0.101 \pm 0.020 \pm 0.156$, $F_{WZ} = 3.225 \pm 0.257 \pm 1.90$, $F_2/F_1 = 0.976 \pm 0.153 \pm 0.051$ is also investigated which in turn helps improve modeling the decays of $\tau \rightarrow KK\pi\nu_\tau$.

Acknowledgments

We like to thank Guangpei Chen and Alan Weinstein for helpful discussions and our paper committee members: David Besson (chair), Alan Weinstein and Hanna Mahlke-Krueger, Jean Duboscq and Jon Urheim for their helpful comments and suggestions.

References


[13] All CLEO results on the branching fractions and resonance contents of the decays $\tau^- \rightarrow (\beta\pi)^-\nu_{\tau}$ have been implemented in the Monte Carlo program. For experimental results, see D. M. Asner et al., Phys. Rev. D 61, 012002 (2000), see also A. Weinstein, hep-ex/0210058, talk given at Tau’02, Santa Cruz, California, September 2002, to appear in the proceedings.


Figure 2: Comparisons of the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses with MC generated with the default model KORALB[9] based on a data sample of 3.26 fb$^{-1}$[7].
Figure 3: Comparisons of the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses from MC (dashed lines) and integrations (full lines) with the same model parameters.
Figure 4: Probability density functions for the $K^+K^−\pi^−$, $K^+\pi^−$ and $K^+K^−$ masses of $\tau^− \rightarrow K^+K^−\pi^−\nu_\tau$ decays with the same model parameters $\alpha = 0.4$, $\beta = -0.365$ and $\delta = 0$, used in the paper [7], while the model KORALB uses $\alpha = -0.2$, $\beta = -0.25$ and $\delta = -0.0385$ (default). The dashed lines represent the contribution from the Wess-Zumino anomaly, dotted lines from the axial vector current, and the full lines are the sum of two. It is clear that the two components are well separated.
Figure 5: Mass spectra for the normalized backgrounds of \( \tau^- \to \pi^- \pi^+ \pi^- \nu_\tau \) (top row), \( K^- \pi^+ \pi^- \nu_\tau \) (middle row) and \( K^+ K^- \pi^0 \nu_\tau \) (bottom row) reconstructed as the signal decay \( \tau^- \to K^+ K^- \pi^- \nu_\tau \).
Figure 6: Mass spectra for the normalized background from the $\tau^- \rightarrow K^+ K^- \pi^- \nu_\tau$ decay (top row) due to mistagging, continuum backgrounds (middle row) and the total backgrounds (bottom row) reconstructed as the signal decay $\tau^- \rightarrow K^+ K^- \pi^- \nu_\tau$. 
Figure 7: Fits to the $K^+K^−\pi^−$ mass with a Breit-Wigner plus a 1st order polynomials (top left), the $K^+K^−$ mass with a 5th order Chebyshev polynomials (top right), and the $K^+\pi^−$ mass with a Breit-Wigner plus a 6th order Chebyshev polynomials $[p_0 + p_1 \cdot (\Delta x) + \cdots + p_5 \cdot ((\Delta x)^5) \cdot ((\Delta x)^{\frac{5}{2}})/0.02$ with $\Delta x = x - of f set$ and bin width=0.02 (bottom). The fitted function will be used to model the background PDFs after normalization.
Figure 8: Fits to the $K^+K^-\pi^-$ mass with a Breit-Wigner plus a 1st order polynomials (top left), the $K^+K^-$ mass with a 5th order Chebyshev polynomials (top right) and the $K^+\pi^-$ mass with a Breit-Wigner plus a 6th order Chebyshev polynomials (bottom) without the background from the decay $\tau^- \rightarrow K^+K^-\pi^-\nu_\tau$ due to mistagging which can be treated as signals with proper corrections (+2.24% higher) to the efficiencies.
\[ \Delta m_{KK} = m_{\text{reconstructed}} - m_{\text{generated}} \text{ (GeV/c}^2) \]

Figure 9: Fits to the reconstructed and generated mass differences with 3 Gaussians for \( K^+K^-\pi^- \) (top left), \( K^+\pi^- \) (top right) and \( K^+K^- \) (bottom) masses.
Figure 10: Fits to the standard deviations of the reconstructed and generated mass differences with 2 Gaussians for $K^+K^−\pi^−$ (top left), $K^+\pi^−\pi^0$ (top right) and $K^+K^−\pi^−\pi^0$ (bottom) masses.
Figure 11: Comparison of the mass errors (full lines: data, dashed lines: MC) obtained from the track error matrix.
Figure 12: Kaon and pion identification efficiencies using $D^{*+} \rightarrow D^{0}\pi^+$ with $D^0 \rightarrow K^-\pi^+$. 
Figure 13: Efficiencies as functions of $K^+K^-\pi^-$ (top left), $K\pi$ (top right) and $K^+K^-$ (bottom) masses. The fits with 1st order polynomials are overlaid. There is large fluctuation in last two $KK\pi$ mass bins due to low statistics (several events observed in these two bins).
Figure 14: Fits to the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses of MC sample.
Figure 15: Fits to the $K^+K^-\pi^-$, $K^+\pi^-$ and $K^+K^-$ masses of the data.
Figure 16: Fits to the $K^-\pi^+$ mass (left) and the standard deviation (right) of the $D^0$ mass from $D^{*+} \rightarrow D^0\pi^+$ decays with $D^0 \rightarrow K^-\pi^+$ in the data with $K/\pi$ identified by combined loglikelihood RICH+dE/dx and $p_{D^0} > 2.0$ GeV/c (corresponding to continuum $D^{*+}$ production).
Figure 17: Fits to the $K^−K^+$ mass (left) and the standard deviation (right) of the $D^0$ mass from $D^{*+} \rightarrow D^0\pi^+$ decays with $D^0 \rightarrow K^-K^+$ in the data with kaons identified by combined loglikelihood RICH+dE/dx.
Figure 18: Fits to the $K^{*0}K^-$ mass (left) and the standard deviation (right) of the $D_s^-$ mass from $D_s^- \rightarrow K^{*0}K^-$ decays with $K^{*0} \rightarrow \pi^- K^+$ in the data with kaons and pion identified by combined loglikelihood RICH+dE/dx and $p_{D_s^-} \geq 3.0$ GeV/c (corresponding to continuum $D_s^-$ production).
Figure 19: Comparison of the MC generated, reconstructed and reweighted mass spectra.
Figure 20: The fitted background PDFs vary by one standard deviation obtained from the error matrix of the fitted parameters.
Figure 21: Comparison of the angular distributions (a-c) between the data with statistical errors only and MC samples with two components (axial and vector, the fitted results are used) and axial vector component only. The distributions of the $K^+$ helicity angle in the $K^{*0} \rightarrow K^+\pi^-$ rest frame evaluated in the three hadrons rest frame with the hadronic mass less or greater than 1.5 GeV/$c^2$ are presented in d). This mass cut us imposed to enhance the vector or axial vector current contributions motivated by Fig. 15 (top left).
Figure 22: The different angular distributions of the $K^+$ in the $K^{*0} \to K^+\pi^-$ rest frame evaluated in the hadronic rest frame for the axial-vector ($J^P = 1^+$, uniform for the $L = 0$ s-wave and $1 + 3\cos^2\theta$ for the $L = 2$ d-wave) and vector ($J^P = 1^-$, $\sin^2\theta$) current processes.
Figure 23: The fits to the angular distributions of the $K^+$ in the $K^*^0 \rightarrow K^+\pi^-$ rest frame evaluated in the hadronic rest frame with $M_{KK\pi} > 1.5 GeV/c^2$ (left) and $M_{KK\pi} < 1.5 GeV/c^2$ (right).