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Spontaneously broken quark helicity symmetry

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Abstract

We show how quark helicity symmetry can be spontaneously broken in $SU(N)$ gauge theory when that theory is quantized in the light-cone representation. The symmetry breaking is implemented by induced operators. These operators result from the fact that the vacuum is not trivial, but involves a condensate of fermionic zero modes. We show that the light-cone eigenvalue problem can be equivalently studied in a trivial vacuum with the induced interactions, which do not conserve quark helicity, included in P^-. These interactions generate a splitting between pi and rho meson masses. As an example, we calculate the meson spectrum in a dimensionally reduced large-N gauge theory. The induced interactions may also provide a linear dependence of the pion mass squared on the quark bare mass which is not manifest in their absence.

I. INTRODUCTION

One of the pillars of hadronic physics is the spontaneous breaking of chiral symmetry. Phenomenologically, one of the most important aspects of that symmetry breaking has been the existence of the chiral condensate, $\langle\bar{\psi}\psi\rangle$. The success of the QCD sum rules [1], for example, suggests that chiral and other condensates play an important role in extracting hadronic physics from QCD. On the other hand, it may also be that an effective theory could be developed in which the effects of the condensates appear in another guise. The 't Hooft model of 2 dimensional QCD [2], for example, has chiral and other condensates [3], yet the spectrum can be obtained from a light-cone hamiltonian in a trivial vacuum. When quantizing on the light-cone, the study of chiral symmetry is difficult. In that representation, chiral symmetry is a dynamical symmetry in that the generator includes the dynamical light-cone hamiltonian operator P^-. Chiral symmetry cannot be checked until the problem is solved. Furthermore, in practical calculations where regulators must be used, it is not easy to ensure that the regulators do not break chiral symmetry explicitly.

In the early 1970's some attempts were made to study the problem in terms of charges and pseudocharges that are kinematical in the light-cone representation. This early literature has been reviewed by Mustaki [4]. The light-cone symmetry studied in that work, whose conserved charge $\tilde{Q}_5$ is quark helicity, can remain a symmetry even when chiral symmetry is explicitly broken by constituent masses. If the quark helicity symmetry is spontaneously broken, the usual pattern of light pseudoscalars could emerge. Moreover, $\tilde{Q}_5$ is not conserved even in chirally symmetry theories. Because of the differences between the operators that are easy to study in the light-cone representation and the operators that are connected with chiral symmetry, the whole subject of chiral symmetry breaking in the light-cone representation has generally involved testing for simple consequences of the spontaneously broken symmetry rather than testing the symmetry itself. One example is to look for the existence of a chiral condensate. If the traditional light-cone quantization procedures are used, there is no chiral condensate. One consequence of the spontaneously broken symmetry, which we shall study in the present paper, is mass splitting between π and ρ mesons. That issue can be formulated in terms of the kinematical light-cone symmetries. We can therefore study this question separately from the more complicated question of chiral symmetry.

One case, where chiral symmetry is broken and the connection between the equal-time and the light-cone representations can be explicitly given, is the Schwinger model. In that case, there exists a complete operator solution which can be evaluated either at $x^0 = 0$ or at $x^+ = 0$ [5]. Since there is one operator solution, if it has a condensate when quantized at equal time there must also be one when quantized on the light cone. In fact there is a chiral condensate in both representations. The way it occurs in the light-cone representation is that the bare light-cone vacuum is dressed by operators that occur in integration constants in the solution of constraint equations. From the operator solution we can learn that these integration constants are not zero and that the vacuum state formed through their inclusion is the same state as the vacuum found if the theory is quantized at equal time. The existence of the condensate ”induces” an operator into the dynamics [6]. The induced operator is not a new operator — it is a part of the P^- found if the theory is quantized at equal time — but it is missed if the traditional form of light-cone quantization is employed. An important feature of the system is that, once the new operator is included, the vacuum can be taken to be the traditional trivial light-cone vacuum without loss of generality.

In this paper, we develop an realization of the idea that QCD contains vacuum structures similar to those that can be explicitly demonstrated in the Schwinger model. As in the case of the Schwinger model, this structure involves a mechanism whereby fermionic light-cone zero modes induce interactions in an effectively trivial vacuum. We shall describe these operators and shall provide example nonperturbative spectrum calculations. The vacuum we use here, and the induced operators which exist due to it, are probably not complete — for instance, there is not yet a gluon condensate. The glue in the vacuum may occur through a mechanism similar to the one which we show here for fermions, but we do not yet know the details.

The results of this paper are as follows. In the next section we briefly review the tools we will need: quark helicity symmetry; the vacuum of the Schwinger model. In section III we derive the form of the induced interactions that spontaneously break quark helicity symmetry in $SU(N)$ gauge theory. We then obtain some exact solutions of simplified models containing only these induced interactions in section IV. Adding gauge interactions in a dimensionally reduced model in section V, we then perform DLCQ calculations of the meson spectrum in this non-trivial example to illustrate the pi-rho splitting that follows from non-conservation of quark helicity. We give some conclusions in section VI.

II. HELICITY SYMMETRY AND THE VACUUM

A. Quark helicity symmetry

Unless otherwise stated, our metric is $x^{\pm} = x^0 \pm x^3$, where x^+ is the light-front time variable. We decompose a spinor $\psi = \psi_+ + \psi_-$ into the projections $\psi_\pm = \frac{1}{2}\gamma_\mp\gamma_\pm\psi$. Typically, the ψ_- component satisfies a constraint equation whose source term is ψ_+. At this stage there arise two possibilities for 'chiral' rotations of the fermi field. The usual chiral transformation is defined by

$$\psi \to \mathrm{e}^{-\mathrm{i}\gamma_5\theta}\psi \ , \tag{1}$$

while a light-cone chiral transformation can be defined by

$$\psi_+ \to \mathrm{e}^{-\mathrm{i}\gamma_5\theta}\psi_+ \ , \tag{2}$$

with the transformation of ψ_- determined by its constraint equation. The transformation (2) may be a symmetry even though (1) is not. A well-known example of this is a free massive fermion, with Lagrangian density

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\gamma_\mu\partial^\mu - \mu\right)\psi \ . \tag{3}$$

The constraint equation in this case is solved formally by

$$\psi_- = \frac{\mu + \mathrm{i}\gamma_\perp\cdot\partial_\perp}{\mathrm{i}\partial_-}\gamma^0\psi_+ \ , \tag{4}$$

where $\partial_\perp = (\partial_1, \partial_2)$. Eq. (4) is inconsistent with (1), though not (2). We consider the boundary condition for the anti-derivative $1/\partial_-$ in the next subsection; it will be the source of vacuum structure that leads to spontaneous breaking of symmetry under (2).

The charges that generate (1)(2) are also different, in particular they're defined with respect to different hypersurfaces. In the absence of derivative interactions, the chiral charge is

$$Q_5 = \int \bar{\psi}\gamma^0\gamma_5\psi \ d^2x_\perp \ dx^3 \ . \tag{5}$$

while the light-cone chiral charge is

$$\tilde{Q}_5 = \int \bar{\psi}\gamma^+\gamma_5\psi \ d^2x_\perp \ dx^- \ . \tag{6}$$

The latter measures (twice) the total fermion and anti-fermion helicity or, equivalently, the spin projection along x^3. In a coordinate system moving at light speed, even a massive particle has maximal helicity. To avoid confusion later and maintain the distinction from chiral symmetry (1), we shall hereafter refer to (2) as *quark helicity symmetry*. A key question is whether the quark helicity is conserved [7] since, if it is, it seems impossible to split the pion and helicity zero component of the rho in the spectrum of $SU(N)$ gauge theory [8, 9, 10]. We will show in section III that spontaneous breaking of symmetry (2) can occur via x^- zero modes. We note that another approach to the problem, taken in ref.[11], was to derive a gap equation (assuming a trivial vacuum) and then gradually remove explicit quark helicity symmetry breaking. The model studied there was not exactly a gauge theory – it was closer to a Yukawa theory – but it also suggested that fermion helicity may be spontaneously non-conserved.

B. The Schwinger Model

The one case for which, to date, zero mode induced operators of the kind that we will use have been carefully derived in detail is the Schwinger model (QED in $1+1$ dimensions). Here, we shall simply review the results, referring to the literature [5, 6, 12] for details and derivations. In the massive case, the abelian action is

$$S = \int d^2x \left[-\frac{1}{4} \operatorname{Tr} F_{\alpha\beta}F^{\alpha\beta} + \mathrm{i}\bar{\psi}\gamma^{\alpha}_{(2)} D_\alpha \psi - \mu\bar{\psi}\psi \right] \ , \tag{7}$$

$\alpha, \beta \in \{+, -\}$, $F_{\alpha\beta}$ is the field strength, $\gamma^{\alpha}_{(2)}$ the two-dimensional representation of the gamma matrices, and $D_\alpha = \partial_\alpha + \mathrm{i}eA_\alpha$. In light-cone gauge $\partial_- A_- = 0$, the constraint relation is still the same as that of a free fermion

$$\partial_- \psi_- = -\mathrm{i}\mu\psi_+, \tag{8}$$

where μ is the bare mass. In general this must be solved as

$$\psi_- = \psi_-^0(x^+) - \mathrm{i}\frac{1}{2}\mu \int \psi_+ dx^-. \tag{9}$$

Here, $\int$ is the antiderivative which just replaces $\mathrm{e}^{\mathrm{i}kx}$ with $\frac{1}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}kx}$ in the Fourier expansion of the field. Note that the x^--zero mode $\psi_-^0(x^+)$ is a field that is independent of ψ_+. $\psi_-^0(x^+)$ has the bosonized form

$$\psi_-^0 = Z_-(\mu)\mathrm{e}^{\Lambda_-^{(-)}(\mu)}\sigma_-\mathrm{e}^{\Lambda_-^{(+)}(\mu)}, \tag{10}$$

where Λ_- is a bosonic field depending on x^+, σ_- is a (space-time independent) spurion and Z_- is a wave function renormalization constant. (+) and (-) refer to positive and negative frequency parts. We write the field ψ_+ similarly in bosonized form as

$$\psi_+ = Z_+ \mathrm{e}^{\Lambda_+^{(-)}} \sigma_+ \mathrm{e}^{\Lambda_+^{(+)}} \ . \tag{11}$$

The physical vacuum is given by

$$|\Omega(\theta)\rangle \equiv \sum_{M=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}M\theta} |\Omega(M)\rangle \quad ; \quad |\Omega(M)\rangle = (\sigma_+^* \sigma_-)^M |0\rangle, \tag{12}$$

where $\sigma^{-1} \equiv \sigma^*$. The existence of these vacua, and their form, can be determined from the fact that we have residual gauge transformations and that $\sigma_+^* \sigma_-$ is the generator of those residual gauge transformations. The mass mass operator then takes the form

$$\begin{aligned} \mu \int \bar{\psi}\psi \, dx^- \;&=\; \mu Z_- Z_+ {:}\mathrm{e}^{\Lambda_-}{:} \int_{-\infty}^{\infty} \left(\sigma_-^* \sigma_+ (\mathrm{e}^{\Lambda_+^{(-)}} \mathrm{e}^{\Lambda_+^{(+)}} - 1) + C.C. \right) dx^- \\ &+\; \frac{1}{2}\mu^2 \int_{-\infty}^{\infty} \left(\psi_+^*(0, x^-) \left[- \int \mathrm{i} \frac{1}{2} \psi_+ \right] + C.C. \right) dx^-, \end{aligned} \tag{13}$$

where $C.C.$ is complex conjugate. The field Λ_- is unphysical, so in the physical subspace we can replace ${:}\mathrm{e}^{\Lambda_-}{:}$ by 1. The term linear in μ is our first example of an induced operator. Fortunately, we do not have to include the full complications of the physical vacuum in practical calculations. The reason is that the combination of spurions $\sigma_-^* \sigma_+$ acts as a c-number in the physical subspace, since $|\Omega(\theta)\rangle$ is an eigenstate of it. Therefore, each component of the vacuum transforms the same under all the dynamical operators. We may therefore just take the bare (trivial) light-cone vacuum $|0\rangle$ as the representative case. If we kept the full vacuum we would just have an infinite number of copies of the exact same algebra. With that understanding, the full dynamics of the system, including the complex vacuum, is obtained by using the bare vacuum but including the induced operator with the appropriate c-number replacing spurions.

Before moving on the the problem of determining the wave function renormalization constants, we should discuss the meaning of ${:}\mathrm{e}^{\Lambda_+}{:}$ that we used in Eqn.'s (13). To discuss the linear growth of the mass of the Schwinger particle with the bare fermion mass due to chiral symmetry breaking (at least in the usual language) we need two versions of the operator $\bar{\psi}\psi$. The first, $\bar{\psi}\psi_1$, is given by $\bar{\psi}\psi_1 = Z_- Z_+ \sigma_-^* \sigma_+ \mathrm{e}^{\Lambda_+^{(-)}} \mathrm{e}^{\Lambda_+^{(+)}} + C.C.$ It is $\bar{\psi}\psi_1$ that has the chiral condensate. For instance, at $\mu = 0$

$$\langle \Omega(\theta) | \bar{\psi}\psi_1 | \Omega(\theta) \rangle = -\frac{e}{2\pi^{3/2}} e^{\gamma} \cos\theta. \tag{14}$$

But the operator that is included in P^- has a further subtraction which removes the expectation in the physical vacuum (this subtraction is not exactly a c-number but it acts like a c-number in the physical subspace). This operator, $\bar{\psi}\psi_2$, with the further correction is the operator in (13). We have $\bar{\psi}\psi_2 = Z_- Z_+ \sigma_-^* \sigma_+ (\mathrm{e}^{\Lambda_+^{(-)}} \mathrm{e}^{\Lambda_+^{(+)}} - 1) + C.C.$ With all this machinery we have the result that the growth of the mass squared of the Schwinger particle with μ is given by $-4\pi\mu\langle\Omega(\theta)|\bar{\psi}\psi_1|\Omega(\theta)\rangle\cos\theta$, but the vacuum remains translationally invariant since it is $\bar{\psi}\psi_2$ that is used in (13). The question of how to extend these considerations to the case of QCD will be important below.

The wave function renormalization constant, Z_-, is determined by the relation

$$\{\psi_-(x), \psi_-^*(y)\} = \delta(x^1 - y^1). \tag{15}$$

This relation necessarily involves the values of ψ_- off the initial value surface, $x^+ = 0$. Roughly speaking, the complex dynamical problem of determining the vacuum in equal-time quantization is replaced by the complex dynamical problem of determining Z_- in light-cone quantization. But Z_- is just a constant and, while in principal it can be determined from the above relation, it can also be fit to data, to a symmetry or to any other property of the correct solution that is sensitive to Z_-. In the case $\mu = 0$, a full operator solution for the Schwinger model can be given and Z_- is known. Starting from that known value, Z_- can be expanded in a power series in μ. With the induced operator in place, all the well verified results from equal-time quantization are reproduced in the light-cone representation [6].

If we now consider the case where the system is periodic in x^- with period L (DLCQ), which is useful for performing numerical calculations, we find that there are two important differences as compared to the continuum case we have been discussing above. The first difference is that both spurions , σ_- and σ_+, become dependent on the space-time variables, in particular, on x^-. The relations are

$$\sigma_+(x) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4Le}(x^- - x^+)Q_+} \sigma_+(0) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4Le}(x^- - x^+)Q_+} \tag{16}$$

$$\sigma_-(x) = \mathrm{e}^{-\mathrm{i}\frac{\pi}{4Le}(x^+ - x^-)Q_-} \sigma_-(0) \mathrm{e}^{-\mathrm{i}\frac{\pi}{4Le}(x^+ - x^-)Q_-}. \tag{17}$$

where Q_+ (Q_-) measures twice the electric charge of ψ_+ (ψ_-) fields. The other important difference is associated with the fact that in the periodic case on the light-cone, the chiral condensate goes to zero as the periodic length, L goes to ∞. More precisely

$$\langle\Omega(\theta)|\bar{\psi}\psi|\Omega(\theta)\rangle = -\frac{1}{L}\cos\theta, \tag{18}$$

where θ is the vacuum angle. To restore equality with the continuum, whether quantized at equal-time or on the light-cone, and with the periodic case quantized at equal time, we must keep the wave function renormalization constant, Z_-. It is not surprising that we must keep this constant: determining it was a complicated dynamical problem in the continuum and we should not expect to avoid that problem by introducing periodicity conditions. The difference here is that Z_- is not determined by a space-like anticommutator as it is in the continuum. The reason is that Z_- is determined by the behavior of the system near $p^+ = 0$ when the singularity is regulated in a way consistent with Lorentz invariance and gauge invariance. Once we have regulated that singularity by introducing the periodicity conditions, which violate both Lorentz invariance and gauge invariance, we have lost the information needed to determine Z_- and we do not regain it by taking the limit $L \to \infty$. Indeed, the only known way to determine Z_- in the periodic case is to compare with the continuum solution. For $\mu = 0$, the solution is

$$Z_- = \frac{eLe^{\gamma}}{2\pi^{3/2}}. \tag{19}$$

Since we cannot in general calculate from first principles constants like this in the periodic case, it must be fit in phenomenological applications that use DLCQ. We need to be aware that in a DLCQ calculation Z_- may depend on the periodicity length, L.

Apart from those two differences, the periodic case is much like the continuum case. In particular, the vacuum still has the form (12). That form can be determined from the fact that in light-cone gauge we have a residual gauge invariance corresponding to the gauge function $\frac{\pi}{L}x^-$ and that the generator of that residual gauge transformation is again the chargeless combination of spurions. We again have an induced operator of the same form as in the continuum.

III. INDUCED INTERACTIONS IN QCD

A. Degrees of freedom

We now consider the case of QCD. The analysis is very similar to the case of the Schwinger model. For some aspects of the analysis we can show that the two cases are the same, while for other aspects we shall have to assume that the two cases are similar. We shall work out the induced operators associated with the integration constant that must be included in the solution of the constraint equation for ψ_-. Actually, since we shall consider the periodic case, the field in question is not strictly an integration constant. The spurions associated with the field have a dependence on x^-, just as they do in the periodic case of the Schwinger model [12]. If we were to work in the continuum, the field in question would be an integration constant and would be independent of x^- [5]. In QCD with one flavour of quarks, the action is

$$S = \int d^4x \left[-\frac{1}{4} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + \mathrm{i}\bar{\psi}\gamma^\mu D_\mu \psi - \mu\bar{\psi}\psi \right] \ , \tag{20}$$

$\mu, \nu \in \{0,1,2,3\}$, $x_\perp = (x^1, x^2)$, $D_\mu = \partial_\mu + \mathrm{i}gA_\mu$. In light-cone gauge there are two induced operators. One comes from the $\mu\bar{\psi}\psi$ mass term, the other from the $J_\perp A^\perp$ interaction of transversely polarized gluons with the transverse part of the quark vector current.

We consider the case where the fermi fields ψ are antiperiodic in the x^- direction but may be continuous in the transverse direction [13]. We initialize the fermi fields at $x^+ = 0$ in the standard way, except that we write their Fourier expansion in such a way as to allow us to factor out the oscillating functions of $x_\perp$:

$$\begin{aligned}
\psi^{(a)}_{+,s}(0, x^-, x^\perp) \ &= \ \frac{1}{\sqrt{\Omega}} \sum_{k_\perp} \mathrm{e}^{ik_\perp x^\perp} \tilde{\psi}^{(a)}_{+,s}(0, x^-, k^\perp) \\
\tilde{\psi}^{(a)}_{+,s}(0, x^-, k^\perp) \ &= \ \sum_{n=1}^{\infty} b^{(a)}_s(n, -k_\perp) \mathrm{e}^{-ik_-(n)x^-} + d^{(a)*}_{-s}(n, k_\perp,)\mathrm{e}^{ik_-(n)x^-} .
\end{aligned} \tag{21}$$

Here

$$k_-(n) = \frac{(n - \frac{1}{2})\pi}{L} \ , \tag{22}$$

(a) is a color index, s is a spin index, and Ω a normalisation factor chosen so that the anti-commutation relations for quarks are

$$\{b^{(a)}_{s_1}(n, k_\perp), b^{(b)*}_{s_2}(m, p_\perp)\} = \delta_{k_\perp - p_\perp} \delta_{nm} \delta_{s_1 s_2} \delta_{ab} \ , \tag{23}$$

with similar relations for anti-quarks d. We initialize the field ψ_-^0, which contains the degrees of freedom in the ψ_- field that are independent of the degrees of freedom in the ψ_+ field, in the same way;

$$\psi_-^{0(a)}(x^+,x^\perp) = \frac{1}{\sqrt{\Omega}}\sum_{s,k_\perp} \mathrm{e}^{ik_\perp x^\perp}\sum_{n=1}^{\infty}\beta_s^{(a)}(n,-k_\perp)\mathrm{e}^{-ik_+(n)x^+} + \delta_{-s}^{(a)*}(n,k_\perp)\mathrm{e}^{ik_+(n)x^+}. \tag{24}$$

Here, we have defined

$$k_+(n) = \frac{(n-\frac{1}{2})\pi}{L}. \tag{25}$$

The quantity in the second sum in each case is just a one dimensional fermi field and can be bosonized in the standard way. We write:

$$\tilde{\psi}_{+,s}^{(a)}(0,x^-,k^\perp) = \mathrm{e}^{-\lambda_s^{(a)(-)}(x^-,k_\perp)}\sigma_{+,s}^{(a)}(x^-,k_\perp)\mathrm{e}^{-\lambda_s^{(a)(+)}(x^-,k_\perp)}, \tag{26}$$

where

$$\lambda_s^{(a)(+)}(x^-,k_\perp) = -\sum_{n=1}^{\infty}\frac{1}{n}C_s^{(a)}(n,k_\perp)\mathrm{e}^{-i\tilde{k}_-(n)x^-}, \tag{27}$$

$$\lambda_s^{(a)(-)}(x^-,k_\perp) = -\lambda_s^{(a)(+)^*} = \sum_{n=1}^{\infty}\frac{1}{n}C_s^{(a)*}(n,k_\perp)\mathrm{e}^{i\tilde{k}_-(n)x^-}, \tag{28}$$

$$\tilde{k}_-(n) = \frac{n\pi}{L}, \tag{29}$$

and

$$\begin{aligned} C_s^{(a)}(n,k_\perp) = {}& \sum_{\ell=0}^{n-1} d_{-s}^{(a)}\left(\ell+\frac{1}{2},k_\perp\right) b_s^{(a)}\left(n-\ell-\frac{1}{2},-k_\perp\right) + \\ & \sum_{\ell=0}^{\infty} b_s^{(a)*}\left(\ell+\frac{1}{2},-k_\perp\right) b_s^{(a)}\left(\ell+n+\frac{1}{2},-k_\perp\right) - \\ & \sum_{\ell=0}^{\infty} d_{-s}^{(a)*}\left(\ell+\frac{1}{2},k_\perp\right) d_{-s}^{(a)}\left(\ell+n+\frac{1}{2},k_\perp\right). \end{aligned} \tag{30}$$

We define a similar bosonization for the ψ_-^0 field, though we shall not need all the resultant operators. The operators corresponding to the C operators (the fusion operators) are unphysical for ψ_-^0 (they are all tachyonic). Just as in the case of the Schwinger model, these operators will not be included in the induced operators when the induced operators act on physical states, although they are needed for the full canonical structure of the theory. We shall not consider those operators further; the operators we need from the ψ_-^0 fields are their spurions σ_-. In the continuum, all the spurions are independent of the space-time coordinates; in the free, periodic case, σ_+ depends on x^- while σ_- depends on x^+. In the periodic, interacting case all the spurions depend on x^-. These facts are all in complete analogy with the Schwinger model. The x^- dependence of the σ_+ spurion is straightforward to work out and is given by

$$\sigma^{(a)}_{+,s}(x^-, k_\perp) = \mathrm{e}^{-i\frac{\pi}{4Lg}\left(Q^{(a)}_{+,s}(k_\perp)x^-\right)}\sigma^{(a)}_{+,s}(0, k_\perp)\mathrm{e}^{-i\frac{\pi}{4Lg}\left(Q^{(a)}_{+,s}(k_\perp)x^-\right)}, \tag{31}$$

where

$$Q^{(a)}_{+,s}(k_\perp) = \sum_{n=1}^{\infty} b^{(a)*}_s(n, -k_\perp)\, b^{(a)}(n, -k_\perp) - \sum_{n=1}^{\infty} d^{(a)*}_{-s}(n, k_\perp)\, d^{(a)}_{-s}(n, k_\perp), \tag{32}$$

The analysis of the x^- dependence of the σ_- spurion is more subtle and we do not have a rigorous derivation. In the work below we shall not need to know the exact x^- dependence; we shall only use the property, which we shall assume holds, that there is no x^- dependence in products of spurions of the type $\sigma_-^*\sigma_+$ when these products are applied to the vacuum. One possibility, suggested by the Schwinger model, is

$$\sigma^{(a)}_{-,s}(x^-) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4Lg}\left(Q^{(a)}_{-,s}(k_\perp)x^-\right)}\sigma^{(a)}_{-,s}(0)\mathrm{e}^{\mathrm{i}\frac{\pi}{4Lg}\left(Q^{(a)}_{-,s}(k_\perp)x^-\right)}, \tag{33}$$

where

$$Q^{(a)}_{-,s}(k_\perp) = \sum_{n=1}^{\infty} \beta^{(a)*}_s(n, -k_\perp)\, \beta^{(a)}(n, -k_\perp) - \sum_{n=1}^{\infty} \delta^{(a)*}_{-s}(n, k_\perp)\, \delta^{(a)}_{-s}(n, k_\perp). \tag{34}$$

In practice, the x^- dependence of the spurions will not cause great complications, since the spurions of the type σ_- will always occur in combination with spurions of the type σ_+ in such a way that, when acting on physical states, the space-time dependence cancels out. Spurions of the type σ_+ alone will act non-trivially on physical states, but we know their x^- dependence (31). When they act to create a particle of momentum $k_-(n)$, their x^- dependence is given by $\mathrm{e}^{ik_-(n)x^-}$; when they act as destruction operators their x^- dependence is given by $\mathrm{e}^{-ik_-(n)x^-}$. All we have done is rewrite the operators using algebraic identities; no dynamics has been done and no approximations have been made. Therefore, the conservation of momentum, which is easier to see in terms of the b and d operators, must also hold for the bosonized form. The rule for σ_- and σ_+ just stated is what is required to implement that; it would also be guaranteed by the x^- dependence shown in (31).

For completeness, we note the commutation relations of the spurions [14],

$$
\begin{aligned}
\sigma_\tau^*(x)\sigma_\tau(x) = \sigma_\tau(x)\sigma_\tau^*(x) &= 1 \\
\{\sigma_\tau^*, (x)\sigma_\rho(x)\} = \{\sigma_\tau(x), \sigma_\rho(x)\} &= 0 \\
[\sigma_\tau^*(x), \sigma_\tau(y)] = [\sigma_\tau(x), \sigma_\tau(y)] &= 0
\end{aligned} \tag{35}
$$

where τ, ρ are differing labels indicating a Lorentz or colour structure, while x, y are differing longitudinal co-ordinates. Spurions commute with the fusion operators (the C's).

We shall also need the Fourier expansion of the transverse components of the gluon fields. Defining $A_\uparrow = (A_1 - \mathrm{i}A_2)/\sqrt{2}$, $A_\downarrow = (A_1 + \mathrm{i}A_2)/\sqrt{2}$,

$$
\begin{aligned}
A_s^{(c)}(0, x^-, x^\perp) &= \frac{1}{\sqrt{\Omega}} \sum_{k_\perp} \mathrm{e}^{ik_\perp x^\perp} \tilde{A}_s^{(c)}(0, x^-, k^\perp) \\
\tilde{A}_s^{(c)}(0, x^-, k^\perp) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\tilde{k}_-(n)}} \left(a_s^{(c)}(n, -k_\perp)\, \mathrm{e}^{-i\tilde{k}_-(n)x^-} + a_s^{(c)*}(n, k_\perp,)\, \mathrm{e}^{i\tilde{k}_-(n)x^-} \right),
\end{aligned}
$$

We must now make an ansatz for the vacuum. On the basis of the Schwinger model we might wish to assume

$$
\sigma_{-,-s}^{(a)*}(x^-, k_\perp)\sigma_{+,s}^{(a)}(x^-, k_\perp)|\Omega\rangle = \kappa|\Omega\rangle, \tag{36}
$$

where κ is unimodular. If the theory is to be C-invariant we must have κ real, so $\kappa = \pm 1$. We can write down the required state explicitly;

$$
|\Omega\rangle = \prod_{s\,;\,a\,;\,k_\perp} \left(\sum_{n=-\infty}^{\infty} (\kappa\, \sigma_{-,-s}^{(a)*}(0, k_\perp)\, \sigma_{+,s}^{(a)}(0, k_\perp))^n \right) |0\rangle \quad . \tag{37}
$$

However, this state is not gauge invariant, so the assumption (36) cannot quite be correct. If we use the state $|\Omega\rangle$ (37) as our ansatz for the vacuum, the structure of the induced operators derived below is nevertheless gauge invariant. We therefore believe that there is a derivation of them that does not involve non-gauge-invariant quantities in intermediate steps. We shall proceed with the derivation using (37) as the vacuum, then afterwards, we shall examine what properties of $|\Omega\rangle$ were essential to the derivation. We shall exhibit a gauge invariant state with the required properties .

B. Induced operators

The existence of ψ_-^0 induces two extra operators, I_1 and I_2, in the QCD Lagrangian, and therefore in P^-, from the mass term and $J_\perp A^\perp$ respectively.

Consider the first induced operator, I_1, from $\bar{\psi}\psi$ [13]. Keeping only the terms that involve ψ_-^0 we find that

$$I_1 = \mu \int dx^- \; d^2x^\perp \left(\psi_{-,\downarrow}^{(a)0^*} \psi_{+,\uparrow}^{(a)} + {\psi_{+,\uparrow}^{(a)}}^* \psi_{-,\downarrow}^{(a)0} + \psi_{-,\uparrow}^{(a)0^*} \psi_{+,\downarrow}^{(a)} + {\psi_{+,\downarrow}^{(a)}}^* \psi_{-,\uparrow}^{(a)0} \right). \tag{38}$$

From this we calculate that

$$\begin{aligned} I_1 \;=\; & \mu g_1 \int dx^- \; d^2k^\perp \Big(\sigma_{-,\downarrow}^{(a)*}(0,k_\perp) \sigma_{+,\uparrow}^{(a)}(0,k_\perp) \left(\mathrm{e}^{-\lambda_\uparrow^{(a)(-)}(x^-,k_\perp)} \mathrm{e}^{-\lambda_\uparrow^{(a)(+)}(x^-,k_\perp)} - 1 \right) \\ & +C.C. + spinflip \Big). \end{aligned} \tag{39}$$

Here, g_1 is an unknown constant coming from a combination of wave function renormalization constants. The minus 1, which removes the 1 in the normal ordered product of exponentials, has the same source as it did in the case of the Schwinger model: it removes the expectation value of I_1 in the physical vacuum. other idearemove the minuIn general, the states that the hamiltonian acts on will be given by a sum of products of ψ_+ spurions and fusion operators acting on the physical vacuum. We now commute the spurions from I_1 through any operators acting on the physical vacuum until they act directly on the physical vacuum. When they act on the physical vacuum, they will just give κ. Commuting them through may introduce a minus sign; it is never more complicated than a minus sign. We could therefore remove the spurions from (39) and multiply by an appropriate $\pm$ but, rather than do that and provide a rule for whether to use the plus or minus sign, we will leave them in the operator, remembering that they have a very simple action on physical states. Once I_1 contains no spurions, it will act equally on all components of the physical vacuum and we can just take the bare vacuum as the representative state. That is all exactly in parallel with the Schwinger model [6] except that, due to now acting on states with particles of different $k_\perp$, color, spin or flavor, there is a question of minus signs that did not arise before. Below, we shall see an example where the question arises.

The operator I_1 may not appear to be gauge invariant. But it does take gauge invariant states into gauge invariant states. The gauge invariance results from special properties of the spurions and the fusion operators and is basically due to the fact that the original form of the operator, (38), is gauge invariant. As an example of that property we consider the action of I_1 on states of the type that would make up the valence states of helicity zero mesons. Specifically, we consider the action of I_1 on states of the type ${b^{(a)}}_\uparrow^*(n,-k_\perp){d^{(a)}}_\downarrow^*(m,k_\perp)|0\rangle$.

We find that

$$I_1\ b^{(a)}{}^{*}_{\uparrow}(n,-k_\perp)d^{(a)}{}^{*}_{\downarrow}(m,k_\perp)|0\rangle = \mu g_1\kappa\Big(b^{(a)}{}^{*}_{\uparrow}(n-1,-k_\perp)d^{(a)}{}^{*}_{\downarrow}(m+1,k_\perp) - 2b^{(a)}{}^{*}_{\uparrow}(n,-k_\perp)d^{(a)}{}^{*}_{\downarrow}(m,k_\perp) + b^{(a)}{}^{*}_{\uparrow}(n+1,-k_\perp)d^{(a)}{}^{*}_{\downarrow}(m-1,k_\perp)\Big)|0\rangle. \quad (40)$$

In this equation, if n or m is equal to 1, then the term on the right hand side that contains an index of zero is zero. The rule, (40), applies only to states containing a quark and an antiquark of opposite spins, opposite transverse momenta, identical color and, if flavor is included, identical flavor (these rules just assure that the pair of quarks could be created from the vacuum by the action of a sum of products of a single type of λ field). If the state contains quarks which do not satisfy any of the rules just stated then I_1 acts successively on each quark. If the quark is in the lowest $(\frac{1}{2})$ longitudinal momentum state the result is zero. Otherwise, we have a diagonal term, which gives $\mu g_1\kappa$ times the original state, plus terms which give states with larger numbers of quarks through pair production. The terms involving pair production are complicated and, since our first intended application is large N gauge theory to which such terms do not contribute, we shall not consider these terms further in the present paper. It may be noticed that if the subtraction of the expectation value of I_1 in the physical vacuum is omitted then the action on all states of the type not considered in (40) is to give a diagonal term of zero while the diagonal part of (40) would be zero.

Now let us consider the other induced operator, I_2, which comes from the $J_\perp A^\perp$ term. I_2 is given by

$$\begin{aligned} I_2 &= I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4} \\ &= g\int dx^-\ d^2x^\perp \sum_{abc}\lambda^c_{ab}\left(\psi^{(a)*}_{+,\downarrow}\psi^{0(b)}_{-,\downarrow}A^{(c)}_{\uparrow} - \psi^{0(a)*}_{-,\uparrow}\psi^{(b)}_{+,\uparrow}A^{(c)}_{\uparrow} + \psi^{0(a)*}_{-,\downarrow}\psi^{(b)}_{+,\downarrow}A^{(c)}_{\downarrow} - \psi^{(a)*}_{+,\uparrow}\psi^{0(b)}_{-,\uparrow}A^{(c)}_{\downarrow}\right), \end{aligned}$$

where g is the QCD coupling constant, and λ^c_{ab} is the colour factor. As before, from the ψ_- field we shall keep only the spurion. Consider the first term

$$I_{2,1} = gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{\psi}^{(a)*}_{+,\downarrow}(x^-,k_\perp)\sigma^{(b)}_{-,\downarrow}(x^-,k_\perp-p_\perp)\tilde{A}^{(c)}_{\uparrow}(x^-,p_\perp). \quad (41)$$

where g_2 is an (unknown) renormalization constant. We insert 1, in the form $\sigma^{(b)*}_{+,\uparrow}(x^-,k_\perp-$

$p_\perp)\sigma^{(b)}_{+,\uparrow}(x^-,k_\perp-p_\perp)$, to rewrite it as

$$I_{2,1}=gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{A}^c_\uparrow(x^-,p_\perp)\tilde{\psi}^{(a)*}_{+,\downarrow}(x^-,k_\perp)$$
$$\sigma^{(b)}_{-,\downarrow}(x^-,k_\perp-p_\perp)\sigma^{(b)*}_{+,\uparrow}(x^-,k_\perp-p_\perp)\sigma^{(b)}_{+,\uparrow}(x^-,k_\perp-p_\perp). \quad (42)$$

We can now commute the spurions among themselves to get a combination, in square brackets below, that will act like a c-number when applied to the vacuum, to the far right:

$$I_{2,1}=-gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{A}^{(c)}_\uparrow(x^-,p_\perp)\tilde{\psi}^{(a)*}_{+,\downarrow}(x^-,k_\perp)\sigma^{(b)}_{+,\uparrow}(x^-,k_\perp-p_\perp)$$
$$\left[\sigma^{(b)}_{-,\downarrow}(x^-,k_\perp-p_\perp)\sigma^{(b)*}_{+,\uparrow}(x^-,k_\perp-p_\perp)\right]. \quad (43)$$

Proceeding in this way we also obtain:

$$I_{2,2}=gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{A}^{(c)}_\uparrow(x^-,p_\perp)\sigma^{(a)*}_{+,\downarrow}(x^-,k_\perp+p_\perp)\tilde{\psi}^{(b)}_{+,\uparrow}(x^-,k_\perp)$$
$$\left[\sigma^{(a)*}_{-,\uparrow}(x^-,k_\perp+p_\perp)\sigma^{(a)}_{+,\downarrow}(x^-,k_\perp+p_\perp)\right] \quad (44)$$

$$I_{2,3}=-gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{A}^{(c)}_\downarrow(x^-,p_\perp)\sigma^{(a)*}_{+,\uparrow}(x^-,k_\perp+p_\perp)\tilde{\psi}^{(b)}_{+,\downarrow}(x^-,k_\perp)$$
$$\left[\sigma^{(a)*}_{-,\downarrow}(x^-,k_\perp+p_\perp)\sigma^{(a)}_{+,\uparrow}(x^-,k_\perp+p_\perp)\right] \quad (45)$$

$$I_{2,4}=gg_2\int d^2p_\perp d^2k_\perp dx^-\sum_{abc}\lambda^c_{ab}\tilde{A}^{(c)}_\downarrow(x^-,p_\perp)\tilde{\psi}^{(a)*}_{+,\uparrow}(x^-,k_\perp)\sigma^{(b)}_{+,\downarrow}(x^-,k_\perp-p_\perp)$$
$$\left[\sigma^{(b)}_{-,\uparrow}(x^-,k_\perp-p_\perp)\sigma^{(b)*}_{+,\downarrow}(x^-,k_\perp-p_\perp)\right]. \quad (46)$$

Spurions combinations in the square brackets turn into a c-number when they act directly on the vacuum according the general rules

$$\begin{array}{ll}
\sigma^*_{-,\downarrow}(x^-,k_\perp)\sigma_{+,\uparrow}(x^-,k_\perp)\to\kappa\ , & \sigma^*_{-,\uparrow}(x^-,k_\perp)\sigma_{+,\downarrow}(x^-,k_\perp)\to\kappa \\
\sigma_{+,\uparrow}(x^-,k_\perp)\sigma^*_{-,\downarrow}(x^-,k_\perp)\to-\kappa\ , & \sigma_{+,\downarrow}(x^-,k_\perp)\sigma^*_{-,\uparrow}(x^-,k_\perp)\to-\kappa \\
\sigma^*_{+,\uparrow}(x^-,k_\perp)\sigma_{-,\downarrow}(x^-,k_\perp)\to\kappa^*\ , & \sigma^*_{+,\downarrow}(x^-,k_\perp)\sigma_{-,\uparrow}(x^-,k_\perp)\to\kappa^* \\
\sigma_{-,\downarrow}(x^-,k_\perp)\sigma^*_{+,\uparrow}(x^-,k_\perp)\to-\kappa^*\ , & \sigma_{-,\uparrow}(x^-,k_\perp)\sigma^*_{+,\downarrow}(x^-,k_\perp)\to-\kappa^*
\end{array}$$

In general, the states that I_2 acts on will be given by a sum of products of ψ_+ spurions and fusion operators acting on the physical vacuum. We can commute the operators in square brackets through the fusion operators and, sometimes with the multiplication by -1 due to (36), through the spurions. Once the bracketed operators act on the physical vacuum they just give a c-number multiplier and we will show that the rest of the operator acts as if in the bare (trivial) light-cone vacuum $|0\rangle$. Rather than try to construct a complicated set of rules to determine when there is a minus sign, we shall leave the bracketed operators in I_2 to help keep the signs straight.

We now illustrate explicitly the action of I_2 on meson states. Consider for example the action of $I_{2,1}$ (43) on the valence states of total momentum $(P^+, P_\perp = 0)$ of the type $\sum_d b_\downarrow^{(d)*}(n, -q_\perp) d_\uparrow^{(d)*}(K-n, q_\perp)|\Omega\rangle$, where we use DLCQ as a regulator with $K = LP^+/2\pi$. The spurions in square brackets (43) do not have the same quantum numbers as any fermionic operator to their right, so we may commute them through to the right as a pair without incuring a minus sign. This leads to

$$\kappa^* g g_2 \int d^2p_\perp d^2k_\perp dx^- \sum_{abc} \lambda_{ab}^c \tilde{A}_\uparrow^{(c)}(x^-, p_\perp) \tilde{\psi}_{+,\downarrow}^{(a)*}(x^-, k_\perp) \sigma_{+,\uparrow}^{(b)}(x^-, k_\perp - p_\perp)$$
$$\sum_d b_\downarrow^{(d)*}(n, -q_\perp) d_\uparrow^{(d)*}(K-n, q_\perp)|\Omega\rangle. \quad (47)$$

We now project out the part of this state containing no ψ_-^0 zero modes, which simply means replacing $|\Omega\rangle$ by $|0\rangle$. Commuting the final σ_+ spurion through to this vacuum, and using the property [14]

$$\sigma_{+,\uparrow}^{(b)}(x^-, k_\perp - p_\perp)|0\rangle = \mathrm{e}^{\frac{\mathrm{i}\pi x^-}{2L}} d_\downarrow^{(b)*}(1, k_\perp - p_\perp)|0\rangle \quad (48)$$

we obtain effectively

$$\kappa^* g g_2 \int d^2p_\perp d^2k_\perp dx^- \sum_{abc} \lambda_{ab}^c \tilde{A}_\uparrow^{(c)}(x^-, p_\perp) \sum_{m=1}^{\infty} \left(b_\downarrow^{(a)*}(m, -k_\perp) \mathrm{e}^{\mathrm{i}k_-(m)x^-} + d_\uparrow^{(a)}(m, k_\perp,) \mathrm{e}^{-\mathrm{i}k_-(m)x^-} \right)$$
$$\sum_d b_\downarrow^{(d)*}(n, -q_\perp) d_\uparrow^{(d)*}(K-n, q_\perp) \mathrm{e}^{\frac{\mathrm{i}\pi x^-}{2L}} d_\downarrow^{(b)*}(1, k_\perp - p_\perp)|0\rangle \quad (49)$$

(Anti)-commuting annihilation operators through to the right and conserving momentum, the final result is

$$-g g_2 \kappa^* \int d^2p_\perp \sqrt{\frac{\pi}{K-n-1}} \sum_{abc} \lambda_{ab}^c b_\downarrow^{(a)*}(n, -q_\perp) d_\downarrow^{(b)*}(1, q_\perp - p_\perp) a_\uparrow^{(c)*}(K-n-1, p_\perp)|0\rangle. \quad (50)$$

Analyzing, in a similar way, the other parts of I_2, the only other non-zero contribution comes from $I_{2,3}$ to give

$$-gg_2\kappa \int d^2p_\perp \sqrt{\frac{\pi}{n-1}} \sum_{abc} \lambda^c_{ab} b^{(b)*}_\uparrow(1,-q_\perp - p_\perp) d^{(a)*}_\uparrow(K-n, q_\perp) a^{(c)*}_\downarrow(n-1, p_\perp)|0\rangle \quad (51)$$

If either n or $K-n$ are 1, then the corresponding term in the result, which has a zero in the denominator, is zero. If we consider the spin flipped case, there are minus signs:

$$\begin{aligned} I_2 \sum_a b^{(a)*}_\uparrow(n,-q_\perp) d^{(a)*}_\downarrow(K-n, q_\perp)|\Omega\rangle = \\ \int d^2p_\perp\, gg_2\kappa \sqrt{\frac{\pi}{n-1}} \sum_{abc} \lambda^c_{ab} b^{(b)*}_\downarrow(1,-q_\perp - p_\perp) d^{(a)*}_\downarrow(K-n, q_\perp) a^{(c)*}_\uparrow(n-1, p_\perp)|0\rangle + \\ \int d^2p_\perp\, gg_2\kappa^* \sqrt{\frac{\pi}{K-n-1}} \sum_{abc} \lambda^c_{ab} b^{(a)*}_\uparrow(n,-q_\perp) d^{(b)*}_\uparrow(1, q_\perp - p_\perp) a^{(c)*}_\downarrow(K-n-1, p_\perp)|0\rangle. \end{aligned} \quad (52)$$

The net effect in all cases is to flip the helicity of the quark which emits a gluon. Thus, quark helicity is no longer conserved. The results look similar to the standard QCD vertices of the massive case $\mu \neq 0$, except that the gluon always absorbs all the available longitudinal momentum (all but one unit) from the quark and the momentum factors are different. We give a few other examples which show that the induced operators include pair production:

$$I_2\, b^{(a)*}_\downarrow(n,-k_\perp) d^{(b)*}_\downarrow(1, k_\perp - p_\perp)|\Omega\rangle = gg_2\kappa\lambda^c_{ab}\sqrt{\frac{\pi}{n}} a^{(c)*}_\downarrow(n,-p_\perp)|\Omega\rangle \quad (53)$$

$$I_2\, b^{(a)*}_\downarrow(1,-k_\perp) d^{(b)*}_\downarrow(n, k_\perp - p_\perp)|\Omega\rangle = gg_2\kappa^*\lambda^c_{ab}\sqrt{\frac{\pi}{n}} a^{(c)*}_\downarrow(n,-p_\perp)|\Omega\rangle \quad (54)$$

$$I_2\, b^{(a)*}_\uparrow(n,-k_\perp) d^{(b)*}_\uparrow(1, k_\perp - p_\perp)|\Omega\rangle = -gg_2\kappa\lambda^c_{ab}\sqrt{\frac{\pi}{n}} a^{(c)*}_\uparrow(n,-p_\perp)|\Omega\rangle \quad (55)$$

$$I_2\, b^{(a)*}_\uparrow(1,-k_\perp) d^{(b)*}_\uparrow(n, k_\perp - p_\perp)|\Omega\rangle = -gg_2\kappa^*\lambda^c_{ab}\sqrt{\frac{\pi}{n}} a^{(c)*}_\uparrow(n,-p_\perp)|\Omega\rangle. \quad (56)$$

Notice that the pair destruction only occurs when one quark is in the lowest longitudinal momentum state, so pair production always creates such a state.

The non-conservation of quark helicity is crucial to obtaining a splitting of the pion and helicity zero component of the rho meson in the chiral limit. The valence quark content of these states

$$\sum_a \left(b_\uparrow^{(a)*}(n,-k_\perp) d_\downarrow^{(a)*}(K-n,k_\perp) \pm b_\downarrow^{(a)*}(n,-k_\perp) d_\uparrow^{(a)*}(K-n,k_\perp) \right) |\Omega\rangle \tag{57}$$

is only distinguished by symmetry under spin exchange. In the absence of I_2 they would be degenerate. We mention that in a $U(N)$ gauge theory the spin exchange in the valence state can proceed by annihilation through the color diagonal component of the gauge field. In the case of QED, for example, this contributes to the hyperfine splitting of positronium. In a chiral $SU(N)$ gauge theory, spontaneous breaking of quark helicity symmetry is the only way to achieve it.

Now we shall discuss the question mentioned above: the problem that (37) is not gauge invariant. Suppose that, instead of (37), we took the vacuum to be

$$|\Omega\rangle = \prod_{s\,;\,k_\perp} \left(\sum_{n=-\infty}^{\infty} (\kappa \sum_a \sigma_{-,-s}^{(a)*}(0,k_\perp)\, \sigma_{+,s}^{(a)}(0,k_\perp))^n \right) |0\rangle \quad . \tag{58}$$

This state is gauge invariant but is no longer an eigenstate of the combination of spurions in square brackets in (43) etc.. But when we project onto the subspace containing no quanta from the ψ_-^0 fields, the only parts from (58) that contribute to that projection are those of (37) (and only some of those states). Therefore, when we project the hamiltonian eigenvalue equation onto the space containing no quanta from the ψ_-^0 fields, we find that the relation (36) is satisfied. Thus we may solve the eigenvalue equation using the operators as we have shown above, acting effectively in the bare vacuum $|0\rangle$. It may be that enforcing the eigenvalue equation in the whole space would lead to no, or an insufficient number of, eigenvectors. In that case we would have to make a physical subspace restriction to the space containing no quanta from the ψ_-^0 fields. We believe that that can consistently be done but we shall not pursue the question any further in this paper.

C. Scaling

In the periodic case of the Schwinger model, we know that the coefficient of the induced operator scales with the periodic length, L, or, with the DLCQ harmonic resolution, K [6]. For the Schwinger model we can work out the coefficient in the continuum using the principle that the fields should be canonical at spacelike separations. We can then fix the scaling for the periodic case by requiring that the periodic case go the the continuum case as $L \to \infty$.

In principle that same procedure could presumably be used for QCD. But given our present ability to calculate that procedure is completely impractical. We anticipate that, until a better idea or better computational methods become available, we shall simply fit g_1 and g_2 to symmetries or, if necessary, to data.

In some simple cases we can see what form the operator will approach, for a given scaling, as the harmonic resolution becomes large and the calculation approaches the continuum calculation. For the case of I_1 applied to the simple meson valence states described in (40) the result is particularly simple. We consider states of the form

$$|K\rangle = \sum_{n=1}^{K-1} f^2(x) b^{(a)*}_{\uparrow}(n, -k_\perp) d^{(a)*}_{\downarrow}(K-n, k_\perp)|0\rangle \tag{59}$$

where

$$x = \frac{(n - \frac{1}{2})}{K} \ , \tag{60}$$

and f^2 is the valence quark wavefunction. If there is no scaling, the operator I_1 goes to zero. If the scaling is linear in K, then (if κ is real) I_1 goes to a pure endpoint operator; that is, $I_1 \; |K\rangle = c(b^{(a)*}_{\uparrow}(1, -k_\perp) d^{(a)*}_{\downarrow}(K-1, k_\perp) + b^{(a)*}_{\uparrow}(K-1, -k_\perp) d^{(a)*}_{\downarrow}(1, k_\perp))|0\rangle$ where c is a c-number depending on the value of f^2 at the endpoints and on the other parameters. (If κ has a nonzero imaginary part, I_1 is a derivative operator proportional to the imaginary part of κ). This linear scaling case is much like the case of the Schwinger model. If the scaling is quadratic, that is, $g_1 \sim K^2$, then I_1 is a second derivative operator, that is

$$I_1 \int_0^1 dx \; f^2(x) \, b^{(a)*}_{\uparrow}(x, -k_\perp) \, d^{(a)*}_{\downarrow}(1-x, k_\perp)|0\rangle \sim \int_0^1 dx \; \frac{d^2 f^2(x)}{dx^2} \; b^{(a)*}_{\uparrow}(x, -k_\perp) \, d^{(a)*}_{\downarrow}(1-x, k_\perp)|0\rangle \tag{61}$$

We know that there is nontrivial scaling in the Schwinger model, but the action of I_1 on states containing quarks not of the form used in (40) may mean that g_1 cannot scale as a positive power of K, since that would seem to lead to divergences. But it may be that those divergences can and should be subtracted in some way. We leave the question of the scaling of both g_1 and g_2 open and, in the next section, give some examples of possible scalings in some simple models.

IV. EXACT SOLUTIONS FOR SIMPLE MODELS

In this section and the next, we look at solutions for simple dimensionally reduced models in 1+1 dimensions obtained by restricting to the $k_\perp = 0$ sector, meaning that we (classically) discard all fields except those satisfying

$$\partial_{x_\perp} A_\mu = \partial_{x_\perp} \psi = 0 \ . \tag{62}$$

We also work in the large N limit and suppress the color indices. These calculations will illustrate the general effects described above and also allow us to investigate how the couplings might scale with the cutoff. In the present section, we find exact solutions for the dimensionally reduced P^- containing only the induced operators. In the next section, we analyze a gauged version of these dimensionally reduced models.

The reduced theories preserve total quark and gluon helicity, identified with the angular momentum projection J_3, a remnant of the 3+1 dimensional rotation symmetry. This leads to doublets consisting of opposite helicities. There is also exact charge conjugation symmetry C and an exact kinematic parity symmetry P of the valence part of wavefunctions: $f^2(x, 1-x) \leftrightarrow f^2(1-x, x)$. This kinematic valence parity only equals parity of the full wavefunction in the free field limit, but is a convenient label. Thus, bound state energy levels can be labelled by $|J_3|^{PC}$. We will be particularly be interested in the quantum numbers $0^{-\pm}$, 1^{--}, which together form the quantum numbers of the pion (or eta') and the three Lorentz components of the rho meson. Generically, the 1^{--} doublet and the 0^{--} are split because of dimensional reduction. We do not address the issue here of how to make a full degenerate Lorentz multiplet for the rho quantum numbers, as this really requires transverse motion. Rather, we wish to study splitting of the $0^{-\pm}$ states, since this occurs only if the quark helicity alone is not conserved. In 1+1 dimensions, to have canonically normalised kinetic terms, we rescale fields $\psi \to \psi\sqrt{V_\perp}$, $A_\perp \to A_\perp\sqrt{V_\perp}$, where $V_\perp = \int dx^1 dx^2$ is the transverse volume factor. The gauge coupling also becomes dimensionful through dimensional reduction, $g^2 \to g^2 V_\perp N$, where we have also absorbed the colour factor N in the large N limit.

A. I_2 only

In this subsection we shall consider the case where the entire P^- is given by I_2. Here, we shall assume that g_2 does not scale with K. We shall expand the wave function as

$$
\begin{aligned}
|\phi\rangle &= \sum_x f^2_{\uparrow\downarrow}(x) b^*_\uparrow(x) d^*_\downarrow(1-x)|0\rangle \\
&+ \sum_{x+y\le 1} f^3_{\downarrow\uparrow\downarrow}(x,y) b^*_\downarrow(x) d^*_\downarrow(1-x-y) a^*_\uparrow(y)|0\rangle \\
&+ \sum_{x+y+z\le 1} f^4_{\uparrow\downarrow\uparrow\downarrow}(z,x,y) b^*_\uparrow(z) d^*_\downarrow(1-x-y-z) a^*_\downarrow(x) a^*_\uparrow(y)|0\rangle \\
&+ \text{spin flip}
\end{aligned} \tag{63}
$$

This expansion includes all the states which can couple to the valence sector in this model, so it is a complete expansion of the wave function in the helicity zero sector. The boundstate equations for the meson invariant mass M are as follows.

$$
\begin{aligned}
M^2 f^2_{\uparrow\downarrow}(x) &= \frac{gg_2}{\sqrt{x}} f^3_{\downarrow\uparrow\downarrow}(0,x) \\
&+ \frac{gg_2}{\sqrt{1-x}} f^3_{\uparrow\downarrow\uparrow}(x,1-x)
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
M^2 f^3_{\downarrow\uparrow\downarrow}(x,y) &= \frac{gg_2}{\sqrt{y}} f^2_{\uparrow\downarrow}(x+y)\delta(x) \\
&- \frac{gg_2}{\sqrt{y}} f^2_{\uparrow\downarrow}(x)\delta(1-x-y) \\
&- \frac{gg_2}{\sqrt{x}} f^4_{\uparrow\downarrow\uparrow\downarrow}(0,x,y) \\
&+ \frac{gg_2}{\sqrt{1-x-y}} f^4_{\downarrow\uparrow\downarrow\uparrow}(x,y,1-x-y)
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
M^2 f^4_{\uparrow\downarrow\uparrow\downarrow}(z,x,y) &= -\frac{gg_2}{\sqrt{x}} f^3_{\downarrow\uparrow\downarrow}(z+x,y)\delta(z) \\
&- \frac{gg_2}{\sqrt{y}} f^3_{\uparrow\downarrow\uparrow}(z,x)\delta(1-y-z-x)
\end{aligned} \tag{66}
$$

with the same equations with $\uparrow\leftrightarrow\downarrow$, $(gg_2) \leftrightarrow -(gg_2)$. 0 is a shorthand for $1/2K$, and 1 for $1-1/2K$; if a denominator vanishes, then that term is excluded from the equation; we define $\delta(x) = 0$ if $x \neq 0$ otherwise 1. The equations can be solved algebraically as simultaneous linear equations, to give the following effective equations in the valence sector.

$$
M^2 f^2_{\uparrow\downarrow}(0) = \frac{(gg_2)^2}{M^2}\left(f^2_{\uparrow\downarrow}(0) - f^2_{\downarrow\uparrow}(1)\right) \tag{67}
$$

$$
M^2 f^2_{\uparrow\downarrow}(x)|_{x\neq 0,1} = \frac{(gg_2)^2}{M^2 x(1-x)}\left(1 + \frac{4}{\frac{M^4}{(gg_2)^2} - \frac{1}{x(1-x)}}\right) f^2_{\uparrow\downarrow}(x)|_{x\neq 0,1} \tag{68}
$$

Solutions of the first of these equations are delta functions at 0,1 and give the solutions for the 0^{-+} ($f_{\uparrow\downarrow}(1) = f_{\uparrow\downarrow}(0)$, $f_{\uparrow\downarrow} = -f_{\downarrow\uparrow}$, $M^2 = \pm\sqrt{2}gg_2$) and 0^{--} ($f_{\uparrow\downarrow}(1) = f_{\uparrow\downarrow}(0)$, $f_{\uparrow\downarrow} = f_{\downarrow\uparrow}$, $M^2 = 0$). The lower lying states would be identified as prototype pion and helicity zero component of the rho; they are split even in the DLCQ continuum limit $K \to \infty$.

There are some artifacts that occur due to keeping only I_2 in this simple model. The lowest lying states are tachyonic. In the large-N limit however, they are absolutely stable. Opposite parity states ($f_{\uparrow\downarrow}(1) = \pm f_{\uparrow\downarrow}(0)$) are degenerate. Also, there are solutions to the second equation which are delta functions at specific values $0 < x < 1/2$ with

$$M^4 = (gg_2)^2 \frac{1 \pm 2\sqrt{x(1-x)}}{x(1-x)} \ . \tag{69}$$

This spectrum is continuous and, given that both signs are possible for M^2, unbounded below in the DLCQ continuum limit $K \to \infty$. The unbound solutions at small $x > 1/2K$ are an artifact due to wee gluon emission. These artifacts will be avoided in a more realistic gauged model (next section).

A similar analysis for the 1^{--} sector produces $M^4 = (gg_2)^2$ for delta function states at $x = 0, 1$. These would partner the 0^{--} state and, although they are not degenerate in the reduced model, would eventually form the rho Lorentz multiplet in higher dimensions.

B. I_1 Scales Linearly with K

In this section we shall consider the possibility that I_1 scales linearly with K, the scaling that we know to be correct for the Schwinger model. We shall again work in the $k_\perp = 0$ sector and at large N. In this section we shall keep both I_1 and I_2 and the quark kinetic energies assuming a quark bare mass of μ. With linear scaling (and real κ) the operator I_1 becomes a pure endpoint operator operating only at x equals 0 or 1. The system is quite singular and we shall use DLCQ as the regulator and specify that the physical limit is the large K limit of DLCQ. Since the continuum limit operators couple the endpoints of the wave function only to themselves and the state of two quarks and one gluon with all the momentum in the gluon, we might therefore expect to find pion-like wave functions of the form

$$\frac{1}{\sqrt{2}}(b^*_\downarrow(0)d^*_\uparrow(1)|0\rangle - b^*_\uparrow(1)d^*_\downarrow(0))|0\rangle + C\left(b^*_\downarrow(1)d^*_\downarrow(0)a^*_\uparrow(1) + b^*_\uparrow(1)d^*_\uparrow(0)a^*_\downarrow(1)\right)|0\rangle. \tag{70}$$

We find that there is an eigenstate of this form and that if we choose $\kappa = -1$ and $g_1 = g_{11}K$, then, to have a finite, nonzero result we must choose

$$g_2 = g_{21}K - g_{22} \tag{71}$$

where

$$g_{21} = \frac{4\mu}{g}\sqrt{g_{11}\mu + 2\mu^2}. \tag{72}$$

g_{22} is a free parameter the sign of which has been chosen for convenience below. The eigenvalue is given by

$$M^2 = \frac{4gg_{22}\sqrt{g_{11}\mu + 2\mu^2}}{g_{11} + 6\mu}. \tag{73}$$

The constant C in (70) is given by

$$C = \frac{\sqrt{g_{11}\mu + 2\mu^2}}{2\mu}. \tag{74}$$

I_1 becomes a pure endpoint operator only in the limit $K \to \infty$. At finite K, I_1 will couple all the x values and we will necessarily have a wave function distributed over all x. One may therefore ask if the limit of calculations done at finite, but increasing, values of K contain an eigenstate which approaches the one described in the previous paragraph. The answer is that they do in the limit $\mu >> g_{11}$. In the opposite limit, $\mu << g_{11}$, we find that the results give

$$g_{21} = \frac{\mu}{g}\sqrt{8(g_{11}\mu + 4\mu^2)}. \tag{75}$$

$$M^2 = \frac{4gg_{22}\sqrt{2(g_{11}\mu + 4\mu^2)}}{g_{11} + 12\mu}. \tag{76}$$

$$C = \frac{\sqrt{g_{11}\mu + 4\mu^2}}{\sqrt{8}\mu}. \tag{77}$$

For values of μ and g_{11} that lie between these extremes the results vary smoothly between those of the previous paragraph and those just given. The reason for the difference is as follows: keeping only the endpoint vectors, as we did in the last paragraph, amounts to setting all the other values of $f^2(x)$ equal to zero. If $\mu >> g_{11}$, then, at finite, increasing K, the wave function does go to a concentration at the endpoint and, for sufficiently large K, looks like an ever narrowing cusp. In the limit the point next to the endpoint completely decouples and we get the same results as we do setting all the values of $f^2(x)$ except for the endpoints equal to zero. But if $\mu << g_{11}$, then the wave function more resembles an ever narrowing Gaussian and the effect of the point next to the endpoint never completely goes away. That leads to the results quoted in the present paragraph which are derived on the assumption that the value at the point next to the endpoint is equal to the value at the endpoint.

C. I_1 Scales Quadratically with K

In this section we shall assume that I_1 scales quadratically with K. We shall keep I_1, I_2 and the quark kinetic energies with mass μ. We shall choose $\kappa = -1$ and take $g_1 = g_{11}K^2$. In the limit of large K, I_1 goes to a second derivative operator and we must have a wave function distributed over all x.

In the present system we consider only the π and the helicity zero ρ.

The wave function for the pion (in the continuum) is of the form

$$|\pi\rangle = C\left(a^*_\downarrow(1)b^*_\uparrow(0)\, d^*_\uparrow(0) + a^*_\uparrow(1)b^*_\downarrow(0)\, d^*_\downarrow(0)\right)|0\rangle + \int_0^1 dx\; \pi(x)\,\frac{1}{\sqrt{2}}\left(b^*_\uparrow(x)\, d^*_\downarrow(1-x) - b^*_\downarrow(1-x)\, d^*_\uparrow(x)\right)|0\rangle\ . \quad (78)$$

The system is very singular and we must give a meaning to the states $a^*_\downarrow(1)b^*_\uparrow(0)\, d^*_\uparrow(0)$ and $a^*_\uparrow(1)b^*_\downarrow(0)\, d^*_\downarrow(0)$. We shall assume that the DLCQ regularization is appropriate and that the correct continuum limit is the large K limit of DLCQ. The equation satisfied by the valence wavefunction $f^2(x)$ is found to be

$$-\mu g_{11}\frac{d^2f^2}{dx^2} + \frac{\mu^2}{x(1-x)}f^2 = M^2 f^2. \quad (79)$$

The boundary conditions are determined by the operator I_2 in the large K limit of DLCQ. The simpler case is the helicity zero ρ, whose wave function is of the form

$$|\rho\rangle = \int_0^1 dx\; \rho(x)\,\frac{1}{\sqrt{2}}\left(b^*_\uparrow(x)\, d^*_\downarrow(1-x) + b^*_\downarrow(1-x)\, d^*_\uparrow(x)\right)|0\rangle. \quad (80)$$

In this case the boundary condition for (79) is that we have the regular solution which is zero at $x = 0$ and at $x = 1$. The kinetic energy can be treated as a perturbation and, for the values of g_{11} and μ we shall consider below, it is negligible. For values of μ much less than g_{11} we have

$$\rho(x) = \sqrt{2}\sin\pi x \quad ; \quad M^2_\rho = \mu\pi^2 g_{11} + 4.88\mu^2. \quad (81)$$

For values of μ which are not so small compared with g_{11}, both the wave function and the mass of the ρ can be expanded in a power series in μ^2

The pion is a much more complicated system. To have a finite value for the pion mass we must choose

$$g_2 = \frac{2}{g}\sqrt{g_{11}\,\mu^3}\,K^{\frac{3}{2}} - \frac{8\mu^3}{3gg_{11}}\sqrt{2K}\ln K - g_{22}\sqrt{K}, \tag{82}$$

where g_{22} is a free parameter. With that choice, the mass of the pion is given by

$$M_\pi^2 = \frac{8\mu^3}{g_{11}} + 4gg_{22}\sqrt{\frac{\mu}{g_{11}}}. \tag{83}$$

It may appear that the mass squared of the pion goes like the square root of the quark bare mass, but either or both of g_{11} and g_{22} could depend on μ so the complete dependence on μ cannot be known until more information on the g's is available. For $g_{11} >> \mu$, and in the normalization where the C in Eq.(78) is 1, $\pi(x)$ is given, to a good approximation by

$$\begin{aligned}\pi(x) \;=\; & \sqrt{\frac{\mu}{g_{11}}} + \left(\frac{\mu}{g_{11}}\right)^{\frac{3}{2}} x(1-x)\ln\left[x(1-x)\right] + \\ & \frac{2gg_{22}}{g_{11}^2}x(1-x) + \left(\frac{\mu}{g_{11}}\right)^{\frac{3}{2}} x^2(1-x)^2\ln\left[x(1-x)\right].\end{aligned} \tag{84}$$

Notice that we now have a singular solution to equation (79), which does not go to 0 at $x = 0$ or at $x = 1$. The mass of the ρ is determined by μ and g_{11}. If we choose $\mu = 5$ and $g_{11} = 1.25 \times 10^4$ we find that the mass of the ρ is about 785. We must choose $gg_{21} = 2500$ and if we then choose $gg_{22} = 2.45 \times 10^5$ we find that the mass of the pion is about 140. For that solution, and in the normalization where $C = 1$, the value of the endpoint of the pion distribution is $\pi(0) \approx .02$. $\pi(x)$ is nearly flat as a function of x. While we cannot be sure until we have done the calculation, we believe that if transverse dimensions are included, the wave function will look something like the one given here for small $k_\perp$, but will become more and more like the wave function given here for the ρ as $k_\perp$ increases. If the model were realistic the (dimensionful) g's would get their dimensions from Λ_{QCD} (and μ). The numbers just given are all consistent with a mass scale of a few hundred MEV but the model is unphysical (or, at least runs counter to common ideas) in that the rho gets its mass entirely from $\mu\bar{\psi}\psi$, whereas, presumably, the rho actually gets is mass from other operators, probably a combination of the induced operators and the more conventional operators.

In many ways the possibility that the induced operators scale in a nontrivial way with K seems attractive. But when they are scaled to increase in strength with increasing K, they also produce singularities when acting on states not included in the model calculations above. We do not know if these can be controlled in higher dimensions, and consider the precise scaling of the couplings to the induced operators to be open.

V. DIMENSIONALLY REDUCED GAUGE THEORY

A. Dimensional reduction

Since the quark helicity violating term I_2 couples to the next-to-zero mode quark only, one might suspect that the effects shown in section IV A disappear in the DLCQ continuum limit $L \to \infty$ of a non-trivial theory. We demonstrate in the following with explicit numerical DLCQ calculations, in a dimensionally reduced gauge theory, that I_2 leads to a splitting in the spectrum between meson-like states with the quantum numbers of the pi and rho mesons.

In this section, to be consistent with previous literature, we use the convention $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}$. The light-cone hamiltonian we use starts from the large N limit of eq. (20), dimensionally reduced to $1+1$ dimensions, with appropriately rescaled fields and coupling. as in the last section. This results in a reduced action [15]

$$\begin{aligned} S \;\to\; \int dx^+ dx^- \; \Bigg\{ & -\frac{1}{4}\operatorname{Tr} F_{\alpha\beta}F^{\alpha\beta} + \frac{\mathrm{i}}{\sqrt{2}}(\bar{u}_\uparrow \gamma^\alpha_{(2)} D_\alpha u_\uparrow + \bar{u}_\downarrow \gamma^\alpha_{(2)} D_\alpha u_\downarrow) + \frac{\mu}{\sqrt{2}}(\bar{u}_\uparrow u_\downarrow + \bar{u}_\downarrow u_\uparrow) \\ & + \operatorname{Tr}\left[-\frac{1}{2}\bar{D}_\alpha A_r \bar{D}^\alpha A^r - \frac{g^2}{4N}[A_r, A_s][A^r, A^s] + \frac{1}{2}m_0^2 A_r A^r\right] \\ & - \frac{g}{\sqrt{2N}}(\bar{u}_\uparrow (A_1 + \mathrm{i}\gamma^5_{(2)} A_2) u_\uparrow - \bar{u}_\downarrow (A_1 - \mathrm{i}\gamma^5_{(2)} A_2) u_\downarrow) \Bigg\} \; . \end{aligned} \tag{85}$$

α and $\beta \in \{+,-\}$, $r, s \in \{1,2\}$, $\gamma^0_{(2)} = \sigma^1$, $\gamma^3_{(2)} = \mathrm{i}\sigma^2$, $\gamma^5_{(2)} = \mathrm{i}\sigma^1\sigma^2$, $\bar{D}_\alpha = \partial_\alpha + \mathrm{i}g[A_\alpha, .]/\sqrt{N}$, $D_\alpha = \partial_\alpha + \mathrm{i}gA_\alpha/\sqrt{N}$. The two-component spinors $u_\uparrow$ and $u_\downarrow$ are related to the original $3+1$ dimensional ψ field by

$$2^{1/4}\psi\sqrt{\int dx^1 dx^2} = \begin{pmatrix} u_{+,\uparrow} \\ u_{-,\uparrow} \\ u_{-,\downarrow} \\ u_{+,\downarrow} \end{pmatrix} \;,\; u_\uparrow = \begin{pmatrix} u_{-,\uparrow} \\ u_{+,\uparrow} \end{pmatrix} \;,\; u_\downarrow = \begin{pmatrix} u_{-,\downarrow} \\ u_{+,\downarrow} \end{pmatrix} \; . \tag{86}$$

The suffices $\uparrow$, $\downarrow$, label helicity, while $+$, $-$ indicates whether the fermions are right or left moving with respect to x^3. Since the gluon mass is not protected by transverse gauge symmetry transformations in a dimensionally reduced model, we must allow a gluon mass m_0 in general. In fact, this will regulate small-x gluon divergences. $u_\uparrow, u_\downarrow, A_1, A_2$ represent the transverse polarizations of the $3+1$ dimensional quarks and gluons. In $1+1$ dimensions, where there is of course no spin, the fields appear as different flavours in fundamental and adjoint representations.

In the massless quark limit, $\mu = 0$, the $u_\uparrow$ and $u_\downarrow$ fields in (85) have separate conserved $U(1)$ fermion numbers, but no axial symmetries (with $\gamma^5_{(2)}$). This $U(1)$x$U(1)$ transcribes to the left and right handed $U(1)$ symmetries in $3+1$ dimensions of a single flavour of massless quarks in QCD. Thus, the dimensionally reduced model has the important property that it inherits the chiral symmetries of QCD with massless quarks. Note that, with a single flavour of quarks, the axial anomaly may spoil one of the $U(1)$ symmetries, but in the large N limit the anomaly is suppressed as it involves fermion pair production. Therefore it is acceptable to work with one $3+1$-dimensional flavour provided we also work at large N. It is also necessary to work at large-N to have spontaneous symmetry breaking of continuous symmetry in $1+1$ dimensions. In $3+1$ dimensions at large N with one flavour, one expects the axial $U(1)_A$ combination to be spontaneously broken and a single Goldstone boson to appear. In the reduced model, this corresponds to the $U(1)$x$U(1)$ flavour symmetry being broken down to its diagonal 'total' fermion number subgroup. The broken $U(1)$ corresponds to the charge $\tilde{Q}_5$ measuring quark helicity. For massless quarks, the dimensionally reduced hamiltonian contains no terms that flip quark helicity if we omit zero modes. In that case, we find that the states $0^{-\pm}$ are degenerate (no pi-rho splitting). Including zero modes and using the non-abelian vacuum ansatz, described in section III, produces effective helicity-flip interactions that split the multiplet. Our calculation later will make approximations that probably break chiral symmetry explicitly, but not quark helicity symmetry, so that we do not obtain an exactly massless Goldstone boson, but this is a separate issue that we do not address in detail in this paper. The chiral properties of the reduced theory (85) have also recently been studied by different methods in ref.[16].

B. Boundstate Equations

In the light-cone gauge, the fields A_+ and non-zero modes of u_- are non-propagating in light-front time $x^+ = (x^0 + x^3)/\sqrt{2}$. We eliminate them using their constraint equations of motion. The expansion in creation and annihilation operators for the dimensionally reduced fermion $u^{(a)}_{+,s}(x^+, x^-)$, where (a) labels colour and s labels helicity, becomes

$$u^{(a)}_{+,s}(0, x^-) = \frac{1}{\sqrt{\Omega}} \sum_{n=1}^{\infty} b^{(a)}_s(n) \mathrm{e}^{-ik_-(n)x^-} + d^{(a)*}_{-s}(n) \mathrm{e}^{ik_-(n)x^-} \ . \tag{87}$$

The expansion for transversely polarized dimensionally reduced gluons is similarly

$$A^{(c)}_s(0, x^-) = \frac{1}{\sqrt{\Omega}} \sum_n \frac{1}{\sqrt{2\tilde{k}_-(n)}} \left(a^{(c)}_s(n) \mathrm{e}^{-i\tilde{k}_-(n)x^-} + a^{(c)*}_{-s}(n) \mathrm{e}^{i\tilde{k}_-(n)x^-} \right) \tag{88}$$

Note: if $s = \uparrow$ then $-s = \downarrow$, while $\uparrow$ indicates a helicity of value $+1/2$ for quarks while $+1$ for gluons.

Introducing momentum fractions $x = \tilde{k}_-(n)/P^+$ or $x = k_-(n)/P^+$, for a helicity-zero meson of total momentum P^+ for example, the state invariant under residual global gauge transformations is written

$$\begin{aligned} &\sum_a \int dx f^2_{\uparrow\downarrow}(x, 1-x) b^{(a)*}_{\uparrow}(x) d^{(a)*}_{\downarrow}(1-x) + \int dx f^2_{\downarrow\uparrow}(x, 1-x) b^{(a)*}_{\downarrow}(x) d^{(a)*}_{\uparrow}(1-x) \\ &+ \sum_{abc} \int dx dy f^3_{\downarrow\uparrow\downarrow}(x, y, 1-x-y) b^{(a)*}_{\downarrow}(x) \lambda^c_{ab} a^{(c)*}_{\uparrow}(y) d^{(b)*}_{\downarrow}(1-x-y) + \\ &\int dx dy f^3_{\uparrow\downarrow\uparrow}(x, y, 1-x-y) b^{(a)*}_{\uparrow}(x) \lambda^c_{ab} a^{(c)*}_{\downarrow}(y) d^{(b)*}_{\uparrow}(1-x-y) + \cdots |\Omega > \end{aligned}$$

where $\cdots$ indicates higher numbers of gluon creation operators a^* and $|\Omega >$ is our vacuum ansatz with $k_\perp$ omitted. The wavefunction components are normalised as

$$\begin{aligned} &\int_0^1 dx \ |f^2_{\uparrow\downarrow}(x, 1-x)|^2 + |f^2_{\downarrow\uparrow}(x, 1-x)|^2 \\ &+ \int_0^1 dx \int_0^{1-x} dy \ |f^3_{\downarrow\uparrow\downarrow}(x, y, 1-x-y)|^2 + |f^3_{\uparrow\downarrow\uparrow}(x, y, 1-x-y)|^2 + \cdots = 1 \end{aligned} \tag{89}$$

Boundstate equations for the wavefunctions f can be obtained by applying the P^- to a meson state, such as eq.(89), and then projecting onto a given vector in the physical Fock space (the sector built on the vacuum ansatz containing no u_- zero modes). The resulting equations are the same as those found in ref.[15], restricted to massless quarks $\mu = 0$, with the addition of I_2 interactions. We write them in terms of the invariant mass of the meson boundstate $M^2 = 2P^+P^-$. When we solve the equations numerically in section V C, we will truncate the Fock space to the sectors of f^2 and f^3 (one-gluon approximation); therefore we will display only the equations for this truncation. Defining $h = gg_2\kappa$, we find

$$\begin{aligned}
M^2 f^2_{\uparrow\downarrow}(x,1-x) &= \frac{m_f^2}{x(1-x)} f^2_{+-}(x,1-x) \\
&+ \frac{g^2}{\pi}\int_0^1 dy \left\{ \frac{f^2_{\uparrow\downarrow}(x,1-x) - f^2_{\uparrow\downarrow}(y,1-y)}{(y-x)^2} \right\} \\
&+ \frac{h}{\sqrt{x}} f^3_{\downarrow\uparrow\downarrow}(0,x,1-x) \\
&+ \frac{h}{\sqrt{1-x}} f^3_{\uparrow\downarrow\uparrow}(x,1-x,0)
\end{aligned} \tag{90}$$

$$\begin{aligned}
M^2 f^3_{-\downarrow\uparrow\downarrow}(x,y,1-x-y) &= \frac{m_b^2}{y} f^3_{\downarrow\uparrow\downarrow}(x,y,1-x-y) \\
&+ \frac{g^2}{\pi}\int_0^{1-x} dz \frac{1+y-x-z}{2(1-x-y-z)^2\sqrt{y(1-x-z)}} \{ f^3_{\downarrow\uparrow\downarrow}(x,y,1-x-y) \\
&- f^3_{\downarrow\uparrow\downarrow}(x,1-x-z,z) \} \\
&+ \frac{g^2}{\pi(1-x-y)} \left(\sqrt{1+\frac{1-x-y}{y}} - 1 \right) f^3_{\downarrow\uparrow\downarrow}(x,y,1-x-y) \\
&+ \frac{g^2}{\pi}\int_0^{x+y} dz \frac{x+2y-z}{2(x-z)^2\sqrt{y(x+y-z)}} \{ f^3_{\downarrow\uparrow\downarrow}(x,y,1-x-y) \\
&- f^3_{\downarrow\uparrow\downarrow}(z,x+y-z,1-x-y) \} \\
&+ \frac{g^2}{\pi x} \left(\sqrt{1+\frac{x}{y}} - 1 \right) f^3_{\downarrow\uparrow\downarrow}(x,y,1-x-y) \\
&+ \frac{g^2}{\pi}\int_0^{1-x} dz \frac{f^3_{\downarrow\uparrow\downarrow}(x,z,1-x-z)}{(1-x)\sqrt{yz}} \\
&+ \frac{g^2}{\pi}\int_0^{x+y} dz \frac{f^3_{\downarrow\uparrow\downarrow}(x+y-z,z,1-x-y)}{(x+y)\sqrt{yz}} \\
&+ \frac{h}{\sqrt{y}} f^2_{\uparrow\downarrow}(x+y,1-x-y)\delta(x) \\
&- \frac{h}{\sqrt{y}} f^2_{\downarrow\uparrow}(x,1-x)\delta(1-x-y)
\end{aligned} \tag{91}$$

and the same equations with $\uparrow \leftrightarrow \downarrow$, $h \leftrightarrow -h$. m_b is m_0 after renormalisation resulting from normal ordering. Although not necessary for our purposes of demonstrating pi-rho splitting, for generality we added a quark 'kinetic' mass m_f in the f^2 sector. This does not break quark helicity symmetry but may violate equal-time chiral symmetry. In fact we expect that

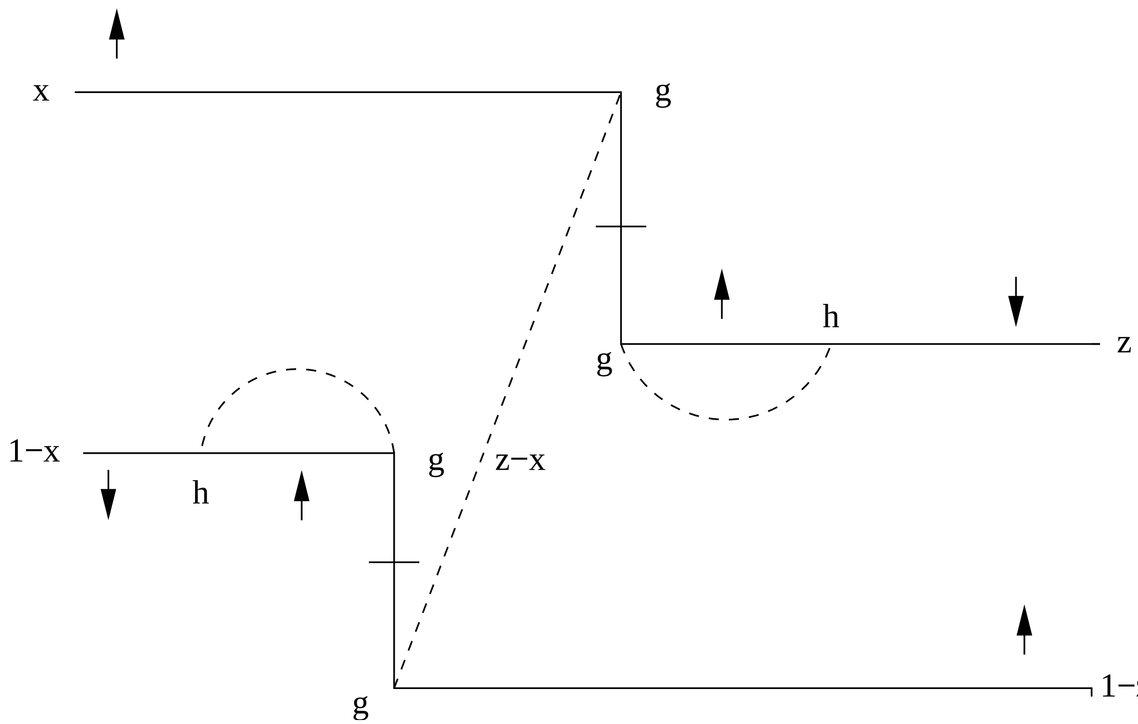

FIG. 1: Quark helicity flip process involving I_2. Solid lines are quarks (x^+-instantaneous when barred), chain lines are gluons. Vertices are labelled by their corresponding couplings.

DLCQ and a one-gluon truncation breaks this chiral symmetry, since it is a dynamical one, so that such counterms in the hamiltonian should be allowed. They can in principle be tuned to partially restore the explicitly broken chiral symmetry, for example by imposing PCAC on vacuum-to-pseudoscalar matrix elements. The corresponding boundstate equations for a meson with non-zero total helicity is the same as above, with suitable modified helicity labels and signs of the h interaction following from eqs.(47)

C. DLCQ gauge theory solution

The endpoint delta functions of momentum, seen in section IV A, now become spread out by the additional interactions. The I_2 interaction, that acts at endpoints only, couples to a part of the wavefunction of measure zero. One therefore expects the direct effects of I_2 in splitting the 0^{-+} and 0^{--} to vanish as DLCQ $K \to \infty$. However, I_2 can combine with other interactions to produce helicity-flip effects away from the endpoints.

An example at order $h^2 g^4$ is illustrated in Figure 1. The value that this process contributes to the expectation value of M^2 can be calculated in light-cone perturbation theory

$$h^2 \left(\frac{g^2}{\pi}\right)^2 \int_0^1 dx \int_x^1 dz \frac{f^2_{\uparrow\downarrow}(x, 1-x) f^2_{\downarrow\uparrow}(z, 1-z)}{(1-x)^2 \left(M^2 - \frac{m_b^2}{1-x}\right)(z-x)\left(M^2 - \frac{m_b^2}{z-x}\right)\left(M^2 - \frac{m_b^2}{z}\right) z^2} \tag{92}$$

If f^2 and M^2 are finite, then this contribution is finite for finite h.

The dimensionally reduced QCD boundstate equations truncated to at most one gluon were solved numerically in DLCQ for the range $K = 5$ to $K = 30$. The particular choice of the relevant parameters m_b, m_f, g, h is not very important since we did not try to tune them to obtain the best phenomenology. However, they were zoned to ensure absence of tachyons, that the $0^{-\pm}$ and 1^{--} states were the lightest in the spectrum, and a reasonable splitting of the $0^{-\pm}$ states compared to their masses. A typical result for the spectrum in this case is shown in Figure 2.

The M^2 eigenvalues have been fit to the form

$$M^2 = A + \frac{B}{\sqrt{K}} + \frac{C}{K} + \frac{D}{K^2} \,. \tag{93}$$

The polynomial dependence on $1/K$ follows from the fact that DLCQ effectively discretizes the integrals in (90)(91) by the trapezium rule. If the integrands are singular then non-polynomial finite-K errors can also occur. In particular there are square root endpoint singularities that give rise to $\frac{1}{\sqrt{K}}$ dependence. The singular longitudinal gauge interactions give rise to a $\frac{1}{K^\beta}$ dependence, where β depends on how the wavefunctions vanish at small x (if at all). β is unknown a priori but is dynamically determined. Although techniques exist for handling this error [17], for simplicity we did not include it because our fits were good enough for demonstrating splitting without it. The graph of fig.2 shows that the splitting of the $0^{-\pm}$ survives the DLCQ continuum limit $K \to \infty$. (The lightness of the 1^{--} is accidental in this example).

VI. CONCLUSIONS

Unphysical modes associated with the (operator valued) integration constants that must be included in the solutions of the constraint equations that arise in quantizing QCD in the light-cone representation, dress the bare light-cone vacuum and induce operators into the dynamics. In this paper, we have derived two such operators, based on an ansatz for the vacuum that comes partly from analysis and partly from analogy with the Schwinger

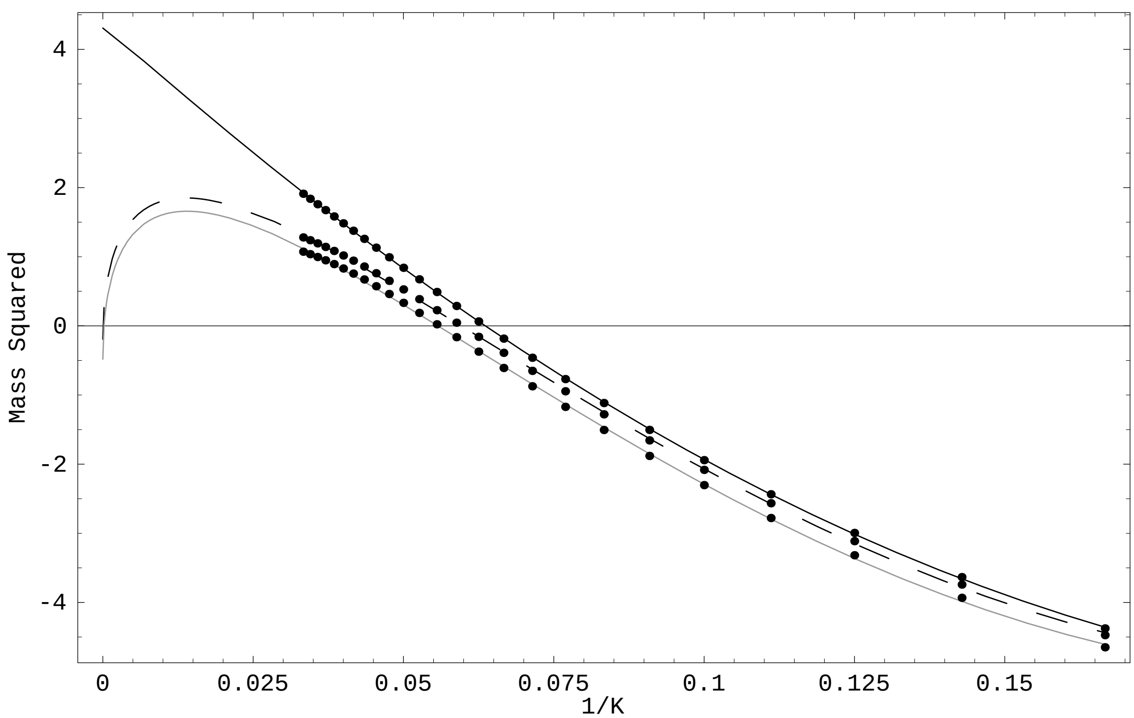

FIG. 2: Variation of mass squared M^2 with DLCQ cutoff K for the lightest three energy levels. The raw DLCQ data is smoothly extrapolated to $K = \infty$: solid line 0^{--}; chain line 0^{-+}; gray line 1^{--}

model. One of these operators breaks quark helicity conservation for zero quark bare mass. We have demonstrated — explicitly in the case of a dimensional reduced gauge theory — how these operators lead to a splitting of the masses of the pion and rho, the helicity zero components of which would be degenerate otherwise.

The vacuum we have used in this paper is probably not the full vacuum, which may well include gluons; even the fermion content we have presented here might not be complete. Furthermore, we have had to make some guesses in the derivation and these might have to be revised after further study. We do believe that the qualitative mechanism for spontaneous symmetry breaking presented in this paper is correct and that the fermion content of the vacuum is either as we have presented it or very similar. While we could obtain the form of induced operators, we do not have reliable estimates for their coefficients, including the possibility that if they are included in a DLCQ calculation these coefficients might scale in a nontrivial way with the DLCQ harmonic resolution, K.

A natural framework for explicit DLCQ calculations in $3+1$ dimensions would be transverse lattice QCD [18], where similar induced operators will arise to spontaneously break quark helicity symmetry. Previous calculations used a trivial vacuum where it was necessary to break it explicitly in the Lagrangian [19]. Another question concerns the perturbation of our results in quark mass μ, which we have only briefly addressed here, and the relationship with the results of ref.[11]. These topics are now under investigation. Finally we draw the readers attention to other recent work on chiral symmetry in light-cone field theory using path integrals [20].

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