You may use Reif or another text of your choice, but no other notes or references.

Required:
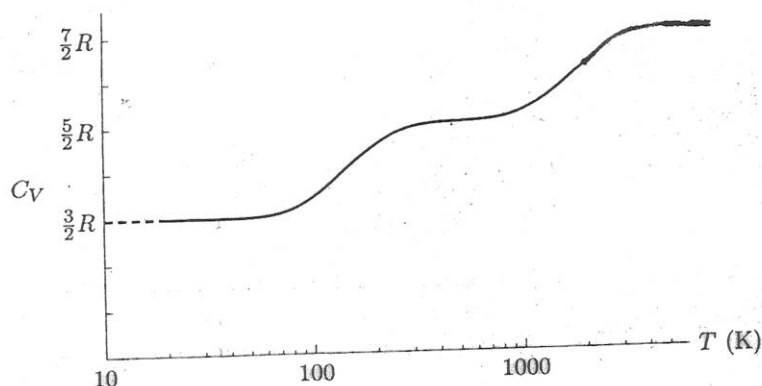
1. The molar specific heat of a diffuse gas made of a certain diatomic molecule as a function of temperature behaves as shown. Estimate the spacings between energy levels for rotational excitations, and for vibrational excitations, for this molecule. (Very simple. You don't need to give numerical values for the estimates.)

2. The magnetization M of a macroscopic magnetic material in a constant magnetic field H is observed to satisfy the equation of state (for sufficiently high temperature T)

$$M = a\frac{H}{T} \, ,$$

while the heat capacity at fixed magnetic field $H = H_0$ satisfies

$$C_H(T, H_0) = \frac{b}{T^2} \, .$$

Assume the system is sufficiently rigid that the volume V is fixed throughout and so may be ignored. From electromagnetic theory, we know that M is the generalized force associated with changes in H.

(a) Give the fundamental thermodynamic relation for this system expressing dS in terms of dE and dH (simple).

(b) Derive a relation between the heat capacity at fixed magnetic field, $C_H(T, H)$, and at fixed magnetization, $C_M(T, H)$, in terms of T and H and the constants a and b.

(c) Compute $(\partial S/\partial H)_T$.

(d) Find $S(T, H) - S(T_0, H_0)$ for any T and H.

(e) If the system is isolated, starts with T_i and H_i, and H is changed quasistatically to H_f, find T_f.

(f) Determine $C_H(T, H)$ for arbitrary magnetic field H.

(g) Compute $S(T, H)$ using the previous result by first computing $S(T, H) - S(T_0, H)$, then $S(T_0, H) - S(T_0, H_0)$.

(h) Now assume the material is made of N spin-1/2 molecules with moments along the direction of H of $\pm\mu$. Give an expression for the partition function $Z(T, H)$, and use it to compute the average energy $\overline{E}(T, H)$.

(i) Give an expression for $\overline{E}$ valid at large T (leading order only) and use it to determine C_H in that limit. Use these to give values for a and b.

(j) Derive from Z an expression for $S(T, H)$ in the same limit, good to order $1/T^2$. Explain the significance of the leading term. Show your result is consistent with that of part (e).

Choose one of the following problems:

3. Consider a classical description of N weakly interacting magnetic atoms at a temperature T. Each magnetic moment μ can make an arbitrary angle θ with respect to a the z axis. In the absence of a magnetic field, the probability that this angle lies between θ and $\theta + d\theta$ is proportional to the solid angle $2\pi \sin\theta d\theta$. In the presence of a magnetic field H in the z direction, this probability must further be proportional to the Boltzmann factor $\exp(-\beta E)$, where E is the magnetic energy of the moment μ at an angle θ.

(a) Use this result to calculate the classical expression for the mean magnetic moment $\overline{M_z}$ of these N atoms.

(b) The correct quantum result, with $\mu_z = g\mu_0 m$, is

$$\overline{M_z} = Ng\mu_0 \left[\left(J + \frac{1}{2}\right) \coth\left(J + \frac{1}{2}\right)\eta - \frac{1}{2}\coth\frac{1}{2}\eta \right]$$

with

$$\eta \equiv \frac{g\mu_0 H}{kT}.$$

Show this reduces to the classical result in (a) in the appropriate limit where the quantum spacings are small, $\eta \ll 1$, and J is large enough, $J\eta \gg 1$, so that the values of $\cos\theta = m/J$ are very closely spaced.

4. Two types of molecules, N_1 of type 1, and N_2 type 2, are originally separated by a wall into two containers of volume V_1 and V_2, but in thermal equilibrium at temperature T. The gases are ideal, and the combined system is isolated. Use the laws of classical thermodynamics to compute the change in entropy which results when you punch a hole in the wall and allow the gases to mix. To do this, you may assume the existence of two semipermeable membranes, one which is transparent to molecules of type 1 but blocks type 2, and the other which is transparent to type 2 but blocks type 1. State the answer in terms of N_1 and N_2.

5. The nuclei of atoms in a certain crystalline solid have spin one. Since the electric charge distribution in the nucleus is not spherically symmetric, but elipsoidal, the energy of a nucleus depends on its spin orientation with respect to the electric field at its location. Thus a nucleus has the same energy $E = \epsilon$ in the states $m = \pm 1$, and $E = 0$ in the state $m = 0$.

(a) Give the nuclear contribution to the molar internal energy of the solid as a function of T.

(b) Give the nuclear contribution to the molar entropy of the solid as a function of T.

(c) By directly counting the total number of accessible states, calculate the nuclear contribution to the molar entropy of the solid at very low temperatures. Calculate it also at very high temperatures. Show these are constistent with the result from (b) in the appropriate limits.

(d) Calculate the molar heat capacity as a function of T. How does it behave for small and large T? Sketch it.

You may use Reif or another text of your choice, no other notes or references.

Choose one of the two following problems:

1. A vertical cylinder contains ν moles of an ideal gas and is closed off by a piston of mass M and area A. The molar specific heat c_V (at constant volume) of the gas is a constant independent of temperature. The heat capacities of the piston and cylinder are negligibly small and any frictional forces between the piston and the cylinder walls can be neglected. The whole system is thermally insulated. Initially, the piston is clamped in position so that the gas has a volume V_0 and a temperature T_0. The piston is now released and, after some oscillations, comes to rest in a final equilibrium configuration where the volume of the gas is larger.

    (a) Does the temperature of the gas increase, decrease, or remain the same? (one sentence)

    (b) Does the entropy of the gas increase, decrease, or remain the same? (one sentence)

    (c) Calculate the final temperature of the gas in terms of T_0, V_0, ν, M, A, c_V, the acceleration due to gravity g, and the gas constant R.

    (d) Calculate the change in entropy in terms of these same constants.
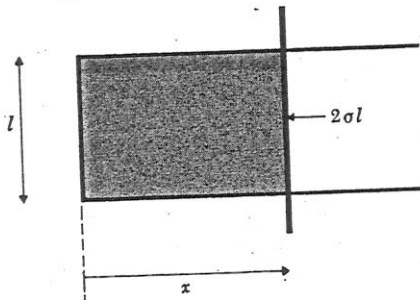
2. The figure illustrates a soap film (shown in gray) supported by a wire frame. Because of surface tension the film exerts a force $2\sigma\ell$ on the cross wire. This force is in such a direction that it tends to move this wire so as to decrease the area of the film. The quantity σ is called the surface tension of the film, and the factor 2 occurs because the film has two surfaces. The temperature dependence of σ is given by

$$\sigma = \sigma_0 - \alpha T \qquad (1)$$

    where σ_0 and α are constants independent of T or x.

    (a) Suppose that the distance x is the only external parameter of significance in the problem. Write a relation expressing the change dE in mean energy of the film in terms of the heat $đQ$

absorbed by it and the work done by it in an infinitesimal quasi-static process in which the distance x is changed by an amount dx.

(b) Calculate the change in entropy $\Delta S = S(x) - S(0)$ and the change in mean energy $\Delta E = E(x) - E(0)$ of the film when it is stretched at a constant temperature T_0 from a length $x = 0$ to a length x.

(c) Calculate the work $W(0 \rightarrow x)$ done and heat $Q(0 \rightarrow x)$ absorbed by the film as it is stretched at this constant temperature from a length $x = 0$ to a length x.

Choose one of the two following problems:

1. A rubber band at absolute temperature T is fastened at one end to a peg, and supports from its other the mass M. Assume as a simple microscopic model of the rubber band that it consists of a linked polymer chain of N segments joined end to end; each segment has length a and can be oriented either parallel or antiparallel to the vertical direction. Find an expression for the partition function, and for the resultant mean length $\bar{l}$ of the rubber band as a function of M. (Neglect the kinetic energies or weights of the segments themselves, or any interaction between the segments.)

2. Consider a diatomic molecule of non-identical atoms with moment of inertia I whose center of mass is fixed, but which is free to rotate. Recall that the Hamiltonian is $L^2/2I$. You will treat this system quantum mechanically.

   (a) Give (but do not evaluate) expressions for the partition function Z and average energy $\overline{E}$ for this system at absolute temperature T (that is, in thermal equilibrium with a heat bath at T) as a function of I.

   (b) Derive an approximation for Z valid at high absolute temperature T. (If you have trouble evaluating the integral, you're probably missing something.) At what values of T is this valid?

   (c) Derive an approximation for Z valid at low T by keeping only the first two terms. Justify this approximation.

   (d) Give the average energy $\overline{E}$ and heat capacity $C(T)$ in both limits.

For Pavel
Sophia

Statistical Mechanics and Thermodynamics
January 2003

You may use Reif, lecture notes, and one book of math tables.

Choose one of the two following problems:

1. Consider a system composed of a very large number N of distinguishable atoms at rest and mutually noninteracting, each of which has only two (nondegenerate) energy levels: 0 and $\epsilon > 0$. Let E/N be the mean energy per atom in the limit $N \to \infty$.

   (a) What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium? What is the maximum attainable value or E/N if the system is in equilibrium (at positive temperature)?

   (b) For thermodynamic equilibrium compute the entropy per atom S/N as a function of E/N.

2. Consider a cylinder 1m long with a thin, massless piston clamped in such a way that it divides the cylinder into two equal parts. The cylinder is in a large heat bath at $T = 300°K$. The left side of the cylinder contains 1 mole of helium gas at 4 atm. The right contains helium gas at a pressure of 1 atm. Let the piston be released. (You may treat the helium as an ideal gas.)

   (a) What is the final equilibrium position?

   (b) How much heat will be transmitted to the bath in the process of equilibration? (Note that $R = 8.3$ J/mole °K, and 1 atm $= 1.013 \times 10^5$ N/m^2)

Choose one of the two following problems:

1. For a photon in mode i, the energy ϵ_i and angular frequency ω_i are related by $\epsilon_i = \hbar\omega_i$.

   (a) State or derive an expression for the partition function Z in terms of ω_i. From this, derive expressions for the average energy E, entropy S, and free energy F.

   (b) From these, derive an expression for the average pressure P, and show that the isothermal work done by the gas is

   $$dW = -\sum_i n_i \hbar \frac{d\omega_i}{dV} dV \tag{1}$$

   where n_i is the average number of photons in the ith mode.

   (c) Show that the radiation pressure is equal to one third of the energy density:

   $$P = \frac{1}{3}\frac{E}{V} . \tag{2}$$

SHOW YOUR WORK IN ALL PROBLEMS

Work 3 of the following 4 problems.

You may take the value of $\hbar$ to be 1 for all problems.

1. For a 2-state system the Hamiltonian is:

$$H = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$$

Another observable, Q, is represented by the operator:

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

At $t = 0$ the system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If Q is measured at time, t, what are the possible outcomes and what are their respective probabilities?

2. The unperturbed Hamiltonian for a harmonic operator is given by

$$H = \omega(a^\dagger a + \frac{1}{2})$$

The system is subjected to the perturbation

$$\lambda(a^\dagger a^\dagger + aa)$$

Find the time evolution operator in the interaction picture through order λ^2.

3. Find the S-wave phase shift for the potential

$$V(r) = \begin{cases} \infty, & \text{if } r \leq a; \\ -V_0, & \text{if } a \leq r \leq b; \\ 0, & \text{otherwise.} \end{cases}$$

4. (a) Let $\hat{H}$ be the Hamiltonian operator of a physical system. Denote by $|\phi_n\rangle$ the eigenvectors of $\hat{H}$ with eigenvalues E_n. For an arbitrary operator $\hat{A}$, what is

$$\langle \phi_n | [\hat{A}, \hat{H}] | \phi_n \rangle$$

Simplify your result as much as possible.

(b) If the above result is true for all n, does $\hat{A}$ commute with $\hat{H}$? Explain in detail.

(c) The three Cartesian components of the angular momentum operator have non-vanishing commutators: $[\hat{L}_x, \hat{L}_y] \neq 0$, $[\hat{L}_y, \hat{L}_z] \neq 0$, $[\hat{L}_z, \hat{L}_x] \neq 0$, but it is still possible to find a state that is a simultaneous eigenfunction of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ (for example, the spherically symmetric S state with eigenvalue 0 for each component). Explain in detail why there is no contradiction here.

Statiscal Mechanics Qualifying Exam Fall 2002

1. Consider N spins in a chain which can be modeled using the one-dimensional Ising Model

$$H = -J \sum_{n=1}^{N-1} s_n s_{n+1},$$

   where the spin has the values $s_n = \pm 1$

   (a) Find the partition function.

   (b) Find the heat capacity per spin.

2. The circuit in the figure below consists of a coil of inductance L and a capacitor of capacitance C. What is the rms noise voltage across AB at temperature τ in the limit where,

   (a) τ is very large?

   (b) τ is very small?

1
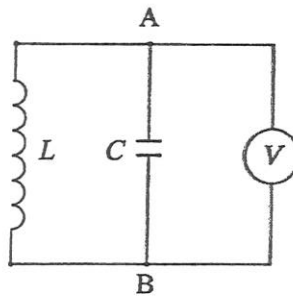
(To do this, consider the gas in a cubic box with periodic boundary conditions.)

(d) Show that for a nonrelativistic Fermi gas the pressure is

$$P = \frac{2}{3}\frac{E}{V} \ .$$
(3)

2. A very sensitive spring balance consists of a quartz spring suspended from a fixed support. The spring constant is α; that is, the restoring force of the spring is $-\alpha x$ if the spring is stretched by an amount x. The balance is at a temperature T in a location where the acceleration due to gravity is g.

(a) If a very small object of mass M is suspended from the spring, what is the mean resultant elongation $\bar{x}$ of the spring.

(b) What is the magnitude $\overline{(x - \bar{x})^2}$ of the thermal fluctuations of the object about its equilibrium position?

(c) It becomes impracticable to measure the mass of an object when the fluctuations are so large that $|\overline{(x - \bar{x})^2}|^{1/2} = \bar{x}$. What is the minimum mass M which can be measured with this balance?

STATISTICAL MECHANICS

MAY 99

SHOW YOUR WORK IN ALL PROBLEMS. CHOOSE ONE OF THESE TWO PROBLEMS.

1. Consider a non-interacting Boltzmann gas of N spin-$\frac{1}{2}$ particles in a cubical box of dimensions $0 \leq x \leq L$, $0 \leq y \leq L$, and $0 \leq z \leq L$. Each particle has magnetic moment μ. There is an applied magnetic field in the z-direction $B = B(z)\hat{z}$ with a gradient in z.

$$B(z) = B_o + B'z$$

(a) What is the z-dependence of the magnetization (dipole moment per unit volume) inside the box?

(b) What is the limit of the magnetization as $B'L \ll B_o$?

(c) What is the average energy $\bar{E}$?

(d) Is $\bar{E}(T \to 0)$ what you would expect? Why?

2. Consider a system of $N \gg 1$ spins fixed to lattice sites; the spins have energy ϵ when pointed in the +z direction, and energy 0 when pointed in the -z direction.

In the microcanonical ensemble, the entropy S is found as a function of the internal energy U and the particle number N. The temperature T is defined by

$$T^{-1} = \frac{\partial S}{\partial U}\bigg|_{N}.$$

(a) Show that

$$S = k\left[N \ln N - \frac{U}{\epsilon}\ln\frac{U}{\epsilon} - \left(N - \frac{U}{\epsilon}\right)\ln\left(N - \frac{U}{\epsilon}\right)\right],$$

and

$$U = N\epsilon\left(1 + e^{\beta\epsilon}\right)^{-1}, \qquad \left(\beta \equiv \frac{1}{kT}\right).$$

[Use Stirling's formula: $\ln(N!) \sim N \ln N - N$, for $N \gg 1$.]

(b) Show that if $U > \frac{1}{2}N\epsilon$, the temperature T is negative. Based on elementary physical considerations of the underlying spin system, is this system with $T < 0$ hotter or colder than the same system with $T > 0$? That is, if a system with N spins, internal energy U_1, and $T_1 < 0$ comes to equilibrium with a system with N spins, internal energy U_2, and $T_2 > 0$, in which direction will heat be transferred?

(c) Find entropy S as a function of temperature and show that $S \to 0$ as $T \to 0$.

3. Consider an ELECTRICALLY NEUTRAL box of quarks, with total number, N, quarks per unit volume.

a) To start, let there be up (u) quarks (charge q=+2/3) and down (d) quarks (q=-1/3) (with a total number N quarks per unit volume). Assume that the mass of the quarks is zero. Find the Fermi energy for each species and the average energy per quark, averaged over both species.

b) Now let there be up quarks, down quarks and strange quarks (q=-1/3), with a total number N quarks per unit volume. Again, find the Fermi energies and the average energy. Compare with part (a).

c) Suppose the strange quark (s) has mass m. Assume that s-quarks and d-quarks can transform into one another (say by the weak interactions). Keep the total number of quarks at N per unit volume. As m goes to zero, we recapture part (b). What happens as m increases? What is the biggest m can be before there will be no s-quarks?

d) Relax the requirement of electrical neutrality and assume that any kind of quark can transform into any other. Keep N the total number of quarks per unit volume. Find the equation or equations for the Fermi energy or energies as a function of N and m. What happens now as m goes to zero and what is the new maximum for m before there will be no more s-quarks? If you can't solve exactly, approximate for m near that maximum value and find the number of up, down and strange quarks per unit volume in terms of N and m in that range.

1. a. The interaction between certain atoms is described by the two-body
        potential

$$V(r) = \begin{cases} \infty, & r < a \\ -V_o \left(\frac{a}{r}\right)^6, & r > a, \quad V_o > 0. \end{cases}$$

    Calculate the classical second virial coefficient correct to order $\frac{1}{T}$
    for a gas of these atoms for temperatures such that $kT \gg V_o$, and
    give the resulting expression for P/nkT (n = number density).

    b. For what range of temperature can the quantum mechanical corrections
        to the classical result be ignored? Give a quantitative criterion.

pick

2. What is meant by a "mean field approximation" in statistical mechanics?
    Explain in the context of the Debye-Hückel theory of screening. When is
    this approximation valid? When is a cluster expansion useful?

3. Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ for an ideal,
    completely degenerate Fermi gas.

4. The canonical partition function for a collection of N atoms distributed
    over M adsorption sites on a solid surface is

$$Z_N = \frac{M^N}{N!} e^{N\beta\epsilon}$$

    where ϵ is the binding energy per occupied site.

    a. Calculate the grand partition function $\mathcal{J}$, and determine the chemical
        potential μ_s of the adsorbed atoms as a function of the fraction of
        occupied sites, x = N/M.

    b. Determine the pressure of an ideal gas of the same atoms which will
        be in equilibrium with the adsorbate for a given value of x (i.e.,
        the vapor pressure of the adsorbed phase).

Pick 1:   2 or 3

Pick 3:   1, 4, 5, 6

3. Consider an ELECTRICALLY NEUTRAL box of quarks, with total number, N, quarks per unit volume.

a) To start, let there be up (u) quarks (charge $q=+2/3$) and down (d) quarks ($q=-1/3$) (with a total number N quarks per unit volume). Assume that the mass of the quarks is zero. Find the Fermi energy for each species and the average energy per quark, averaged over both species.

b) Now let there be up quarks, down quarks and strange quarks ($q=-1/3$), with a total number N quarks per unit volume. Again, find the Fermi energies and the average energy. Compare with part (a).

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15. STATISTICAL MECHANICS (20 points)

Consider a non-interacting Boltzmann gas of N spin 1/2 particles in a cubical box of dimensions $0 \leq x \leq L$, $0 \leq y \leq L$ and $0 \leq z \leq L$. Each particle has magnetic moment μ. There is an applied magnetic field in the z-direction $B=B(z)\hat{z}$ with a gradient in z

$$B(z) = B_0 + B'z.$$

(A) What is the z-dependence of the magnetization (dipole moment per unit volume) inside the box? What is the limit of the magnetization as $B'L \ll B_0$?

(B)

(C) WHAT IS AVERAGE ENERGY, $\bar{E}$?

(D) IS $\bar{E}$ (T→α) WHAT $\hbar\omega$ WOULD EXPECT? WHY?

16. STATISTICAL MECHANICS (20 points)

Consider a system of $N \gg 1$ spins fixed to lattice sites; the spins have energy ε when pointed in the $+z$ direction, and energy 0 when pointed in the $-z$ direction.

In the microcanonical ensemble, the entropy S is found as a function of the internal energy U and the particle number N. The temperature T is defined by

$$T^{-1} = \partial S / \partial U \big|_N.$$

(a) Show that

$$S = k \left\{ N \ln N - \frac{U}{\varepsilon} \ln \frac{U}{\varepsilon} - \left(N - \frac{U}{\varepsilon} \right) \ln \left(N - \frac{U}{\varepsilon} \right) \right\},$$

and

$$U = N\varepsilon \left(1 + e^{\beta \varepsilon} \right)^{-1}, \qquad (\beta = 1/kT).$$

[Use Stirling's formula: $\ln(N!) \sim N \ln N - N$, for $N \gg 1$.]

(b) Show that if $U > \frac{1}{2} N\varepsilon$, the temperature T is negative. Based on elementary physical considerations of the underlying spin system, is this system with $T<0$ hotter or colder than the same system with $T>0$? That is, if a system with N spins, internal energy U_1, and $T_1<0$ comes to equilibrium with a system with N spins, internal energy U_2, and $T_2>0$, in which direction will heat be transferred?

(C) Find ∅ ENTROPF S AS FUNCTION OF TEMPERATURE AND SHOW S → 0 AS T → 0

1. a. The interaction between certain atoms is described by the two-body potential

$$V(r) = \begin{cases} \infty, & r < a \\ -V_o \left(\frac{a}{r}\right)^6, & r > a, \quad V_o > 0. \end{cases}$$

Calculate the classical second virial coefficient correct to order $\frac{1}{T}$ for a gas of these atoms for temperatures such that $kT \gg V_o$, and give the resulting expression for P/nkT (n = number density).

b. For what range of temperature can the quantum mechanical corrections to the classical result be ignored? Give a quantitative criterion.

Pick 1 of 2

2. What is meant by a "mean field approximation" in statistical mechanics? Explain in the context of the Debye-Hückel theory of screening. When is this approximation valid? When is a cluster expansion useful?

3. Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$ for an ideal, completely degenerate Fermi gas.

4. The canonical partition function for a collection of N atoms distributed over M adsorption sites on a solid surface is

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where ε is the binding energy per occupied site.

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b. Determine the pressure of an ideal gas of the same atoms which will be in equilibrium with the adsorbate for a given value of x (i.e., the vapor pressure of the adsorbed phase).

Pick 1:   2 or 3

Pick 3:   1, 4, 5, 6

5. a. What is meant by the "order" of a phase transition?

   b. Give an equation which determines the location of the transition line in, for example, a liquid-gas transition (P,T diagram), and explain how this result is derived. What is meant in this context by a critical point?

   c. A typical $\mathcal{H}$, T diagram for a type I superconductor is shown below. The total magnetization is zero in the shaded (superconducting) region (Meissner effect) and non-zero outside. Determine the order of the superconducting phase transition when $\mathcal{H}$ is varied with T fixed, $0 < T < T_c$. Explain (or show) how you obtain your result.
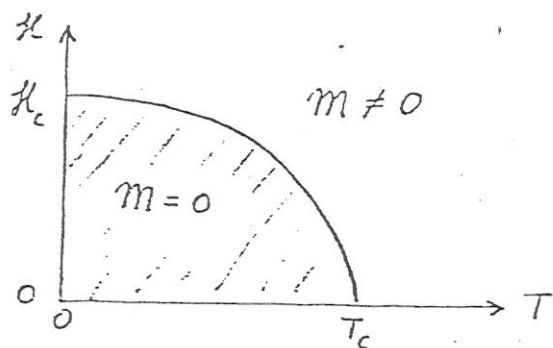
6. A system of N distinguishable spin-1 particles on a lattice is placed in a magnetic field $\mathcal{H}$. The possible energies for the individual spins are $-\mu\mathcal{H}$, 0, $+\mu\mathcal{H}$ for spin projections +1, 0, -1 along $\mathcal{H}$.

   a. Calculate the canonical partition function for the system.

   b. Calculate the entropy of the system, and determine its high temperature limit. What is the interpretation of this limit in terms of Boltzmann's definition of S?
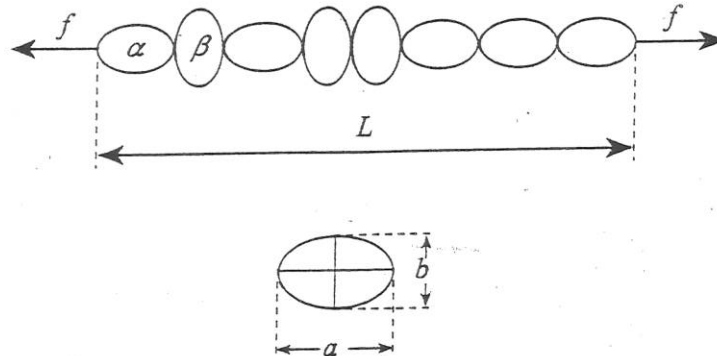
Statistical Mechanics and Thermodynamics
May 2000

Do one of the following two problems, showing all work:

1. Consider a one-dimensional chain consisting of N molecules which exist in two configurations, α and β, with corresponding energies ε_α and ε_β, and lengths a and b. The chain is subject to a constant tensile force f.

(a) Write the partition function Z_N for the system. (Account for the potential energy associated with f in addition to the individual energies ε_α and ε_β.)

(b) Calculate the average length $\langle L \rangle$ as a function of f and the temperature T.

(c) Assume that $\varepsilon_\alpha > \varepsilon_\beta$ and $a > b$. Give the average length $\langle L \rangle$ in the absence of the tensile force $f = 0$. What are the high- and low-temperature limits, and what is the characteristic temperature at which the changeover between the two limits occurs?

(d) Calculate the linear response function

$$\chi = \left(\frac{\partial \langle L \rangle}{\partial f} \right)_{f=0} .$$

2. A vertical cylinder contains ν moles of an ideal gas and is closed off by a piston of mass M and area A. The molar specific heat c_V (at constant volume) of the gas is a constant independent of temperature. The heat capacities of the piston and cylinder are negligibly small and any frictional forces between the piston and the cylinder walls can be neglected. The whole system is thermally insulated. Initially, the piston is clamped in position so that the gas has a volume V_0 and a temperature T_0. The piston is now released and, after some oscillations, comes to rest in a final equilibrium configuration where the volume of the gas is larger.

(a) Does the temperature of the gas increase, decrease, or remain the same?

(b) Does the entropy of the gas increase, decrease, or remain the same?

(c) Calculate the final temperature of the gas in terms of T_0, V_0, ν, M, A, c_V, the acceleration due to gravity g, and the gas constant R.

(d) Calculate the change in entropy in terms of these same constants.

### Statistical Mechanics and Thermodynamics.
Aug 2000

Do one of the following two problems, showing all work. You may use Reif and class notes.

1. A horizontal zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ϵ. We require that the zipper only unzip from one side (say from the left) and that a link can only open if all links to the left of it $(1, 2, \ldots, n-1)$ are already open. (This model is sometimes used for DNA molecules.)
(a) Find the partition function.
(b) Find the average number of open links $<n>$ and show that for low temperatures $kT \ll \epsilon$, $<n>$ is independent of N.

2. The cycle of a highly idealized gasoline engine can be approximated by the Otto cycle, illustrated. The paths $1 \to 2$ and $3 \to 4$ are adiabatic compression and expansion, respectively; $2 \to 3$ and $4 \to 1$ are constant-volume processes. Treat the working medium as an ideal gas with constant $\gamma = c_p/c_v$.
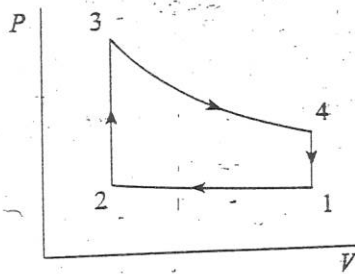
(a) Compute the efficiency of this cycle for $\gamma = 1.4$ and compression ratio $r = V_i/V_f = 10$.
(b) Calculate the work done on the gas in the compression process $1 \to 2$, assuming initial volume $V_i = 2L$ (liters) and $p_i = 1$ atm.

STATISTICAL MECHANICS

MAY 99

SHOW YOUR WORK IN ALL PROBLEMS. CHOOSE ONE OF THESE TWO PROBLEMS.

1. Consider a non-interacting Boltzmann gas of N spin-$\frac{1}{2}$ particles in a cubical box of dimensions $0 \leq x \leq L$, $0 \leq y \leq L$, and $0 \leq z \leq L$. Each particle has magnetic moment μ. There is an applied magnetic field in the z-direction $B = B(z)\hat{z}$ with a gradient in z.

$$B(z) = B_o + B'z$$

   (a) What is the z-dependence of the magnetization (dipole moment per unit volume) inside the box?

   (b) What is the limit of the magnetization as $B'L \ll B_o$?

   (c) What is the average energy $\bar{E}$?

   (d) Is $\bar{E}(T \to 0)$ what you would expect? Why?