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Exact solutions to Pauli–Villars-regulated field theories*

Stanley J. Brodsky
Stanford Linear Accelerator, Stanford University, Stanford, California 94309

John R. Hiller
Department of Physics, University of Minnesota-Duluth, Duluth, Minnesota 55812

Gary McCartor
Department of Physics, Southern Methodist University, Dallas, TX 75275

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Abstract

We present a new class of quantum field theories which are exactly solvable. The theories are generated by introducing Pauli–Villars fermionic and bosonic fields with masses degenerate with the physical positive metric fields. An algorithm is given to compute the spectrum and corresponding eigensolutions. We also give the operator solution for a particular case and use it to illustrate some of the tenets of light-cone quantization. Since the solutions of the solvable theory contain ghost quanta, these theories are unphysical. However, we also discuss how perturbation theory in the difference between the masses of the physical and Pauli–Villars particles could be developed, thus generating physical theories. The existence of explicit solutions of the solvable theory also allows one to study the relationship between the equal-time and light-cone vacua and eigensolutions.

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I. INTRODUCTION

Exact solutions to quantum field theories in physical space-time with non-trivial interactions are rare. In this paper we shall show how one can obtain the complete eigenspectrum and eigensolutions of a quantum field theory of interacting massive fermions and bosons in 3+1 space-time dimensions. The basic format of the solvable theory is the conventional Yukawa theory with $g\phi\overline{\psi}\psi$ interactions, accompanied by negative-metric Pauli–Villars (PV) boson and fermion fields with masses degenerate with the physical quanta. An algorithm is then given which generates the complete eigenspectrum and the corresponding eigensolutions, with respect to a light-cone-quantized Fock basis.

Since the solutions of the solvable theory contain ghost quanta, these theories are unphysical. However, we will discuss how perturbation theory in the difference between the masses of the physical and PV particles can be developed, thus ultimately generating physical theories in which the wave functions allow one to compute space-like and time-like form factors and other quantities of phenomenological interest. Conversely, the exact solutions provide boundary conditions for the wave functions of the physical theory in the limit of degenerate masses. The explicit solutions of the solvable theory also allow one to study the relationship between the equal-time and light-cone vacua and eigensolutions, and they display properties due to covariance, such as light-cone spin conservation, which are characteristic of physical nonperturbative eigensolutions. In addition, such solutions provide important checks of computer codes for discretized light-cone quantization (DLCQ) [1,2] of nondegenerate theories [3].

Pauli–Villars regularization [4] is an important method for regulating the ultraviolet divergences of light-cone Hamiltonian theories. We have previously shown that the use of PV regulation provides a correct renormalization of DLCQ, for Yukawa theory at least to one loop; in contrast, a momentum cutoff of DLCQ does not preserve the chiral properties of the theory [5]. We have also made a number of studies showing the practicality of using PV regulation in 3+1 nonperturbative DLCQ calculations [5,6,3]. In an important development, Paston and Franke [7], and Paston, Franke and Prokhvatilov [8] have now shown that regulation with the correct combination of PV fields always gives perturbative agreement with Feynman theory, and they have given a complex set of rules for deciding which set of PV fields are sufficient to regulate a given theory at all orders.

In the present paper we will show that if a theory is regulated with PV fields and the masses of the PV fields are set equal to the masses of the physical fields, the resulting theory is easy to solve. After some discussion of PV-regulated Yukawa theory in Sec. II, we give the general procedure for finding eigenvalues and eigenvectors in Sec. III. We then give in Sec. IV an operator solution for Yukawa theory and use it to find the relation between the light-cone basis states and the equal-time basis states. We show that, not only is the light-cone perturbative vacuum equal to the physical vacuum while the equal-time perturbative vacuum is not, but that all the eigenstates are much simpler when expressed in the light-cone representation than when expressed in the equal-time representation. In Sec. V we show that the procedure works for the case of several PV fields of the same type and give an explicit example.

Since the masses of the ghost metric quanta are degenerate with the masses of physical particles, the solutions of the solvable theories will violate unitarity, and they are thus

not physical. Since the exact solutions exist for any value of the coupling constant, one can construct the solution of a theory with large values of the PV masses as a perturbative expansion, not in powers of the coupling constant, but in powers of the difference between the PV masses and the physical masses. Such a perturbation theory would require the expansion parameter to have large values, so its practical utility will depend on the analytic properties of the solution in the mass differences. The calculation of scattering matrix elements and other physical quantities as a perturbation theory in powers of the mass differences involves polynomials in the coupling constant of no higher order than the order of the expansion in the masses. Therefore, although it might at first seem otherwise, low orders of the new series will not contain information from higher order Feynman graphs. The solutions may be of use in studying various general properties of quantum field theories even if the convergence of the new perturbation series is not very good. We do know that the solutions have at least one use: we have used them to help debug the computer code used in the calculations of [3]. Additional conclusions and applications are discussed in Sec. VI. Our light-cone conventions and definitions are collected in an appendix.

II. PAULI–VILLARS REGULARIZATION OF YUKAWA THEORY

We begin our discussion with the light-cone quantization of Yukawa theory in $3+1$ dimensions. In order to keep the notation as simple as possible, we shall first introduce just one PV boson and one PV fermion although this is not sufficient to regulate the full Yukawa theory; the results of Paston and Franke [7] show that one bosonic PV field and two PV fermion fields will regulate the theory in such a way that it is perturbatively equivalent to Feynman theory. However, for the purposes of this section, one can omit fermion loops (the theory studied in [3]), or introduce an additional transverse momentum cutoff. Paston *et al.* [9] have suggested that one bosonic PV field and one PV fermion field plus a transverse cutoff regulates the theory in such a way as to generate only mass, coupling constant, and wave function renormalization.

Taking the physical fields to be ψ_1 and ϕ_1 and the PV (negative-metric) fields to be ψ_2 and ϕ_2, the action becomes

$$\begin{aligned} S = \int d^4x \Bigg[& \frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}\mu_1^2 \phi_1^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2}\mu_2^2 \phi_2^2 \\ & + \frac{i}{2}\Big(\overline{\psi}_1 \gamma^\mu \partial_\mu - (\partial_\mu \overline{\psi}_1)\gamma^\mu\Big)\psi_1 - m_1 \overline{\psi}_1 \psi_1 - \frac{i}{2}\Big(\overline{\psi}_2 \gamma^\mu \partial_\mu - (\partial_\mu \overline{\psi}_2)\gamma^\mu\Big)\psi_2 + m_2 \overline{\psi}_2 \psi_2 \\ & - g\phi\overline{\psi}\psi \Bigg], \end{aligned} \tag{2.1}$$

where the Yukawa three-point interaction is expressed in terms of

$$\psi \equiv \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)\,, \quad \phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + \phi_2)\,. \tag{2.2}$$

For simplicity, we have not considered here a ϕ^4 term; it is, however, easily included. We also define

$$\eta \equiv \frac{1}{\sqrt{2}}(\psi_1 - \psi_2)\,, \quad \zeta \equiv \frac{1}{\sqrt{2}}(\phi_1 - \phi_2)\,. \tag{2.3}$$

Notice that η and ζ are zero-norm fields. In terms of these fields we have

$$\begin{aligned} S = \int d^4x \Bigg[& \frac{1}{2}(\partial_\mu\phi\partial^\mu\zeta + \partial_\mu\zeta\partial^\mu\phi) - \frac{1}{2}(\frac{1}{2}(\mu_1^2+\mu_2^2))(\phi\zeta+\zeta\phi) + \frac{1}{2}(\frac{1}{2}(\mu_2^2-\mu_1^2))(\phi^2+\zeta^2) \\ & + \frac{i}{2}\Big(\overline{\psi}\gamma^\mu\partial_\mu - (\partial_\mu\overline{\psi})\gamma^\mu\Big)\eta + \frac{i}{2}\Big(\overline{\eta}\gamma^\mu\partial_\mu - (\partial_\mu\overline{\eta})\gamma^\mu\Big)\psi \\ & - \frac{1}{2}(m_1+m_2)(\overline{\psi}\eta + \overline{\eta}\psi) + \frac{1}{2}(m_2-m_1)(\overline{\psi}\psi + \overline{\eta}\eta) \\ & - g\phi\overline{\psi}\psi \Bigg]\,. \end{aligned} \tag{2.4}$$

The light-cone Hamiltonian P^- can be constructed using the methods of [10]:

$$P^- = \frac{1}{2}\int dx^- d\boldsymbol{x}_\perp\; T^{+-}\,, \tag{2.5}$$

$$\begin{aligned} T^{+-} = & -\partial_i\phi\partial^i\zeta - \partial_i\zeta\partial^i\phi + (\mu^2 + \frac{1}{2}\delta^2)(\phi\zeta+\zeta\phi) - \frac{1}{2}\delta^2(\phi^2+\zeta^2) \\ & -2i(\psi_-^\dagger\partial_-\eta_- - (\partial_-\psi_-)^\dagger\eta_- + \eta_-^\dagger\partial_-\psi_- - (\partial_-\eta_-)^\dagger\psi_-) \\ & -i(\psi_-^\dagger\alpha^i\partial_i\eta_+ + \psi_+^\dagger\alpha^i\partial_i\eta_- - (\partial_i\psi_+)^\dagger\alpha^i\eta_- - (\partial_i\psi_-)^\dagger\alpha^i\eta_+ \\ & +\eta_-^\dagger\alpha^i\partial_i\psi_+ + \eta_+^\dagger\alpha^i\partial_i\psi_- - (\partial_i\eta_+)^\dagger\alpha^i\psi_- - (\partial_i\eta_-)^\dagger\alpha^i\psi_+) \\ & +2(m+\frac{1}{2}\Delta)(\psi_+^\dagger\gamma^0\eta_- + \psi_-^\dagger\gamma^0\eta_+ + \eta_+^\dagger\gamma^0\psi_- + \eta_-^\dagger\gamma^0\psi_+) \\ & -\Delta(\psi_+^\dagger\gamma^0\psi_- + \psi_-^\dagger\gamma^0\psi_+ + \eta_+^\dagger\gamma^0\eta_- + \eta_-^\dagger\gamma^0\eta_+) \\ & +2g\phi(\psi_+^\dagger\gamma^0\psi_- + \psi_-^\dagger\gamma^0\psi_+)\,, \end{aligned} \tag{2.6}$$

where we have defined

$$\mu^2 \equiv \mu_1^2\,, \quad \delta^2 \equiv \mu_2^2 - \mu_1^2\,, \quad m \equiv m_1\,, \quad \Delta \equiv m_2 - m_1\,, \tag{2.7}$$

and $\alpha^i \equiv \gamma^0\gamma^i$ are the original Dirac matrices. The fields ψ_- and η_- are nondynamical and must be eliminated via the constraint relations; these take the form

$$2i\partial_-\eta_- = \Big[-i\alpha^i\partial_i + (m+\frac{1}{2}\Delta)\gamma^0\Big]\eta_+ + \Big[-\frac{1}{2}\Delta\gamma^0 + g\gamma^0\phi\Big]\psi_+ \tag{2.8}$$

and

$$2i\partial_-\psi_- = \Big[-i\alpha^i\partial_i + (m+\frac{1}{2}\Delta)\gamma^0\Big]\psi_+ + \Big[-\frac{1}{2}\Delta\gamma^0\Big]\eta_+\,. \tag{2.9}$$

The mode expansions are

$$\psi_+(\underline{x}) = \frac{1}{\sqrt{16\pi^3}} \sum_s \int d\underline{k}\, \chi_s \left[b_{s,\underline{k}} e^{-i\underline{k}\cdot\underline{x}} + d^\dagger_{-s,\underline{k}} e^{+i\underline{k}\cdot\underline{x}} \right], \tag{2.10}$$

$$\eta_+(\underline{x}) = \frac{1}{\sqrt{16\pi^3}} \sum_s \int d\underline{k}\, \chi_s \left[\beta_{s,\underline{k}} e^{-i\underline{k}\cdot\underline{x}} + \delta^\dagger_{-s,\underline{k}} e^{+i\underline{k}\cdot\underline{x}} \right], \tag{2.11}$$

$$\phi(\underline{x}) = \frac{1}{\sqrt{16\pi^3}} \int d\underline{q} \frac{1}{\sqrt{q^+}} \left[a_{\underline{q}} e^{-i\underline{q}\cdot\underline{x}} + a^\dagger_{\underline{q}} e^{+i\underline{q}\cdot\underline{x}} \right], \tag{2.12}$$

$$\zeta(\underline{x}) = \frac{1}{\sqrt{16\pi^3}} \int d\underline{q} \frac{1}{\sqrt{q^+}} \left[\alpha_{\underline{q}} e^{-i\underline{q}\cdot\underline{x}} + \alpha^\dagger_{\underline{q}} e^{+i\underline{q}\cdot\underline{x}} \right]. \tag{2.13}$$

The integration measure is $d\underline{k} = d^2k_\perp dk^+$. The light-cone four-spinors χ_s are defined in the Appendix. The canonical commutation relations derived from the Lagrangian are

$$\left\{ \psi_{1_+\alpha}(x^+, \underline{x}), \psi^\dagger_{1_+\beta}(x^+, \underline{x}') \right\} = (\Lambda_+)_{\alpha\beta} \delta^{(3)}(\underline{x} - \underline{x}'), \tag{2.14}$$

$$\left\{ \psi_{2_+\alpha}(x^+, \underline{x}), \psi^\dagger_{2_+\beta}(x^+, \underline{x}') \right\} = -(\Lambda_+)_{\alpha\beta} \delta^{(3)}(\underline{x} - \underline{x}'), \tag{2.15}$$

$$\left[\phi_1(x^+, \underline{x}), \partial^+ \phi_1(x^+, \underline{x}') \right] = i\delta^{(3)}(\underline{x} - \underline{x}'), \tag{2.16}$$

$$\left[\phi_2(x^+, \underline{x}), \partial^+ \phi_2(x^+, \underline{x}') \right] = -i\delta^{(3)}(\underline{x} - \underline{x}'), \tag{2.17}$$

the others being zero. These are realized by the Fock space relations

$$\{ b_{s,\underline{k}}, \beta^\dagger_{s',\underline{k}'} \} = \{ d_{s,\underline{k}}, \delta^\dagger_{s',\underline{k}'} \} = \delta_{ss'} \delta^{(3)}(\underline{k} - \underline{k}'), \tag{2.18}$$

$$[a_{\underline{q}}, \alpha^\dagger_{\underline{q}'}] = \delta^{(3)}(\underline{q} - \underline{q}'). \tag{2.19}$$

All others are zero, including

$$\{ b, b^\dagger \} = \{ d, d^\dagger \} = \{ \beta, \beta^\dagger \} = \{ \delta, \delta^\dagger \} = [a, a^\dagger] = [\alpha, \alpha^\dagger] = 0. \tag{2.20}$$

We can now (following [10]) give P^-. We write

$$P^- \equiv P^-_{(0)} + g P^-_{(1)}, \tag{2.21}$$

where

$$\begin{aligned} P^-_{(0)} = & \int d\underline{q} \left[\frac{\boldsymbol{q}_\perp^{\,2} + (\mu^2 + \delta^2/2)}{q^+} \right] (a^\dagger_{\underline{q}} \alpha_{\underline{q}} + \alpha^\dagger_{\underline{q}} a_{\underline{q}}) \\ & + \sum_s \int d\underline{k} \left[\frac{\boldsymbol{k}_\perp^{\,2} + (m^2 + D^2/2)}{k^+} \right] \left(b^\dagger_{s,\underline{k}} \beta_{s,\underline{k}} + \beta^\dagger_{s,\underline{k}} b_{s,\underline{k}} + d^\dagger_{s,\underline{k}} \delta_{s,\underline{k}} + \delta^\dagger_{s,\underline{k}} d_{s,\underline{k}} \right) \\ & + \int d\underline{q} \left[\frac{-\delta^2}{2q^+} \right] (a^\dagger_{\underline{q}} a_{\underline{q}} + \alpha^\dagger_{\underline{q}} \alpha_{\underline{q}}) \\ & + \sum_s \int d\underline{k} \left[\frac{-D^2}{2k^+} \right] \left(b^\dagger_{s,\underline{k}} b_{s,\underline{k}} + \beta^\dagger_{s,\underline{k}} \beta_{s,\underline{k}} + d^\dagger_{s,\underline{k}} d_{s,\underline{k}} + \delta^\dagger_{s,\underline{k}} \delta_{s,\underline{k}} \right), \end{aligned} \tag{2.22}$$

and $D^2 \equiv m_2^2 - m_1^2$. The first-order interaction Hamiltonian separates naturally into two pieces, describing boson emission with and without a spin flip:

$$P^-_{(1)} \equiv V_{\rm flip} + V_{\rm noflip}\,. \tag{2.23}$$

We have found it convenient to express the spin-flip part of $P^-_{(1)}$ in terms of transverse "polarization" vectors

$$\boldsymbol{\epsilon}_{\perp,+1} \equiv \frac{-1}{\sqrt{2}}(1,i)\,, \qquad\qquad \boldsymbol{\epsilon}_{\perp,-1} \equiv \frac{1}{\sqrt{2}}(1,-i)\,, \tag{2.24}$$

satisfying

$$\epsilon_l^{i\,*}\epsilon^i_{l'} = \delta_{ll'}\,, \qquad \sum_l \epsilon^i_l \epsilon_l^{j\,*} = \delta^{ij}\,, \qquad \epsilon^i_l\epsilon^i_{l'} = \epsilon_l^{i\,*}\epsilon_{l'}^{i\,*} = -\delta_{l,-l'}\,. \tag{2.25}$$

Of course, the boson has no such degree of freedom; this is merely a way of writing the Hamiltonian more compactly. Other useful relations satisfied by the $\boldsymbol{\epsilon}_{\perp,l}$ include:

$$\boldsymbol{\epsilon}^{\,*}_{\perp,l} = -\boldsymbol{\epsilon}_{\perp,-l}\,, \tag{2.26}$$

$$v^i\chi^\dagger_s\beta\alpha^i\chi_{s'} = \sqrt{2}\,\boldsymbol{\epsilon}^{\,*}_{\perp,2s}\cdot\boldsymbol{v}_\perp\,\delta_{s,-s'}\,, \tag{2.27}$$

where $\boldsymbol{v}_\perp$ is any transverse vector. We have

$$\begin{aligned} V_{\rm flip} = \frac{1}{\sqrt{8\pi^3}}\sum_s\int d\underline{k}d\underline{l}d\underline{q}\;\frac{(\boldsymbol{\epsilon}_{\perp,2s}{}^*\cdot\boldsymbol{l}_\perp)}{l^+\sqrt{q^+}} \\ \times\Big\{b^\dagger_{s,\underline{k}}b_{-s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q}) + b^\dagger_{s,\underline{k}}d^\dagger_{s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{q}-\underline{k}-\underline{l}) \\ +b^\dagger_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) - d^\dagger_{s,\underline{l}}d_{-s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) \\ +d_{-s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) - d^\dagger_{s,\underline{l}}d_{-s,\underline{k}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q})\Big\} + \text{h.c.}\,, \end{aligned} \tag{2.28}$$

$$\begin{aligned} V_{\rm noflip} = \frac{(m+\Delta/2)}{\sqrt{16\pi^3}}\sum_s\int d\underline{k}d\underline{l}d\underline{q}\;\frac{1}{l^+\sqrt{q^+}} \\ \times\Big\{b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q}) + d^\dagger_{-s,\underline{l}}b^\dagger_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) \\ +b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) + d^\dagger_{s,\underline{l}}d_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) \\ +d_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) + d^\dagger_{s,\underline{l}}d_{s,\underline{k}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q})\Big\} + \text{h.c.} \\ -\frac{\Delta/2}{\sqrt{16\pi^3}}\sum_s\int d\underline{k}d\underline{l}d\underline{q}\;\frac{1}{l^+\sqrt{q^+}} \\ \times\Big\{b^\dagger_{s,\underline{k}}\beta_{s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q}) + \delta^\dagger_{-s,\underline{l}}b^\dagger_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) \\ +b^\dagger_{s,\underline{k}}\beta_{s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) + \delta^\dagger_{s,\underline{l}}d_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) \\ +d_{s,\underline{k}}\beta_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) + \delta^\dagger_{s,\underline{l}}d_{s,\underline{k}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q})\Big\} + \text{h.c.} \end{aligned} \tag{2.29}$$

The structure of these interactions reflects the conservation of J_z in each interaction; the light-cone spin-flip $\Delta S_z = \pm 1$ of the fermions is compensated by a unit change $\Delta L_z = \mp 1$ in orbital angular momentum [11].

Notice that no four-point interactions arise in the light-cone Hamiltonian from the elimination of the dependent fermionic fields. The absence of such instantaneous interactions follows from the lack of mass dependence in such interactions and from the opposite signature of the PV fermion; the interactions associated with an instantaneous "physical" fermion then cancel against those of the instantaneous PV fermion.

III. CONSTRUCTION OF THE EXACT SOLUTIONS

If δ and Δ are equal to zero, the system is exactly solvable. To see this, we define an index for each state as the number of $\alpha^\dagger$ type quanta plus the number of $\beta^\dagger$ and $\delta^\dagger$ type quanta minus the number of $a^\dagger$ type quanta minus the number of $b^\dagger$ and $d^\dagger$ type quanta. States with a definite index are then eigenstates of the operator $\mathcal{I} = [\alpha^\dagger a + \beta^\dagger b + \delta^\dagger d] - [a^\dagger \alpha + b^\dagger \beta + d^\dagger \delta]$ with the value of the index being the eigenvalue; note, however, that matrix elements of $\mathcal{I}$ between such states may not be equal to the index, due to the indefinite metric. States with a definite value of the index span the space. The kinetic energy part of P^- (for zero δ and Δ) is diagonal in $\mathcal{I}$, whereas the interacting part of P^-, when acting on a state of given index, produces only states with *lower* index. Thus the light-cone Hamiltonian is triangular, allowing its eigensolutions to be constructed as a combination of Fock states in a finite number of sectors. In particular, each eigenvector of the system will contain a state of highest index, and its eigenvalue will be equal to the free eigenvalue of the highest state.

We define the functions

$$U(\underline{k},\underline{l}) \equiv \frac{1}{\sqrt{16\pi^3}}\frac{1}{\sqrt{k^+ - l^+}}\left(\frac{1}{l^+} + \frac{1}{k^+}\right), \tag{3.1}$$

$$V(\underline{k},\underline{l}) \equiv \frac{1}{\sqrt{16\pi^3}}\frac{1}{\sqrt{k^+ - l^+}}\left(\frac{-l_1 - il_2}{l^+} + \frac{k_1 + ik_2}{k^+}\right), \tag{3.2}$$

$$T(\underline{k},\underline{l}) \equiv \frac{1}{\sqrt{16\pi^3}}\frac{1}{\sqrt{k^+}}\left(\frac{-(k_1 - l_1) + i(k_2 - l_2)}{k^+ - l^+} + \frac{l_1 - il_2}{l^+}\right), \tag{3.3}$$

$$S(\underline{k},\underline{l}) \equiv \frac{1}{\sqrt{16\pi^3}}\frac{1}{\sqrt{k^+}}\left(\frac{-1}{k^+ - l^+} + \frac{1}{l^+}\right), \tag{3.4}$$

and

$$E_{n,m}(\underline{l_1},\ldots,\underline{l_n},\underline{l_{n+1}},\ldots,\underline{l_{n+m}}) \equiv \sum_{i=1}^{n}\Big[\frac{\boldsymbol{l}_{\perp,i}^2 + m^2}{l_i^+}\Big] + \sum_{i=n+1}^{n+m}\Big[\frac{\boldsymbol{l}_{\perp,i}^2 + \mu^2}{l_i^+}\Big]. \tag{3.5}$$

With this notation, we can write the eigenvector whose highest state is $\beta^\dagger_{+,\underline{k}}|0\rangle$ as

$$\begin{aligned}\beta^\dagger_{+,\underline{k}}|0\rangle &+ mg\int_0^{k^+} d\underline{l}\frac{U(\underline{k},\underline{l})}{E_{1,0}(\underline{k}) - E_{1,1}(\underline{l},\underline{k}-\underline{l})}b^\dagger_{+,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}|0\rangle \\ &+ g\int_0^{k^+} d\underline{l}\frac{V(\underline{k},\underline{l})}{E_{1,0}(\underline{k}) - E_{1,1}(\underline{l},\underline{k}-\underline{l})}b^\dagger_{-,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}|0\rangle\,.\end{aligned} \tag{3.6}$$

The eigenvalue of the state is $E_{1,0}(\underline{k})$. Not all states are this simple. Another example: The eigenstate whose highest state is $\beta^\dagger_{+,\underline{l}}\alpha^\dagger_{\underline{k}-\underline{l}}|0\rangle$ is

$$\begin{aligned}&\beta^\dagger_{+,\underline{l}}\alpha^\dagger_{\underline{k}-\underline{l}}|0\rangle + mg\frac{U(\underline{k},\underline{l})}{E_{1,1}(\underline{l},\underline{k}-\underline{l}) - E_{1,0}(\underline{k})}b^\dagger_{+,\underline{k}}|0\rangle - g\frac{V(\underline{k},\underline{l})}{E_{1,1}(\underline{l},\underline{k}-\underline{l}) - E_{1,0}(\underline{k})}b^\dagger_{-,\underline{k}}|0\rangle \\ &+mg\int_0^{l^+} d\underline{t}\frac{U(\underline{l},\underline{t})}{E_{1,1}(\underline{l},\underline{k}-\underline{l}) - E_{1,2}(\underline{t},\underline{l}-\underline{t},\underline{k}-\underline{l})}b^\dagger_{+,\underline{t}}a^\dagger_{\underline{l}-\underline{t}}\alpha^\dagger_{\underline{k}-\underline{l}}|0\rangle\end{aligned}$$

$$
\begin{aligned}
&+g\int_0^{l^+} d\underline{t}\frac{V(\underline{l},\underline{t})}{E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{1,2}(\underline{t},\underline{l}-\underline{t},\underline{k}-\underline{l})}b^\dagger_{-,\underline{t}}a^\dagger_{\underline{l}-\underline{t}}\alpha^\dagger_{\underline{k}-\underline{l}}|0\rangle\\
&+g\int_0^{k^+-l^+} d\underline{t}\frac{T(\underline{k}-\underline{l},\underline{t})}{E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l})}b^\dagger_{+,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}\beta^\dagger_{+,\underline{l}}|0\rangle\\
&-g\int_0^{k^+-l^+} d\underline{t}\frac{T^*(\underline{k}-\underline{l},\underline{t})}{E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l})}b^\dagger_{-,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}}\beta^\dagger_{+,\underline{l}}|0\rangle \qquad (3.7)\\
&+mg\int_0^{k^+-l^+} d\underline{t}\frac{S(\underline{k}-\underline{l},\underline{t})}{E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l})}(b^\dagger_{-,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}+b^\dagger_{+,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}})\beta^\dagger_{+,\underline{l}}|0\rangle\\
&+g^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{T(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&b^\dagger_{+,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\\
&-g^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{T^*(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&b^\dagger_{-,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}}(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\\
&+mg^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{S(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,0}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&(b^\dagger_{-,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}+b^\dagger_{+,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}})(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\\
&+g^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{T(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{1,2}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&b^\dagger_{+,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\\
&-g^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{T^*(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{1,2}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&b^\dagger_{-,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}}(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\\
&+mg^2\int_0^{k^+-l^+} d\underline{t}\int_0^{l^+} d\underline{q}\frac{S(\underline{k}-\underline{l},\underline{t})}{(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{1,2}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{l}))(E_{1,1}(\underline{l},\underline{k}-\underline{l})-E_{3,1}(\underline{t},\underline{k}-\underline{l}-\underline{t},\underline{q},\underline{l}-\underline{q}))}\\
&(b^\dagger_{-,\underline{t}}d^\dagger_{+,\underline{k}-\underline{l}-\underline{t}}+b^\dagger_{+,\underline{t}}d^\dagger_{-,\underline{k}-\underline{l}-\underline{t}})(mU(\underline{l},\underline{q})b^\dagger_{+,\underline{q}}a^\dagger_{\underline{l}-\underline{q}}+V(\underline{l},\underline{q})b^\dagger_{-,\underline{q}}a^\dagger_{\underline{l}-\underline{q}})|0\rangle\,.
\end{aligned}
$$

Notice that any state containing only $a^\dagger$'s, $b^\dagger$'s and $d^\dagger$'s is an eigenstate of the full P^- with the same free eigenvalue.

In these examples the eigenvectors have zero norm, but this is not always the case. If we add the vector $b^\dagger_{+,\underline{k}}|0\rangle$ to the state given in (3.6), the resulting vector is an eigenvector with the same eigenvalue but nonzero norm. Two such states with momenta p and k satisfy $\langle p|k\rangle = 2\delta(p-k)$. An eigenvector whose highest state (as per the index) is a state composed purely of physical quanta corresponds to that physical state in the free theory; all other states are unphysical.

Even when δ or Δ is nonzero, there is interesting structure. The interacting part of P^- is still a move-down operator in terms of the index. In fact the only operators which move a state up to a higher index (and thereby fill in the upper triangle of P^-) are the operators

$$\int d\underline{q}\Big[\frac{-\delta^2}{2q^+}\Big]\alpha^\dagger_{\underline{q}}\alpha_{\underline{q}} + \sum_s \int d\underline{k}\Big[\frac{-D^2}{2k^+}\Big]\Big(\beta^\dagger_{s,\underline{k}}\beta_{s,\underline{k}} + \delta^\dagger_{s,\underline{k}}\delta_{s,\underline{k}}\Big)\,. \tag{3.8}$$

These operators are still diagonal in momentum space and are independent of the transverse momenta.

If we chose an unperturbed P^- containing all the terms in the full P^- except those in (3.8), we obtain a triangular system but the operator is very deficient and probably not very useful. On the other hand, if we forget for the moment that Δ and D are related and treat them formally as independent parameters, we can set $D = 0$ while allowing Δ to become nonzero. If we chose as the unperturbed P^-, the P^- which results from setting δ and D equal to zero but allowing Δ to be nonzero, then the perturbing P^- (the terms proportional to δ and D) has no dependence on the coupling constant, g. (If Δ is a perturbation parameter, the perturbing P^- does depend on g through the $V_{\rm noflip}$ interaction.) The P^- which includes Δ, but not δ or D, is not deficient and may form a good starting point for calculations. In that case the eigenvectors project onto an infinite number of sectors. For instance, the eigenvector whose highest state is $\beta^\dagger_{+,\underline{k}}|0\rangle$ projects onto all sectors containing a $\beta^\dagger_+$ or a $b^\dagger$ particle plus an arbitrary number of $a^\dagger$ particles. We can write the eigenvector as

$$\begin{aligned}\beta^\dagger_{+,\underline{k}}|0\rangle + \sum_{i=1}^{\infty}\int_0^{k^+} d\underline{l}_1^+ \int_0^{l_1^+} d\underline{l}_2^+ \ldots \int_0^{l_{i-1}^+} d\underline{l}_i^+ \\ \Big[X_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)\beta^\dagger_{+,\underline{l}_i}a^\dagger_{\underline{k}-\underline{l}_1}a^\dagger_{\underline{l}_1-\underline{l}_2}\ldots a^\dagger_{\underline{l}_{i-1}-\underline{l}_i}|0\rangle \\ +Y_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)b^\dagger_{+,\underline{l}_i}a^\dagger_{\underline{k}-\underline{l}_1}a^\dagger_{\underline{l}_1-\underline{l}_2}\ldots a^\dagger_{\underline{l}_{i-1}-\underline{l}_i}|0\rangle \\ +Z_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)b^\dagger_{-,\underline{l}_i}a^\dagger_{\underline{k}-\underline{l}_1}a^\dagger_{\underline{l}_1-\underline{l}_2}\ldots a^\dagger_{\underline{l}_{i-1}-\underline{l}_i}|0\rangle\Big]\,.\end{aligned} \tag{3.9}$$

The X, Y, and Z's satisfy the following recursion relations:

$$X_{i+1}(\underline{k},\underline{l}_1,\ldots,\underline{l}_{i+1}) = \frac{-\frac{1}{2}\Delta g U(\underline{l}_i,\underline{l}_{i+1})X_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)}{E_{1,0}(\underline{k}) - E_{1,i+1}(\underline{l}_{i+1},\underline{k}-\underline{l}_1,\underline{l}_1-\underline{l}_2,\ldots,\underline{l}_i-\underline{l}_{i+1})}\,, \tag{3.10}$$

$$Y_{i+1}(\underline{k},\underline{l}_1,\ldots,\underline{l}_{i+1}) = \frac{-\frac{1}{2}\Delta g U(\underline{l}_i,\underline{l}_{i+1})Y_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i) + (m+\frac{1}{2}\Delta)gU(\underline{l}_i,\underline{l}_{i+1})X_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)}{E_{1,0}(\underline{k}) - E_{1,i+1}(\underline{l}_{i+1},\underline{k}-\underline{l}_1,\underline{l}_1-\underline{l}_2,\ldots,\underline{l}_i-\underline{l}_{i+1})}\,, \tag{3.11}$$

$$Z_{i+1}(\underline{k},\underline{l}_1,\ldots,\underline{l}_{i+1}) = \frac{-\frac{1}{2}\Delta g U(\underline{l}_i,\underline{l}_{i+1})Z_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i) + gV(\underline{l}_i,\underline{l}_{i+1})X_i(\underline{k},\underline{l}_1,\ldots,\underline{l}_i)}{E_{1,0}(\underline{k}) - E_{1,i+1}(\underline{l}_{i+1},\underline{k}-\underline{l}_1,\underline{l}_1-\underline{l}_2,\ldots,\underline{l}_i-\underline{l}_{i+1})}\,. \tag{3.12}$$

These are subject to the initial conditions

$$X_1(\underline{k},\underline{l}_1) = -\frac{1}{2}g\Delta U(\underline{k},\underline{l}_1)\,, \tag{3.13}$$

$$Y_1(\underline{k},\underline{l}_1) = (m+\frac{1}{2}\Delta)gU(\underline{k},\underline{l}_1)\,, \tag{3.14}$$

$$Z_1(\underline{k},\underline{l}_1) = gV(\underline{k},\underline{l}_1)\,. \tag{3.15}$$

IV. THE OPERATOR SOLUTION, THE VACUUM AND THE EQUAL-TIME REPRESENTATION

We now give the operator solution for the case $\delta = \Delta = D = 0$. The simplest field to obtain is the Bose field. If we define the operator, $A_{\underline{q}}$, to be

$$
\begin{aligned}
&A_{\underline{q}} \equiv \alpha_{\underline{q}} \\
&+g\int_{q^+}^{\infty} d\underline{k}\frac{1}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(V(\underline{k},\underline{k}-\underline{q})b^{\dagger}_{-,\underline{k}-\underline{q}}b_{+,\underline{k}}-V^{*}(\underline{k},\underline{k}-\underline{q})b^{\dagger}_{+,\underline{k}-\underline{q}}b_{-,\underline{k}}) \\
&+g\int_{0}^{q^+} d\underline{k}\frac{1}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(T^{*}(\underline{q},\underline{k})d_{+,\underline{q}-\underline{k}}b_{+,\underline{k}}-T(\underline{q},\underline{k})d_{-,\underline{q}-\underline{k}}b_{-,\underline{k}}) \\
&+g\int_{q^+}^{\infty} d\underline{k}\frac{1}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(V(\underline{k},\underline{k}-\underline{q})d^{\dagger}_{-,\underline{k}-\underline{q}}d_{+,\underline{k}}-V^{*}(\underline{k},\underline{k}-\underline{q})d^{\dagger}_{+,\underline{k}-\underline{q}}d_{-,\underline{k}}) \\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(b^{\dagger}_{+,\underline{k}-\underline{q}}b_{+,\underline{k}}+b^{\dagger}_{-,\underline{k}-\underline{q}}b_{-,\underline{k}}) \\
&+mg\int_{0}^{q^+} d\underline{k}\frac{S(\underline{q},\underline{k})}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(d_{+,\underline{q}-\underline{k}}b_{-,\underline{k}}+d_{-,\underline{q}-\underline{k}}b_{+,\underline{k}}) \\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(d^{\dagger}_{+,\underline{k}-\underline{q}}d_{+,\underline{k}}+d^{\dagger}_{-,\underline{k}-\underline{q}}d_{-,\underline{k}})\,, \qquad (4.1)
\end{aligned}
$$

we can use the relation

$$[P^-,\zeta] = -i\partial^-\zeta\,, \qquad (4.2)$$

along with the initial condition (2.13), to show that

$$
\begin{aligned}
&\zeta(x^+,\underline{x}) = \frac{1}{\sqrt{16\pi^3}}\int d\underline{q}\frac{1}{\sqrt{q^+}}A_{\underline{q}}e^{-i(\frac{1}{2}E_{0,1}(\underline{q})x^+ + \underline{q}\cdot\underline{x})} - \frac{1}{\sqrt{16\pi^3}}\int d\underline{q}\frac{1}{\sqrt{q^+}}e^{-i\underline{q}\cdot\underline{x}}\Bigg[\\
&g\int_{q^+}^{\infty} d\underline{k}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k}))x^+}}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(V(\underline{k},\underline{k}-\underline{q})b^{\dagger}_{-,\underline{k}-\underline{q}}b_{+,\underline{k}}-V^{*}(\underline{k},\underline{k}-\underline{q})b^{\dagger}_{+,\underline{k}-\underline{q}}b_{-,\underline{k}}) \\
&+g\int_{0}^{q^+} d\underline{k}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k}))x^+}}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(T^{*}(\underline{q},\underline{k})d_{+,\underline{q}-\underline{k}}b_{+,\underline{k}}-T(\underline{q},\underline{k})d_{-,\underline{q}-\underline{k}}b_{-,\underline{k}}) \\
&+g\int_{q^+}^{\infty} d\underline{k}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k}))x^+}}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(V(\underline{k},\underline{k}-\underline{q})d^{\dagger}_{-,\underline{k}-\underline{q}}d_{+,\underline{k}}-V^{*}(\underline{k},\underline{k}-\underline{q})d^{\dagger}_{+,\underline{k}-\underline{q}}d_{-,\underline{k}}) \\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k}))x^+}U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(b^{\dagger}_{+,\underline{k}-\underline{q}}b_{+,\underline{k}}+b^{\dagger}_{-,\underline{k}-\underline{q}}b_{-,\underline{k}}) \\
&+mg\int_{0}^{q^+} d\underline{k}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k}))x^+}S(\underline{q},\underline{k})}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(d_{+,\underline{q}-\underline{k}}b_{-,\underline{k}}+d_{-,\underline{q}-\underline{k}}b_{+,\underline{k}}) \\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k}))x^+}U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(d^{\dagger}_{+,\underline{k}-\underline{q}}d_{+,\underline{k}}+d^{\dagger}_{-,\underline{k}-\underline{q}}d_{-,\underline{k}})\Bigg] + \text{h.c.} \qquad (4.3)
\end{aligned}
$$

By evaluating this expression at $t = 0$ (or any other equal-time surface) we can work out the relation between the light-cone operators and the equal-time operators. We shall indicate three-vectors in the three spatial dimensions with a hat: $\hat{k} \equiv (k_1, k_2, k_3)$, and will indicate the equal-time operators with a breve: $\breve{a}$. We write as usual

$$\zeta(0,\hat{x}) = \frac{1}{\sqrt{8\pi^3}} \int d\hat{k} \frac{1}{\sqrt{2\omega_{\hat{k}}}} \left(\breve{a}_{\hat{k}} e^{i\hat{k}\cdot\hat{x}} + \breve{a}^\dagger_{\hat{k}} e^{-i\hat{k}\cdot\hat{x}} \right), \tag{4.4}$$

where

$$\breve{a}_{\hat{k}} = \frac{1}{\sqrt{8\pi^3}} \frac{1}{\sqrt{2}} \int d\hat{x} e^{-i\hat{k}\cdot\hat{x}} \sqrt{\omega_{\hat{k}}} \left(\zeta(0,\hat{x}) + \frac{i}{\omega_{\hat{k}}} \partial_0 \zeta(0,\hat{x}) \right). \tag{4.5}$$

In these and later formulas we use $\omega_{\hat{k}} \equiv \sqrt{\mu^2 + \hat{k}^2}$. It is also useful to define the quantities

$$p^-(\underline{q},\underline{k}) \equiv -E_{1,0}(\underline{k}-\underline{q}) + E_{1,0}(\underline{k}), \qquad r^-(\underline{q},\underline{k}) \equiv E_{1,0}(\underline{q}-\underline{k}) + E_{1,0}(\underline{k}), \tag{4.6}$$

with which we can define

$$p_0(\underline{q},\underline{k}) \equiv \frac{1}{2}(p^-(\underline{q},\underline{k}) + q^+), \qquad p_3(\underline{q},\underline{k}) \equiv \frac{1}{2}(p^-(\underline{q},\underline{k}) - q^+), \tag{4.7}$$

$$r_0(\underline{q},\underline{k}) \equiv \frac{1}{2}(r^-(\underline{q},\underline{k}) + q^+), \qquad r_3(\underline{q},\underline{k}) \equiv \frac{1}{2}(r^-(\underline{q},\underline{k}) - q^+). \tag{4.8}$$

These allow us to define the spatial three-vectors

$$\hat{p}(\underline{q},\underline{k}) \equiv (\boldsymbol{q}_\perp, p_3), \qquad \hat{r}(\underline{q},\underline{k}) \equiv (\boldsymbol{q}_\perp, r_3). \tag{4.9}$$

We now find

$$\begin{aligned}
\breve{a}_{\hat{t}} = {} & \sqrt{\frac{\omega_{\hat{t}} - t_3}{\omega_{\hat{t}}}} A_{\omega_{\hat{t}}-t_3, \boldsymbol{t}_\perp} - \int d\underline{q} \frac{1}{2\sqrt{q^+}} \Bigg[\\
& g \int_{q^+}^{\infty} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} + \frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} - \hat{p}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) + E_{1,0}(\underline{k}-\underline{q}) - E_{1,0}(\underline{k})} (V(\underline{k},\underline{k}-\underline{q}) b^\dagger_{-,\underline{k}-\underline{q}} b_{+,\underline{k}} - V^*(\underline{k},\underline{k}-\underline{q}) b^\dagger_{+,\underline{k}-\underline{q}} b_{-,\underline{k}}) \\
& + g \int_{q^+}^{\infty} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} - \frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} + \hat{p}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) + E_{1,0}(\underline{k}-\underline{q}) - E_{1,0}(\underline{k})} (V^*(\underline{k},\underline{k}-\underline{q}) b^\dagger_{+,\underline{k}} b_{-,\underline{k}-\underline{q}} - V(\underline{k},\underline{k}-\underline{q}) b^\dagger_{-,\underline{k}} b_{+,\underline{k}-\underline{q}}) \\
& + g \int_{0}^{q^+} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} + \frac{r_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} - \hat{r}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) - E_{1,0}(\underline{q}-\underline{k}) - E_{1,0}(\underline{k})} (T^*(\underline{q},\underline{k}) d_{+,\underline{q}-\underline{k}} b_{+,\underline{k}} - T(\underline{q},\underline{k}) d_{-,\underline{q}-\underline{k}} b_{-,\underline{k}}) \\
& + g \int_{0}^{q^+} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} - \frac{r_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} + \hat{r}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) - E_{1,0}(\underline{q}-\underline{k}) - E_{1,0}(\underline{k})} (T(\underline{q},\underline{k}) b^\dagger_{+,\underline{k}} d^\dagger_{+,\underline{q}-\underline{k}} - T^*(\underline{q},\underline{k}) b^\dagger_{-,\underline{k}} d^\dagger_{-,\underline{q}-\underline{k}}) \\
& + g \int_{q^+}^{\infty} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} + \frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} - \hat{p}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) + E_{1,0}(\underline{k}-\underline{q}) - E_{1,0}(\underline{k})} (V(\underline{k},\underline{k}-\underline{q}) d^\dagger_{-,\underline{k}-\underline{q}} d_{+,\underline{k}} - V^*(\underline{k},\underline{k}-\underline{q}) d^\dagger_{+,\underline{k}-\underline{q}} d_{-,\underline{k}}) \\
& + g \int_{q^+}^{\infty} d\underline{k} \frac{(\sqrt{\omega_{\hat{t}}} - \frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}}) \delta(\hat{t} + \hat{p}(\underline{q},\underline{k}))}{E_{0,1}(\underline{q}) + E_{1,0}(\underline{k}-\underline{q}) - E_{1,0}(\underline{k})} (V^*(\underline{k},\underline{k}-\underline{q}) d^\dagger_{+,\underline{k}} d_{-,\underline{k}-\underline{q}} - V(\underline{k},\underline{k}-\underline{q}) d^\dagger_{-,\underline{k}} d_{+,\underline{k}-\underline{q}})
\end{aligned}$$

$$
\begin{aligned}
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}+\frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}-\hat{p}(\underline{q},\underline{k}))U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(b^{\dagger}_{+,\underline{k}-\underline{q}}b_{+,\underline{k}}+b^{\dagger}_{-,\underline{k}-\underline{q}}b_{-,\underline{k}})\\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}-\frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}+\hat{p}(\underline{q},\underline{k}))U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(b^{\dagger}_{+,\underline{k}}b_{+,\underline{k}-\underline{q}}+b^{\dagger}_{-,\underline{k}}b_{-,\underline{k}-\underline{q}})\\
&+mg\int_{0}^{q^+} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}+\frac{r_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}-\hat{r}(\underline{q},\underline{k}))S(\underline{q},\underline{k})}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(d_{+,\underline{q}-\underline{k}}b_{-,\underline{k}}+d_{-,\underline{q}-\underline{k}}b_{+,\underline{k}})\\
&+mg\int_{0}^{q^+} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}-\frac{r_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}+\hat{r}(\underline{q},\underline{k}))S(\underline{q},\underline{k})}{E_{0,1}(\underline{q})-E_{1,0}(\underline{q}-\underline{k})-E_{1,0}(\underline{k})}(b^{\dagger}_{-,\underline{k}}d^{\dagger}_{+,\underline{q}-\underline{k}}+b^{\dagger}_{+,\underline{k}}d^{\dagger}_{-,\underline{q}-\underline{k}})\\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}+\frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}-\hat{p}(\underline{q},\underline{k}))U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(d^{\dagger}_{+,\underline{k}-\underline{q}}d_{+,\underline{k}}+d^{\dagger}_{-,\underline{k}-\underline{q}}d_{-,\underline{k}})\\
&+mg\int_{q^+}^{\infty} d\underline{k}\frac{(\sqrt{\omega_{\hat{t}}}-\frac{p_0(\underline{q},\underline{k})}{\sqrt{\omega_{\hat{t}}}})\delta(\hat{t}+\hat{p}(\underline{q},\underline{k}))U(\underline{k},\underline{k}-\underline{q})}{E_{0,1}(\underline{q})+E_{1,0}(\underline{k}-\underline{q})-E_{1,0}(\underline{k})}(d^{\dagger}_{+,\underline{k}}d_{+,\underline{k}-\underline{q}}+d^{\dagger}_{-,\underline{k}}d_{-,\underline{k}-\underline{q}})\Bigg]. \qquad (4.10)
\end{aligned}
$$

The relationship between the light-cone representation and the equal-time representation is quite complicated even for this relatively simple case. We also see that the equal-time perturbative vacuum is not the physical vacuum. The physical vacuum — the ground state of the system, which we shall call $|\Omega\rangle$ — is equal to the light-cone perturbative vacuum. That is, the state we have called $|0\rangle$, which is destroyed by all the light-cone destruction operators, is the physical vacuum: $|\Omega\rangle = |0\rangle$. We shall call the equal-time perturbative vacuum — the state destroyed by all the equal-time destruction operators — $|\breve{0}\rangle$. From the presence in (4.10) of the terms proportional to $b^{\dagger}_{s,\underline{k}}d^{\dagger}_{-s,\underline{q}-\underline{k}}$ and $b^{\dagger}_{s,\underline{k}}d^{\dagger}_{s,\underline{q}-\underline{k}}$ we see that $|\breve{0}\rangle \neq |\Omega\rangle$. Of course, this conclusion immediately follows from the fact that the Hamiltonian contains terms proportional to $\breve{b}^{\dagger}_{s,\hat{k}}\breve{d}^{\dagger}_{t,\hat{q}}\breve{a}^{\dagger}_{-\hat{k}-\hat{q}}$ — the usual way of seeing that $|\breve{0}\rangle \neq |\Omega\rangle$.

The same procedures which we have used to find the eigenstates in the light-cone representation can be used in the equal-time representation. However, the eigensolutions are much more complicated when expressed in the equal-time representation than when expressed in the light-cone representation. For instance, although $|\Omega\rangle$ is just given by $|0\rangle$, $|\Omega\rangle$ projects onto any state containing N equal-time fermion-anti fermion pairs along with N equal-time boson quanta as long as the total momentum is zero. Similarly, the state given in (3.6) projects onto an infinite number of sectors of equal-time basis states.

We return to working out the operator solution. We define

$$
\begin{aligned}
&B_{s,\underline{k}} \equiv \beta_{s,\underline{k}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int_{k^+}^{\infty} d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_{\perp}}{k^+\sqrt{l^+-k^+}}+\frac{\boldsymbol{\epsilon}^{*}_{\perp,2s}\cdot\boldsymbol{l}_{\perp}}{l^+\sqrt{l^+-k^+}}\right)b_{-s,\underline{l}}a^{\dagger}_{\underline{l}-\underline{k}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int_{0}^{k^+} d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_{\perp}}{k^+\sqrt{k^+-l^+}}+\frac{\boldsymbol{\epsilon}^{*}_{\perp,2s}\cdot\boldsymbol{l}_{\perp}}{l^+\sqrt{k^+-l^+}}\right)b_{-s,\underline{l}}a_{\underline{k}-\underline{l}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int d\underline{l}\frac{1}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_{\perp}}{k^+\sqrt{k^++l^+}}+\frac{\boldsymbol{\epsilon}^{*}_{\perp,2s}\cdot\boldsymbol{l}_{\perp}}{l^+\sqrt{k^++l^+}}\right)d^{\dagger}_{s,\underline{l}}a_{\underline{l}+\underline{k}} \qquad (4.11)
\end{aligned}
$$

$$
+\frac{1}{\sqrt{16\pi^3}}mg\int_{k^+}^{\infty}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{1}{k^+\sqrt{l^+-k^+}}+\frac{1}{l^+\sqrt{l^+-k^+}}\right)b_{s,\underline{l}}a^\dagger_{\underline{l}-\underline{k}}
$$

$$
+\frac{1}{\sqrt{16\pi^3}}mg\int_{0}^{k^+}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{1}{k^+\sqrt{k^+-l^+}}+\frac{1}{l^+\sqrt{k^+-l^+}}\right)b_{s,\underline{l}}a_{\underline{k}-\underline{l}}
$$

$$
+\frac{1}{\sqrt{16\pi^3}}mg\int d\underline{l}\frac{1}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{1}{k^+\sqrt{k^++l^+}}-\frac{1}{l^+\sqrt{k^++l^+}}\right)d^\dagger_{-s,\underline{l}}a_{\underline{l}+\underline{k}}\,,
$$

and

$$
\begin{aligned}
&D^\dagger_{s,\underline{k}}\equiv\delta^\dagger_{s,\underline{k}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int_{k^+}^{\infty}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{l^+-k^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{l^+-k^+}}\right)d^\dagger_{-s,\underline{l}}a_{\underline{l}-\underline{k}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int_{0}^{k^+}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^+-l^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^+-l^+}}\right)d^\dagger_{-s,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}\\
&+\frac{1}{\sqrt{8\pi^3}}g\int d\underline{l}\frac{1}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^++l^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^++l^+}}\right)b_{s,\underline{l}}a^\dagger_{\underline{l}+\underline{k}}\\
&+\frac{1}{\sqrt{16\pi^3}}mg\int_{k^+}^{\infty}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{1}{k^+\sqrt{l^+-k^+}}+\frac{1}{l^+\sqrt{l^+-k^+}}\right)d^\dagger_{s,\underline{l}}a_{\underline{l}-\underline{k}}\\
&+\frac{1}{\sqrt{16\pi^3}}mg\int_{0}^{k^+}d\underline{l}\frac{1}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{1}{k^+\sqrt{k^+-l^+}}+\frac{1}{l^+\sqrt{k^+-l^+}}\right)d^\dagger_{s,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}\\
&+\frac{1}{\sqrt{16\pi^3}}mg\int d\underline{l}\frac{1}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{1}{k^+\sqrt{k^++l^+}}-\frac{1}{l^+\sqrt{k^++l^+}}\right)b_{-s,\underline{l}}a^\dagger_{\underline{l}+\underline{k}}.
\end{aligned}
\tag{4.12}
$$

With these definitions we can write the full space-time dependence of the η_+ field as

$$
\begin{aligned}
&\eta_+(x^+,\underline{x})=\frac{1}{\sqrt{16\pi^3}}\sum_s\int d\underline{k}\chi_s\Bigg\{e^{-i\underline{k}\cdot\underline{x}}\Bigg[B_{s,\underline{k}}e^{-i\frac{1}{2}E_{1,0}(\underline{k})x^+}\\
&-\frac{1}{\sqrt{8\pi^3}}g\int_{k^+}^{\infty}d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{l^+-k^+}}+\frac{\boldsymbol{\epsilon}^*_{\perp,2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{l^+-k^+}}\right)b_{-s,\underline{l}}a^\dagger_{\underline{l}-\underline{k}}\\
&-\frac{1}{\sqrt{8\pi^3}}g\int_{0}^{k^+}d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^+-l^+}}+\frac{\boldsymbol{\epsilon}^*_{\perp,2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^+-l^+}}\right)b_{-s,\underline{l}}a_{\underline{k}-\underline{l}}\\
&-\frac{1}{\sqrt{8\pi^3}}g\int d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k}))x^+}}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{\boldsymbol{\epsilon}_{\perp,-2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^++l^+}}+\frac{\boldsymbol{\epsilon}^*_{\perp,2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^++l^+}}\right)d^\dagger_{s,\underline{l}}a_{\underline{l}+\underline{k}}\\
&-\frac{1}{\sqrt{16\pi^3}}mg\int_{k^+}^{\infty}d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{1}{k^+\sqrt{l^+-k^+}}+\frac{1}{l^+\sqrt{l^+-k^+}}\right)b_{s,\underline{l}}a^\dagger_{\underline{l}-\underline{k}}\\
&-\frac{1}{\sqrt{16\pi^3}}mg\int_{0}^{k^+}d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{1}{k^+\sqrt{k^+-l^+}}+\frac{1}{l^+\sqrt{k^+-l^+}}\right)b_{s,\underline{l}}a_{\underline{k}-\underline{l}}\\
&-\frac{1}{\sqrt{16\pi^3}}mg\int d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k}))x^+}}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{1}{k^+\sqrt{k^++l^+}}-\frac{1}{l^+\sqrt{k^++l^+}}\right)d^\dagger_{-s,\underline{l}}a_{\underline{l}+\underline{k}}\Bigg]\\
&+e^{+i\underline{k}\cdot\underline{x}}\Bigg[D^\dagger_{-s,\underline{k}}e^{i\frac{1}{2}E_{1,0}(\underline{k})x^+}
\end{aligned}
\tag{4.13}
$$

$$
-\frac{1}{\sqrt{8\pi^3}}g\int_{k^+}^{\infty}d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})-E_{0,1}(\underline{l}-\underline{k}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{l^+-k^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{l^+-k^+}}\right)d^\dagger_{-s,\underline{l}}a_{\underline{l}-\underline{k}}
$$

$$
-\frac{1}{\sqrt{8\pi^3}}g\int_{0}^{k^+}d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})+E_{0,1}(\underline{k}-\underline{l}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^+-l^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^+-l^+}}\right)d^\dagger_{-s,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}
$$

$$
-\frac{1}{\sqrt{8\pi^3}}g\int d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}+\underline{k}))x^+}}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{-\boldsymbol{\epsilon}_{\perp,2s}\cdot\boldsymbol{k}_\perp}{k^+\sqrt{k^++l^+}}+\frac{-\boldsymbol{\epsilon}^*_{\perp,-2s}\cdot\boldsymbol{l}_\perp}{l^+\sqrt{k^++l^+}}\right)b_{s,\underline{l}}a^\dagger_{\underline{l}+\underline{k}}
$$

$$
-\frac{1}{\sqrt{16\pi^3}}mg\int_{k^+}^{\infty}d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})-E_{0,1}(\underline{l}-\underline{k}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}-\underline{k})}\left(\frac{1}{k^+\sqrt{l^+-k^+}}+\frac{1}{l^+\sqrt{l^+-k^+}}\right)d^\dagger_{s,\underline{l}}a_{\underline{l}-\underline{k}}
$$

$$
-\frac{1}{\sqrt{16\pi^3}}mg\int_{0}^{k^+}d\underline{l}\frac{e^{i\frac{1}{2}(E_{1,0}(\underline{l})+E_{0,1}(\underline{k}-\underline{l}))x^+}}{E_{1,0}(\underline{k})-E_{1,0}(\underline{l})-E_{0,1}(\underline{k}-\underline{l})}\left(\frac{1}{k^+\sqrt{k^+-l^+}}+\frac{1}{l^+\sqrt{k^+-l^+}}\right)d^\dagger_{s,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}
$$

$$
-\frac{1}{\sqrt{16\pi^3}}mg\int d\underline{l}\frac{e^{i\frac{1}{2}(-E_{1,0}(\underline{l})+E_{0,1}(\underline{l}+\underline{k}))x^+}}{E_{1,0}(\underline{k})+E_{1,0}(\underline{l})-E_{0,1}(\underline{l}+\underline{k})}\left(\frac{1}{k^+\sqrt{k^++l^+}}-\frac{1}{l^+\sqrt{k^++l^+}}\right)b_{-s,\underline{l}}a^\dagger_{\underline{l}+\underline{k}}\Bigg]\Bigg\}.
$$

The fields ψ_+ and ϕ are free fields:

$$
\psi_+(x^+,\underline{x})=\frac{1}{\sqrt{16\pi^3}}\sum_s\int d\underline{k}\chi_s\left[b_{s,\underline{k}}e^{-i(\frac{1}{2}E_{1,0}(\underline{k})x^+-\underline{k}\cdot\underline{x})}+d^\dagger_{-s,\underline{k}}e^{+i(\frac{1}{2}E_{1,0}(\underline{k})x^+-\underline{k}\cdot\underline{x})}\right], \tag{4.14}
$$

$$
\phi(x^+,\underline{x})=\frac{1}{\sqrt{16\pi^3}}\int d\underline{q}\frac{1}{\sqrt{q^+}}\left[a_{\underline{q}}e^{-i(\frac{1}{2}E_{0,1}(\underline{q})x^+-\underline{q}\cdot\underline{x})}+a^\dagger_{\underline{q}}e^{+i(\frac{1}{2}E_{0,1}(\underline{q})x^+-\underline{q}\cdot\underline{x})}\right]. \tag{4.15}
$$

Thus we have obtained solutions for all the independent degrees of freedom. By use of equations (2.9) and (2.8) we can reconstruct the Fermi fields. One can then evaluate the Fermi fields on the surface $t=0$ and work out the relation between the equal-time Fermi modes and the light-cone Fermi modes just as we did above for the Bose field.

V. SEVERAL PAULI–VILLARS FIELDS OF THE SAME TYPE

In general the UV regularization of a renormalizable theory requires the introduction of more than one PV field of each type. For example, the full Yukawa theory requires two PV fermions, and QCD requires several PV fermions for each color and flavor [8]. Our method is easily extendable to such cases. We will illustrate this for the theory studied in Ref. [3], Yukawa theory without fermion loops regulated with three PV Bose fields. The theory includes one (physical) Fermi field and four Bose fields: the physical field, which we will call ϕ_1; two negative metric PV fields, which we will call ϕ_2 and ϕ_4; and a positive metric PV field, which we will call ϕ_3. From these we define the following four zero-norm fields:

$$
\zeta\equiv N(\xi_1\phi_1+\xi_2\phi_2+\xi_3\phi_3+\xi_4\phi_4)\,, \tag{5.1}
$$

$$
\zeta_1\equiv N(\xi_3\phi_1+\xi_4\phi_2-\xi_1\phi_3-\xi_2\phi_4)\,, \tag{5.2}
$$

$$
\zeta_2\equiv N(\xi_2\phi_1-\xi_1\phi_2+\xi_4\phi_3-\xi_3\phi_4)\,, \tag{5.3}
$$

$$
\zeta_3\equiv N(\xi_4\phi_1-\xi_3\phi_2-\xi_2\phi_3+\xi_1\phi_4)\,. \tag{5.4}
$$

The ξ_i are relative coupling strengths for the different fields and are chosen to satisfy constraints that accomplish designated cancellations. In particular, we have $\xi_1=1$ to retain

g as the ordinary bare coupling and $\sum_i (-1)^{i+1}\xi_i^2 = 0$ to give the ζ fields zero norm. The factor N is chosen such that the ζ fields have the following commutation relations:

$$\Big[\zeta(x^+,\underline{x}),\partial^+\zeta_2(x^+,\underline{x}')\Big] = i\delta^{(3)}(\underline{x}-\underline{x}')\,, \tag{5.5}$$

$$\Big[\zeta_1(x^+,\underline{x}),\partial^+\zeta_3(x^+,\underline{x}')\Big] = i\delta^{(3)}(\underline{x}-\underline{x}')\,. \tag{5.6}$$

All other commutators are zero. The value of N is then given by

$$N = 1/\sqrt{2(\xi_1\xi_2+\xi_3\xi_4)}\,. \tag{5.7}$$

We make the mode expansions

$$\psi_+(\underline{x}) = \frac{1}{\sqrt{16\pi^3}}\sum_s\int d\underline{k}\;\chi_s\,\Big[b_{s,\underline{k}}e^{-i\underline{k}\cdot\underline{x}} + d^\dagger_{-s,\underline{k}}e^{+i\underline{k}\cdot\underline{x}}\Big]\,, \tag{5.8}$$

$$\zeta(\underline{x}) = \frac{1}{\sqrt{16\pi^3}}\int d\underline{q}\frac{1}{\sqrt{q^+}}\,\Big[a_{\underline{q}}e^{-i\underline{q}\cdot\underline{x}} + a^\dagger_{\underline{q}}e^{+i\underline{q}\cdot\underline{x}}\Big]\,, \tag{5.9}$$

$$\zeta_i(\underline{x}) = \frac{1}{\sqrt{16\pi^3}}\int d\underline{q}\frac{1}{\sqrt{q^+}}\,\Big[a_{i,\underline{q}}e^{-i\underline{q}\cdot\underline{x}} + a^\dagger_{i,\underline{q}}e^{+i\underline{q}\cdot\underline{x}}\Big]\,,\quad i=1,2,3\,. \tag{5.10}$$

If we take the masses of the fields, ϕ_i, to be μ_i, we find the operator P^- to be of the form

$$P^- \equiv P^-_{(0)} + gP^-_{(1)} + g^2P^-_{(2)}\,, \tag{5.11}$$

where

$$\begin{aligned}
P^-_{(0)} = &\sum_s\int d\underline{k}\Big[\frac{\boldsymbol{k}_\perp^{\,2}+m^2}{k^+}\Big]\Big(b^\dagger_{s,\underline{k}}b_{s,\underline{k}} + d^\dagger_{s,\underline{k}}d_{s,\underline{k}}\Big)\\
&+\int d\underline{q}\Big[\frac{\boldsymbol{q}_\perp^{\,2}}{q^+}\Big](a^\dagger_{\underline{q}}a_{2\underline{q}} + a^\dagger_{2\underline{q}}a_{\underline{q}} + a^\dagger_{1,\underline{q}}a_{3\underline{q}} + a^\dagger_{3\underline{q}}a_{1\underline{q}})\\
&+N^2\int d\underline{q}\Big[\frac{\xi_2^2\mu_1^2-\xi_1^2\mu_2^2+\xi_4^2\mu_3^2-\xi_3^2\mu_4^2}{q^+}a^\dagger_{\underline{q}}a_{\underline{q}} + \frac{\xi_4^2\mu_1^2-\xi_3^2\mu_2^2+\xi_2^2\mu_3^2-\xi_1^2\mu_4^2}{q^+}a^\dagger_{2\underline{q}}a_{2\underline{q}}\\
&+\frac{\xi_1^2\mu_1^2-\xi_2^2\mu_2^2+\xi_3^2\mu_3^2-\xi_4^2\mu_4^2}{q^+}a^\dagger_{3\underline{q}}a_{3\underline{q}} + \frac{\xi_3^2\mu_1^2-\xi_4^2\mu_2^2+\xi_1^2\mu_3^2-\xi_2^2\mu_4^2}{q^+}a^\dagger_{4\underline{q}}a_{4\underline{q}}\\
&+\frac{\xi_1\xi_2(\mu_1^2+\mu_2^2)+\xi_3\xi_4(\mu_3^2+\mu_4^2)}{q^+}(a^\dagger_{\underline{q}}a_{3\underline{q}} + a^\dagger_{3\underline{q}}a_{\underline{q}})\\
&+\frac{\xi_2\xi_4(\mu_1^2-\mu_3^2)-\xi_1\xi_3(\mu_2^2-\mu_4^2)}{q^+}(a^\dagger_{\underline{q}}a_{2\underline{q}} + a^\dagger_{2\underline{q}}a_{\underline{q}})\\
&+\frac{\xi_2\xi_3(\mu_1^2-\mu_4^2)+\xi_1\xi_4(\mu_2^2-\mu_3^2)}{q^+}(a^\dagger_{\underline{q}}a_{4\underline{q}} + a^\dagger_{4\underline{q}}a_{\underline{q}})\\
&+\frac{\xi_1\xi_4(\mu_1^2-\mu_4^2)+\xi_2\xi_3(\mu_2^2+\mu_3^2)}{q^+}(a^\dagger_{3\underline{q}}a_{2\underline{q}} + a^\dagger_{2\underline{q}}a_{3\underline{q}})\\
&+\frac{\xi_1\xi_3(\mu_1^2-\mu_3^2)-\xi_2\xi_4(\mu_2^2-\mu_4^2)}{q^+}(a^\dagger_{3\underline{q}}a_{4\underline{q}} + a^\dagger_{4\underline{q}}a_{3\underline{q}})\\
&+\frac{\xi_3\xi_4(\mu_1^2+\mu_2^2)+\xi_1\xi_2(\mu_3^2+\mu_4^2)}{q^+}(a^\dagger_{2\underline{q}}a_{4\underline{q}} + a^\dagger_{4\underline{q}}a_{2\underline{q}})\Big]\,,
\end{aligned} \tag{5.12}$$

and

$$P^-_{(1)} = \frac{1}{\sqrt{8\pi^3}}\sum_s \int d\underline{k}d\underline{l}d\underline{q}\ \frac{(\boldsymbol{\epsilon}_{\perp,2s}{}^*\cdot\boldsymbol{l}_\perp)}{l^+\sqrt{q^+}}$$
$$\times\Big\{b^\dagger_{s,\underline{k}}b_{-s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q}) + b^\dagger_{s,\underline{k}}d^\dagger_{s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{q}-\underline{k}-\underline{l})$$
$$+b^\dagger_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) - d^\dagger_{s,\underline{l}}d_{-s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l})$$
$$+d^\dagger_{-s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) - d^\dagger_{s,\underline{l}}d_{-s,\underline{k}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q})\Big\} + \text{h.c.}$$
$$+\frac{m}{\sqrt{16\pi^3}}\sum_s \int d\underline{k}d\underline{l}d\underline{q}\ \frac{1}{l^+\sqrt{q^+}}$$
$$\times\Big\{b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q}) + d^\dagger_{-s,\underline{l}}b^\dagger_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q})$$
$$+b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l}) + d^\dagger_{s,\underline{l}}d_{s,\underline{k}}a_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{q}-\underline{l})$$
$$+d_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{q}) + d^\dagger_{s,\underline{l}}d_{s,\underline{k}}a^\dagger_{\underline{q}}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{q})\Big\} + \text{h.c.} \qquad (5.13)$$

In this case we do have four point interactions. They are given by

$$P^-_{(2)} = \frac{1}{16\pi^3}\sum_s \int d\underline{k}d\underline{l}d\underline{p}d\underline{q}\ \frac{1}{\sqrt{p^+q^+}}$$
$$\times\Big\{b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^+-p^+}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{p}-\underline{q}) + b^\dagger_{s,\underline{k}}d^\dagger_{-s,\underline{l}}a_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^+-p^+}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{p}-\underline{q})$$
$$+b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^+-q^+}\,\delta^{(3)}(\underline{k}+\underline{p}-\underline{l}-\underline{q}) + b^\dagger_{s,\underline{k}}d^\dagger_{-s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^+-q^+}\,\delta^{(3)}(\underline{k}+\underline{l}+\underline{p}-\underline{q})$$
$$+b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^++p^+}\,\delta^{(3)}(\underline{k}+\underline{p}-\underline{l}-\underline{q}) + b^\dagger_{s,\underline{k}}d^\dagger_{-s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^++p^+}\,\delta^{(3)}(\underline{k}+\underline{l}+\underline{p}-\underline{q})$$
$$+b^\dagger_{s,\underline{k}}b_{s,\underline{l}}a^\dagger_{\underline{p}}a^\dagger_{\underline{q}}\,\frac{1}{k^++p^+}\,\delta^{(3)}(\underline{k}+\underline{p}+\underline{q}-\underline{l}) + d^\dagger_{s,\underline{k}}d_{s,\underline{l}}a_{\underline{p}}a_{\underline{q}}\,\frac{1}{l^++p^+}\,\delta^{(3)}(\underline{k}-\underline{l}-\underline{p}-\underline{q})$$
$$-d_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^++q^+}\,\delta^{(3)}(\underline{k}+\underline{l}+\underline{q}-\underline{p}) + d^\dagger_{s,\underline{k}}d_{s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{l^++q^+}\,\delta^{(3)}(\underline{k}+\underline{p}-\underline{l}-\underline{q})$$
$$-d_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{k^+-p^+}\,\delta^{(3)}(\underline{k}+\underline{l}+\underline{q}-\underline{p}) + d^\dagger_{s,\underline{k}}d_{s,\underline{l}}a^\dagger_{\underline{p}}a_{\underline{q}}\,\frac{1}{l^+-p^+}\,\delta^{(3)}(\underline{k}+\underline{p}-\underline{l}-\underline{q})$$
$$-d_{s,\underline{k}}b_{-s,\underline{l}}a^\dagger_{\underline{p}}a^\dagger_{\underline{q}}\,\frac{1}{k^+-p^+}\,\delta^{(3)}(\underline{k}+\underline{l}-\underline{p}-\underline{q}) + d^\dagger_{s,\underline{k}}d_{s,\underline{l}}a^\dagger_{\underline{p}}a^\dagger_{\underline{q}}\,\frac{1}{l^+-p^+}\,\delta^{(3)}(\underline{k}+\underline{p}+\underline{q}-\underline{l})\Big\}. \qquad (5.14)$$

Again, if we define an index given by minus the number of $a^\dagger$ type quanta in the state, we find that, for $\mu_1 = \mu_2 = \mu_3 = \mu_4$, all the terms in P^- either leave the index unchanged or lower it. The system is again triangular and easy to solve. In the present case, unlike the situation we found above with PV Fermi fields, the eigenstates project onto an infinite number of sectors. Nevertheless, the coefficients can be obtained recursively.

We shall illustrate this with one example. If we define a new vertex amplitude by

$$X(\underline{k},\underline{p},\underline{q}) \equiv \frac{1}{16\pi^3}\frac{1}{\sqrt{p^+q^+}}\left(\frac{1}{k^+-p^+}\right), \qquad (5.15)$$

we find that the projection of the eigenstate whose highest state is $\beta^\dagger_{+,\underline{k}}|0\rangle$, onto the sectors with not more than two Bose quanta, is given by

$$
\begin{aligned}
&\beta^\dagger_{+,\underline{k}}|0\rangle \\
&+gm\int_0^{k^+} d\underline{l}\frac{U(\underline{k},\underline{l})}{E_{1,0}(\underline{k})-E_{1,1}(\underline{l},\underline{k}-\underline{l})}b^\dagger_{+,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}|0\rangle + g\int_0^{k^+} d\underline{l}\frac{V(\underline{k},\underline{l})}{E_{1,0}(\underline{k})-E_{1,1}(\underline{l},\underline{k}-\underline{l})}b^\dagger_{-,\underline{l}}a^\dagger_{\underline{k}-\underline{l}}|0\rangle \\
&+g^2\int_0^{k^+} d\underline{p}\int_0^{k^+-p^+} d\underline{q} \\
&\frac{m^2U(\underline{k},\underline{k}-\underline{p})U(\underline{k}-\underline{p},\underline{k}-\underline{p}-\underline{q})+V(\underline{k},\underline{k}-\underline{p})V(\underline{k}-\underline{p},\underline{k}-\underline{p}-\underline{q})+X(\underline{k},\underline{p},\underline{q})}{E_{1,0}(\underline{k})-E_{1,2}(\underline{k}-\underline{p}-\underline{q},\underline{p},\underline{q})}b^\dagger_{+,\underline{k}-\underline{p}-\underline{q}}a^\dagger_{\underline{p}}a^\dagger_{\underline{q}}|0\rangle \\
&+g^2\int_0^{k^+} d\underline{p}\int_0^{k^+-p^+} d\underline{q} \\
&\frac{mU(\underline{k},\underline{k}-\underline{p})V(\underline{k}-\underline{p},\underline{k}-\underline{p}-\underline{q})+mV(\underline{k},\underline{k}-\underline{p})U(\underline{k}-\underline{p},\underline{k}-\underline{p}-\underline{q})}{E_{1,0}(\underline{k})-E_{1,2}(\underline{k}-\underline{p}-\underline{q},\underline{p},\underline{q})}b^\dagger_{-,\underline{k}-\underline{p}-\underline{q}}a^\dagger_{\underline{p}}a^\dagger_{\underline{q}}|0\rangle\,. \qquad (5.16)
\end{aligned}
$$

VI. CONCLUSIONS

If quantum field theories are regulated with PV fields, and the masses of the PV fields are set equal to the masses of the physical fields, the resulting theories can be solved explicitly. The eigenstates are nontrivial but can be constructed either in closed form or in terms of recursion relations. The spectrum of these theories is identical to that of the corresponding noninteracting theory. However, many eigenstates have zero norm, and the S matrix is trivial; these properties make the fully degenerate case difficult to interpret physically, as should be expected in a theory that violates unitarity so severely.

We have also shown that, in some cases, exact operator solutions can be obtained. These solutions can be used to study those general properties of quantum field theories which depend on covariance, but not on unitarity. These properties allow us to also investigate the relation between the light-cone representation and the equal-time representation. Since these theories have a highly structured equal-time vacuum, they may provide insight into the nontrivial equal-time vacuum structure of theories such as QCD.

The existence of the solution for arbitrary coupling constant, but for zero values of the mass differences δ and Δ (or their equivalents in more complicated theories), opens the possibility of doing perturbation expansions, not in the coupling constant, but in the mass differences between the Pauli–Villars and physical hadrons, much like the case of broken supersymmetry. We have shown that the wave functions of theories can be obtained in exact form for degenerate physical and PV masses. Thus, if evolution equations in the mass differences can be derived, we know how to initialize the solutions. We plan to consider this approach in future work.

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APPENDIX: LIGHT-CONE CONVENTIONS

We define light-cone coordinates by

$$x^{\pm} \equiv x^0 \pm x^3 \,, \tag{A1}$$

with the transverse coordinates $\boldsymbol{x}_\perp \equiv (x^1, x^2)$ unchanged. Covariant four-vectors are written as *e.g.* $x^\mu = (x^+, x^-, \boldsymbol{x}_\perp)$, with the spacetime metric

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{A2}$$

Explicitly, we have

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu = \frac{1}{2}(x^+ y^- + x^- y^+) - \boldsymbol{x}_\perp \cdot \boldsymbol{y}_\perp \,. \tag{A3}$$

We also make use of an underscore notation: for position-space variables we write

$$\underline{x} \equiv (x^-, \boldsymbol{x}_\perp) \,, \tag{A4}$$

while for momentum-space variables

$$\underline{k} \equiv (k^+, \boldsymbol{k}_\perp) \,. \tag{A5}$$

Then the dot product becomes

$$\underline{k} \cdot \underline{x} \equiv \frac{1}{2} k^+ x^- - \boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp \,. \tag{A6}$$

Spatial derivatives are defined by

$$\partial_+ \equiv \frac{\partial}{\partial x^+} \,, \qquad \partial_- \equiv \frac{\partial}{\partial x^-} \,, \qquad \partial_i \equiv \frac{\partial}{\partial x^i} . \tag{A7}$$

The gamma matrices $\gamma^\pm \equiv \gamma^0 \pm \gamma^3 = (\gamma^\mp)^\dagger$ satisfy the familiar relation

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \,, \tag{A8}$$

with $g^{\mu\nu}$ the light-cone metric. It is simple to verify that the (hermitian) matrices

$$\Lambda_\pm \equiv \frac{1}{2} \gamma^0 \gamma^\pm \tag{A9}$$

satisfy

$$\Lambda_\pm^2 = \Lambda_\pm \,, \qquad \Lambda_\pm \Lambda_\mp = 0 \,, \qquad \Lambda_+ + \Lambda_- = 1 \,, \tag{A10}$$

so that they serve as projectors on spinor space. In the Dirac representation of the γ-matrices, Λ_+ is given by

$$\Lambda_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \tag{A11}$$

which has two eigenvectors, both with eigenvalue +1. They are

$$\chi_{+\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{A12}$$

These serve as a convenient spinor basis for the expansion of the field $\psi_+ \equiv \Lambda_+\psi$ on the light cone.

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