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Lessons From Gauge (and Nongauge) Theories in Low (< 12) Dimensions

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In this contribution I shall review some of the lessons that I think have been learned over the years from studies of model problems using the light-cone representation. The selection of works which I shall discuss is made to illustrate certain specific points and is not meant to be a review of all the valuable work done in this area. There has been a very large amount of work in the area, much of it very valuable, and a complete review would exceed the time and space allocations for the contribution and also the competence of the author.

1. Introduction

In trying to organize my thoughts regarding the lessons that have been learned over the years from studies of model problems I was somehow reminded of the three promises the apparitions made to Macbeth. I have in mind something like this:

- SOME CORRECT RESULTS CAN BE OBTAINED FROM CALCULATIONS WHICH ARE MUCH SIMPLER THAN EQUAL-TIME CALCULATIONS
- LIGHT-CONE WAVE FUNCTIONS ARE OFTEN MUCH SIMPLER THAN EQUAL-TIME WAVE FUNCTIONS
- MACBETH! MACBETH! BEWARE THE $P^+ = 0$ SINGULARITY

I am afraid that, as was true of the Thane of Cawdor, we have sometimes been guilty of focusing our attention too much on the first two items and ignoring (or denying) the last one. Therefore, in this contribution I am going to discuss the last item. In making that choice I do not in any way mean to deny or belittle the many successes of the light-cone methods in model calculations. But these successes have been well discussed at the meetings and are generally well known to most of the participants here. Furthermore, while it may be more fun to call fourth and discuss our successes, further progress requires that we identify problem areas for the light-cone methods and solve those problems.

In any realistic calculation and most model calculations done in the light-cone representation one encounters a singularity at $p^+ = 0$. To proceed it is necessary to regulate that singularity in some way. No particular way of performing the regulation has been shown to be both generally applicable and correct (whatever, exactly, correct might be taken to mean). We may therefore wonder how significant a problem the $p^+ = 0$ singularity represents. To put the blackest light on it we might think:

THERE ARE AN INFINITE NUMBER OF WAYS TO REGULATE THE $P^+ = 0$ SINGULARITY, SO A RANDOMLY CHOSEN WAY WILL GIVE A CALCULATION WHICH IS WRONG, NOT CERTAINLY, BUT WITH PROBABILITY 1

On the other hand, the situation might not be that bad. Perhaps it matters little just how we regulate the singularity; perhaps all ways of regulating it, or at least a large number of them, give the same (correct) answer.

The purpose of this contribution is to explore what model calculations have taught us about this question. I shall concentrate on calculations for three types of theories: the Schwinger model; ϕ^4 theories; and Yukawa theories. I shall also have something to say about how, in my view, some of the contributions in this session show progress in learning to regulate the $p^+ = 0$ singularity. I shall also mention a few cases where I believe further model calculations could provide valuable additional insights into the nature of the

$p^+ = 0$ singularity and help devise ways to regulate it in realistic calculations.

2. The Schwinger Model

Probably the best studied case is the Schwinger model. For the Schwinger model , as usual, we must regulate the $p^+ = 0$ singularity. Let us begin with the comforting observation that there is at least one way to regulate the singularity which allows us to give a completely correct solution. If we regulate ultraviolet singularities with gauge invariant point splitting, regulate infrared singularities with a Klaiber subtraction and include in the calculations two nonphysical fields, one of which is a ghost, we can give an operator solution which is equivalent to standard solutions in other gauges and quantization schemes [1]. In particular, we get the correct spectrum and eigenstates, the correct θ-structure for the vacua, the correct anomaly and the correct chiral condensate. The theory can be quantized either on the light-cone or at equal-time and the same operator solution will be found in either case. Furthermore, if that regularization scheme is used and the solution is used as a starting point for mass perturbation theory, the standard first order correction the the Schwinger particle is found [2], that is, we get

$$\Delta M^2 \sim -4\pi\mu\langle\Omega|\bar{\Psi}\Psi|\Omega\rangle \qquad (1)$$

If the theory is quantized on the light-cone there are subtleties which require special methods [3] but I will not discuss the point further here. The relation (1) comes about from the fact that the two auxiliary fields induce an operator into P^- which will be missed if those fields are omitted (the operator is proportional to the chiral condensate). The same operator was found by entirely different methods by the St. Petersburg group and was discussed by Prokhvatilov in his talk here [4]. In full QCD there will be operators similar to the one in the Schwinger model [3].

So, the problem can be solved exactly, but the regularization scheme discussed above is rather complicated. What happens if we choose to regularize the theory in a simpler way? Surely one of the simplest ways to regulate the theory is to impose periodicity conditions on the light-cone and include no zero modes or auxiliary fields. That case is studied in Refs. [5,6]. Many correct properties are obtained in that regularization scheme: the spectrum is correct and the qualitative properties of the eigenstates are correct. But there is no θ-structure, no chiral condensate and it is not possible to define the pseudocurrent and so there is no way to give meaning to the anomaly. As one would expect from the fact that there is no chiral condensate, if one uses the solution as a starting point for mass perturbation theory, the physical mass grows as the second power of the bare mass for small values of the bare mass (as opposed to the standard result given in eq. (1)). For larger values of the bare mass the results agree pretty well with lattice calculations.

If one wants to do mass perturbation theory and just wants to find the shift in the mass and wave function one can regulate that calculation by imposing a zero on the wave function at $p^+ = 0$. The calculations are done in the continuum (no periodicity conditions) and no use is made of auxiliary fields or of any additional infrared regulator. It may even seem that one has not imposed a regularization but that is not correct. Such calculations have a long history [7,8,6]. The results have lead to what is sometimes called the two percent problem. The source of that term is that if the results of the calculation are compared with the results from standard mass perturbation theory, the results of the light-cone calculation differ from the standard result *evaluated at* $\theta = 0$ by about two percent. The vacuum in the light-cone calculation is not the $\theta = 0$ vacuum or a θ-state for any value of θ but rather the ground state in the $Q_5 = 0$ sector of the theory so it is not clear that such a comparison is very meaningful but that is the origin of the two percent problem. Burkardt and Harada [9] improved the situation somewhat recently by attempting to give the calculation a θ-structure by adding a term to P^- intended to simulate the effect of a background field (the term is not the same term as that found in [2] or [10]). In that calculation there is at least a parameter, θ, which can be set equal to zero or any other value; it is not clear that the full θ structure of the theory is present. What they find is that the mass shift has the correct dependence on (their) θ but the two percent problem persists. Much work (and computer time) has been expended trying to show that the two percent problem is a numerical problem with the light-cone calculation due to the truncation of higher Fock sectors or some other inaccuracy. So far as I know, these efforts have not succeeded. In fact, one of the experts told me that he did not believe that the problem could be resolved this way [11].

I do not know what the resolution of the two percent problem will be (if it is ever resolved) but from the point of view of this contribution it is possible that the light-cone calculation does not give the same answer as standard mass perturbation theory. Imposing a zero on the wave function is regulating the theory in the infrared and from the results of that single calculation it is not possible to say what the effect of that regulation has been on the covariance or gauge invariance of the full theory. In the absence of arguments that that regulator maintains all those symmetries (including invariance under the large gauge transformations) it must be considered possible that the regulator has changed the answer.

That that can happen is shown by the study of another possible regulator. One can impose periodicity conditions on the light-cone and include all the zero modes and auxiliary fields necessary to provide gauge invariance and the restricted Lorentz invariance [12]. That scheme is somewhat like the first scheme I discussed except that there is no ghost field (it's use is incompatible with the periodicity conditions) and the Klaiber subtraction is replaced by the periodicity conditions. In that case one gets the correct spectrum, the correct θ-structure, the correct anomaly and a nonzero value of the chiral condensate. But the value of the chiral condensate does not go the the correct limit at the periodicty length goes to infinity (the value of the chiral condensate goes to zero in that limit). The source of the problem can be traced to the fact that the use of the periodicity conditions is simply too severe a regulation of the $p^+ = 0$ singularity. The damage is not repaired by taking the limit where full Lorentz invariance is formally restored.

There has been a lot of good work on light-cone aspects of the Schwinger model which I have not discussed because the results did not serve to illustrate the specific points I wished to make (as far as I could understand). In some cases authors quantized on space-like surfaces then rotated to the light-cone, quantized on a space-like surface in light-cone gauge or did other studies; some references are given in [13].

3. ϕ^4

The problem of ϕ^4 in two dimensions with a wrong sign mass has been considered by a number of authors. The theory is known to have a condensate for sufficiently large values of the coupling so it is natural to wonder how this physics is expressed in the light-cone representation. As usual, to do any calculations one must regulate the $p^+ = 0$ singularity. Most of the calculations have been done in the context of regulating the singularity with periodicity conditions on $x^+ = 0$. If no zero modes are kept, and the field is simply expanded in modes, there is no way to get a condensate. Somewhat of an improvement was made by shifting the field to the classical minimum then expanding the shifted field in modes [14]. That gives a nonzero value for the condensate but the full dynamics of the problem is clearly missing. A further improvement was to keep a (constrained) zero mode, which can be shown to exist from the equations of motion [15,16]. The zero mode is an operator which has a c-number part; keeping just the c-number part is equivalent to shifting the field. Maintaining the full dynamics of the zero mode is a more complex problem and a considerable amount of numerical work was done to study the system [17].

While the calculations did give a condensate and a proper renormalization was shown to exist [16], the results of the light-cone calculations never agreed with known results for the theory. In particular, all the calculations discussed above gave mean field critical exponents and I think it is clear that with that regularization for the $p^+ = 0$ singularity one cannot get anything but mean field critical exponents. It has been suggested a number of times in the past that the source of the problem is the regularization of the $p^+ = 0$ singularity and that the solution must involve dynamical zero modes. Prior to today there were few calculations to back up these statements; I know only of the rather preliminary results presented in [18]. Thus the results presented here [19] surely represent an advance. I have not yet been able to study the new results in sufficient detail to see whether I believe that they represent a completely satisfactory solution to the problem but they surely are interesting: the problem is certainly important.

Another possible avenue by which the ϕ^4 problem might be addressed has been proposed by Rozowski and Thorn [20]. They propose giving up translational invariance. In that case the field can have a condensate without any zero modes. They attempt to recover translational invariance in a limit while still retaining the condensate.

Before leaving the scalar field questions I should point out that even in free theory there is a problem. Long ago Nakanishi and Yamawaki [21] showed that the two point function for the free scalar field in four dimensions is not properly calculated for space-like separations of the points if the theory is quantized on the light-cone and any natural regularization of the $p^+ = 0$ singularity (such as principal value or the use of periodicity conditions) is used. To get the correct answer one must employ the regularizaton:

$$\frac{1}{p^+} \longrightarrow \frac{1}{p^+} e^{\frac{im\epsilon}{4p^+}} \qquad (2)$$

where m is the mass of the field. While the need for this regularization is clear upon a little thought, it might not be the first thing that would occur to someone solving the free scalar field for the first time and thus unaware of the details of the solution.

4. Yukawa Theory

A number of very useful studies have been made on Yukawa theory. As usual, the issue is regulating the $p^+ = 0$ singularity. An interesting study was done by Burkardt and Langnau [22] based on perturbation theory performed in the light-cone representation. They made perturbative calculations for Yukawa theory using a variety of momentum cutoffs, checked the results for covariance and compared them to standard, Feynman calculations. The results show that any momentum cutoff induces counter terms. Similar studies were made by Schoonderwoerd and Baaker [23]. The momentum cutoffs break covariance and so do not allow a direct comparison with Feynman methods except in the final results. A method which does allow a direct comparison at each stage of the calculation is Pauli-Villars.

The idea of using Pauli-Villars to regulate perturbative light-cone calculations goes back at least to the work of Chang and Yan [24]. They regulated the one loop self energy in Yukawa with Pauli-Villars fields and found that they needed three Pauli-Villars bose fields and that the third Pauli-Villars condition must be chosen to be different from the standard one. They then state that that procedure will properly regulate all orders. Their results raise a number of questions. It is well known that in Feynman methods one Pauli-Villars field regulates the Feynman graph. Why can we not take that convergent integral, perform the p^- integral and get a finite light-cone integral regulated with that same Pauli-Villars field. Well, if we perform the p^- integral in the Feynman diagram we do get a finite result for all values of $p^\perp$ and p^+ but the remaining integral is not finite, it is linearly divergent. What has happened? The answer is that the original Feynman integral is not really convergent; it is conditionally convergent and thus, any value assigned to it is a prescription [25,26]. Since the integral is conditionally convergent, including the points in the domain of integration in a different order might lead to a different answer and that is what has happened. The two additional Pauli-Villars fields are needed to correct a linear divergence and a finite error; these differences as compared to the Feynman answer result from the different order of performing the integrations. The final correction is finite, which explains why the final Pauli-Villars condition must be different from the standard one. In their paper Chang and Yan state that one should add enough Pauli-Villars fields to render the Feynman integral absolutely convergent but there is no number of Pauli-

Villars fields which will do that. Furthermore, we have shown that, at best, with the three Pauli-Villars bose fields, the final, nonstandard Pauli-Villars condition must be modified in higher order perturbative or in nonperturbative calculations [27]. For these reasons, it has not been clear whether Pauli-Villars regulation would really allow a proper renormalization to all orders of a light-cone Yukawa calculation (or of a nonperturbative calculation).

I therefore think that the work of the St. Petersburg group, reported here by Prokhvatilov [28,29,4] represents a substantial advance. They have given a measure which can be calculated for an arbitrary theory with an arbitrary set of Pauli-Villars fields proposed to regulate the theory. If the measure is positive they show that, at least perturbatively, the results of a light-cone calculation will be identical with the results of a similarly regulated Feynman calculation. In the case of Yukawa theory the use of three Pauli-Villars bose fields does not pass the test, so while they do not exactly predict that the problems we described above must occur they do predict that they might occur; more importantly, they show that these problems will not occur if the theory is regulated with one Pauli-Villars bose field and two Pauli-Villars fermi fields. Notice the consequences of their result: if a theory is properly regulated with Pauli-villars fields, then, at least at the level of perturbation theory, all the symmetries which are kinematically imposed in Feynman theory will be preserved in the light-cone methods. These symmetries might include ones hard to impose in the light-cone representation such as chiral symmetry and rotational symmetry. In addition to the Yukawa work, the St. Petersburg group has proposed a regulation procedure for QCD which involves Pauli-Villars fields and other elements.

5. Free Fields, Auxiliary Fields and Other Matters

We saw in the case of the Schwinger model that a completely covariant, gauge invariant regulation of the Schwinger model requires that auxiliary fields be included in the solution. Such fields will be needed for any gauge theory. The need for such fields was first shown by Bassetto and coworkers [30]. They found the fields by quantizing in light-cone gauge at equal-time. Such fields will be needed whether quantized at equal-time or on the light-cone and are almost surely needed in all gauges even gauges usually considered "physical" gauges including the Coulomb gauge [31].

The thing that is particular to light-cone quantization is that, in light-cone gauge, these fields cannot be initialized on the principal quantization surface, $x^+ = 0$; so special techniques are needed to include them in the formulation [32–34]. The auxiliary fields will produce important physical effects any time condensates are an important part of the theory under consideration so it is important that we learn more about them than is currently understood. They are the subject of Yuji Nakawaki's contribution to these proceedings [35]. If the anti-light-cone gauge is used, the auxiliary fields can be initialized on $x^+ = 0$; in that case they are static fields. That framework has been discussed at past light-cone meetings [36] but was not discussed here.

I wish to remark briefly on another model which has been studied recently but was not discussed here. There has been a puzzle about an inconsistency in the 't Hooft model: the spectrum 't Hooft obtained implies a condensate which is not contained in his solution. 't Hooft found his solution using the light-cone representation and the source of the trouble is almost surely his rough treatment of the $p^+ = 0$ singularity. Shifman discussed the problem at the Lutsen meeting but the work has not been written up. The Italian group, Bassetto, Nardeli, Guiguolo and Vian, who comprise our hosts here, have published a number of studies on that model and particularly on the related model without fermions. The work can be traced from Vian's talk at the last Heidelberg meeting [37]. In spite of all the work done, the fundamental inconsistency in the 't Hooft solution has never been resolved. No one has ever given a solution whose spectrum implies the condensate contained in the solution. It would be valuable to resolve this problem.

6. Remarks

We have learned many things from studies of model problems in low dimensions. We have learned that the $p^+ = 0$ singularity is usually a very complicated object and that if it is not treated correctly our calculations will not be successful. While I do not believe that a completely general method for treating the $p^+ = 0$ singularity is currently available, I believe that much progress has been made, that we have learned to do it in some specific cases and that we have determined many properties that a general solution must have. Also, I need to repeat what I said at the beginning: studies of model problems in low dimensions have provided us with many successes and have shown that the light-cone representation can have significant advantages over the equal-time representation.

I would like to take this opportunity to thank the organizers for hosting such a successful conference in such a beautiful place.

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