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Perturbative Formulation of Pure Space-Like Axial Gauge QED with Infrared Divergences Regularized by Residual Gauge Fields

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Abstract

We construct a new perturbative formulation of pure space-like axial gauge QED in which the inherent infrared divergences are regularized by residual gauge fields. For that purpose we perform our calculations in coordinates $x^\mu = (x^+, x^-, x^1, x^2)$, where $x^+ = x^0 \sin\theta + x^3 \cos\theta$ and $x^- = x^0 \cos\theta - x^3 \sin\theta$. $A_- = A^0 \cos\theta + A^3 \sin\theta = n{\cdot}A = 0$ is taken as the gauge fixing condition. That framework allows us to show that the residual gauge fields are not canonical variables in a pure space-like axial gauge formulation, although they are necessary ingredients of the system. To overcome that difficulty we first construct, in the region $0 \leqq \theta < \frac{\pi}{4}$, a temporal gauge formulation in which x^- is taken as the evolution parameter and $A_+, A_i (i = 1, 2)$ and their canonical conjugate momenta are independent canonical variables so that we can quantize them by canonical methods. As a result we can obtain the commutation relations of the x^--independent Nakanishi-Lautrup field B, which we extrapolate into the axial region $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, where x^+ is taken as the evolution parameter. We obtain a formal solution of the gauge field equations with the relevant integration constants specified on the basis of the free gauge fields in the temporal formulation. Then we make use of the solution to specify the integration constants appearing when we solve the axial gauge constraint equations. It turns out that, because the residual gauge fields are multiplied by the inverse Laplace operator $\frac{1}{n_- \partial_+^2 + \partial_\perp^2}$, which becomes singular in the axial region, they give rise to vanishing contributions to the equal x^+-time canonical commutation relations. For this reason, we can uniquely quantize the physical degrees of freedom and we do not obtain any extra terms in the equation for the Fermi field in spite of having introduced the residual gauge fields. We have also obtained the conserved Hamiltonian by integrating the canonical energy-momentum tensor $T_{\mu\nu}$ around a suitable contour. The Hamiltonian consists of the usual terms, with the residual gauge fields in A_μ as regulators, plus the Hamiltonian for the residual gauge fields which is determined by integrating a density over the hyperplane $x^- =$ constant. We show in detail that, in perturbation theory, infrared divergences resulting from the residual gauge fields cancel infrared divergences resulting from the physical parts of the gauge field. As a result we obtain the gauge field propagator prescribed by Mandelstam and Leibbrandt. By taking the limit $\theta{\to}\frac{\pi}{4}$ we can construct the light-cone formulation which is free from infrared difficulty. With that analysis complete, we perform a successful calculation of the one loop electron self energy, something not previously done in light-cone quantization and light-cone gauge.

§1. Introduction

Axial gauges, $n^\mu A_\mu = 0$, specified by a constant vector n^μ, have been used recently in spite of their lack of manifest Lorentz covariance.[1)] One reason is that the Faddeev-Popov ghosts decouple in the axial gauge (AG) formulations.[2)] For that reason we will not need to discuss the Faddeev-Popov ghosts in this paper and the term ghost will always refer to residual gauge (RG) fields which are introduced as integration constants in solutions to constraint equations. The case $n^2 = 0$, the light-cone gauge, has been extensively used in the light-cone field theory in attempts to find nonperturbative solutions to QCD.[3)] The same formulation is often used in phenomenological calculations where the low energy properties of the theory are parameterized in light-cone wave functions. Traditionally, constraints in AG formulations are solved to eliminate dependent fields in terms of physical fields. That elimination requires one to utilize inverse derivatives which introduce so-called spurious singularities. It was first pointed out by Nakanishi[4)] that there exists an intrinsic difficulty in the AG formulations, the solution to which requires an indefinite metric; even in QED. It was also noticed that in order to bring perturbative calculations done in the light-cone gauge into agreement with calculations done in covariant gauges, spurious singularities of the free gauge field propagator have to be regularized, not as principal values, but according to the Mandelstam-Leibbrandt (ML) prescription[5)] in such a way that causality is preserved. Shortly afterwards, Bassetto et al.[6)] found that the ML form of the propagator is obtained in the light-cone gauge canonical formalism in ordinary coordinates if one introduces a Lagrange multiplier field and its conjugate as RG degrees of freedom.

It is now established[7)] that AG formulations are not ghost free, contrary to what was originally expected, and is still sometimes claimed. Nevertheless, the problem of finding the correct method by which to introduce RG fields into pure space-like AG formulations has not been completely solved. A first step toward solving this problem was made by McCartor and Robertson in the light-cone formulation of QED.[8)] In previous papers,[9)] we considered the solvable Schwinger model and learned ways to introduce and to quantize RG fields. In this paper we apply these methods to the more realistic problem of QED and construct an extended Hamiltonian formalism of QED in the $\pm$-coordinates $x^\mu = (x^+, x^-, x^1, x^2)$, where $x^+ = x^0 \sin\theta + x^3 \cos\theta$, $x^- = x^0 \cos\theta - x^3 \sin\theta$ and the space-like constant vector n is specified to be $n^\mu = (n^+, n^-, n^1, n^2) = (0, 1, 0, 0)$ so that the gauge fixing condition is pure space-like. The same framework was used by others to analyze two-dimensional models.[10)] We show that if we properly choose constituent fields, then we can construct the extended Hamiltonian formalism of QED in quite the same manner as the Schwinger model.

In spite of the common use of light-cone gauge (with A_+ treated as a constrained field) and light-cone quantization, the one loop electron self energy has never been successfully calculated in that formulation. To our knowledge, that most fundamental of loop calculations has been successfully calculated in three formulations of light-cone quantization: Morara and Soldati[7] calculated it in the gauge $A_+ = 0$; Langnau and Burkardt[11] calculated it in Feynman gauge but did not start the calculation on the light-cone; in[12] it was calculated in Feynman gauge starting from the light-cone and also calculated in light-cone gauge, but with a higher derivative regulator so that A_+ was a degree of freedom, not a constraint. But it has not been calculated in light-cone gauge with A_+ treated as a constraint. The problems that prevent such a calculation can be glimpsed in[11] and are discussed in detail in.[12] Basically, the infrared singularities are too strong to allow a successful calculation. In the present paper we shall show that with the RG fields in place, the infrared singularities are softened in such a way as to allow a successful calculation.

The paper is organized as follows. In §2, we construct the temporal gauge (TG) formulation and obtain the commutation relations of the Nakanishi-Lautrup field B, which are then extrapolated into the axial region. In §3 we consider the dipole ghost field and the constituent fields corresponding to those in the $n{\cdot}A = 0$ gauge Schwinger model. We then express A_μ in terms of these constituent fields. In § 4 we obtain the translational generators by applying McCartor's method.[13] In § 5 perturbation theory is developed and we show that if we define the singularities resulting from inversion of a hyperbolic Laplace operator as principal values, the worst infrared divergences resulting from physical parts of the gauge fields are cancelled by infrared divergences from the RG parts. We then find that the remaining infrared divergences are regularized as the ML form of gauge field propagator. In that section we also perform a calculation of the one loop electron self energy. § 6 is devoted to concluding remarks.

We use the following conventions:
Greek indices $\mu, \nu, \cdots$ will take the values $+, -, 1, 2$ and label the component of a given four-vector (or tensor) in the $\pm$ coordinates;
Latin indices $i, j, \cdots$ will take the values $1, 2$ and label the $1, 2$ component of a given four-vector (or tensor) in the $\pm$ coordinates;
the Einstein convension of sum over repeated indices will be always used;

$$\boldsymbol{x}^\pm = (x^\pm, x^1, x^2),\ \boldsymbol{x}_\perp = (x^1, x^2),\ d^2\boldsymbol{x}_\perp = dx^1dx^2,\ d^3\boldsymbol{x}^\pm = dx^1dx^2dx^\pm$$

$$\boldsymbol{k}_\pm = (k_\pm, k_1, k_2),\ d^3\boldsymbol{k}_\pm = dk_1dk_2dk_\pm$$

$$g_{--} = \cos 2\theta, \quad g_{-+} = g_{+-} = \sin 2\theta, \quad g_{++} = -\cos 2\theta$$

$$g_{-i} = g_{i-} = g_{+i} = g_{i+} = 0, \quad g_{ij} = -\delta_{ij}.$$

$$\gamma^+ = \gamma^0 \sin\theta + \gamma^3 \cos\theta, \quad \gamma^- = \gamma^0 \cos\theta - \gamma^3 \sin\theta.$$

§2. Temporal gauge formulation of $\boldsymbol{n\cdot A}=0$ gauge QED

In this section we keep θ in the temporal region $0\leqq\theta<\frac{\pi}{4}$ and choose x^- as the evolution parameter. This enables us to show that RG fields are necessary ingredients in the canonical quantization of $n\cdot A=0$ gauge QED.

The $n\cdot A=0$ gauge QED is defined by the Lagrangian

$$L=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-B(n\cdot A)+\bar{\varPsi}(i\gamma^\mu D_\mu-m)\varPsi \tag{2·1}$$

where $D_\mu=\partial_\mu+ieA_\mu$ and B is the Lagrange multiplier field, that is, the Nakanishi-Lautrup field in noncovariant formulations.[14] From (2·1) we derive the field equations

$$\partial_\mu F^{\mu\nu}=n^\nu B+J^\nu,\quad J^\nu=e\bar{\varPsi}\gamma^\nu\varPsi \tag{2·2}$$

$$(i\gamma^\mu D_\mu-m)\varPsi=0, \tag{2·3}$$

and the gauge fixing condition $n\cdot A=0$. The field equation of B,

$$(n\cdot\partial)B=\partial_- B=0, \tag{2·4}$$

is obtained by operating on (2·2) with ∂_ν.

Canonical conjugate momenta are defined to be

$$\pi^+=\frac{\delta L}{\delta\partial_- A_+}=F_{-+},\quad \pi^-=\frac{\delta L}{\delta\partial_- A_-}=0,\quad \pi^i=\frac{\delta L}{\delta\partial_- A_i}=F^-{}_i,$$
$$\pi_B=\frac{\delta L}{\delta\partial_- B}=0,\quad \pi_\varPsi=\frac{\delta L}{\delta\partial_-\varPsi}=i\bar{\varPsi}\gamma^- \tag{2·5}$$

and the Gauß law constraint is described in terms of them as

$$\partial_+\pi^+ +\partial_i\pi^i=B+J^-. \tag{2·6}$$

We see from (2·5) and (2·6) that in the TG formulation we have three pairs of bosonic canonical variables, $A_+,\pi^+,\ A_i,\pi^i$ $(i=1,2)$, which indicates indispensability of degrees of freedom other than the physical ones in constructing the canonical formulation. By using the canonical equal x^--time quantization conditions imposed on the independent canonical variables and the expression of B obtained from (2·6) we easily obtain the commutation relations of B:

$$[B(x),A_\mu(y)]=i\partial_\mu\delta^{(3)}(\boldsymbol{x}^+-\boldsymbol{y}^+),\quad [B(x),\pi^+(y)]=[B(x),\pi^i(y)]=0,$$
$$[B(x),B(y)]=0,\quad [B(x),\varPsi(y)]=e\delta^{(3)}(\boldsymbol{x}^+-\boldsymbol{y}^+)\varPsi(y). \tag{2·7}$$

It is important to note here that, due to (2·4), these commutation relations hold even when $x^- \neq y^-$.

The Hamiltonian, that is, the translational generator for the x^--direction, is given by

$$P_- = \int d^3\boldsymbol{x}^+ T_-{}^- = \int d^3\boldsymbol{x}^+ \{2^{-1}((\pi^+)^2 + n_-^{-1}(\pi^i - n_+ F_{+i})^2 + n_-(F_{+i})^2 + (F_{12})^2) + \bar{\Psi}(m - i\gamma^+\partial_+ - \gamma^i\partial_i)\Psi + J^\mu A_\mu\} \tag{2·8}$$

so that it is straightforward to develop an x^--time ordered perturbation theory. It was shown previously that the free gauge field is described as[15)]

$$A_\mu = T_\mu - \frac{n_\mu}{\nabla_T{}^2} B + \partial_\mu \Lambda \tag{2·9}$$

where T_μ is a free field satisfying

$$\Box T_\mu = 0, \quad \partial^\mu T_\mu = 0, \quad n{\cdot}T = 0, \tag{2·10}$$

$$[T_\mu(x), T_\nu(y)] = i(-g_{\mu\nu} + \frac{n_\mu\partial_\nu + n_\nu\partial_\mu}{\partial_-} - n^2 \frac{\partial_\mu\partial_\nu}{\partial_-^2}) D(x-y) \tag{2·11}$$

and

$$\nabla_T{}^2 = \partial_1^2 + \partial_2^2 + n_-\partial_+^2, \tag{2·12}$$

$$\Lambda = -\frac{1}{\nabla_T{}^2}\left(C - n_- x^- B - \frac{n_- n_+}{\nabla_T{}^2}\partial_+ B\right). \tag{2·13}$$

Note in (2·11) that $D(x-y)$ is the commutation function of the free massless field and that the inverse derivative ∂_-^{-1} is defined by

$$\partial_-^{-1} f(x) = \frac{1}{2}\int_{-\infty}^{\infty} dy^- \varepsilon(x^- - y^-) f(x^+, y^-, \boldsymbol{x}_\perp) \tag{2·14}$$

which imposes, in effect, the principal value regularization. Note furthermore that C in (2·13) is the conjugate of B and satisfies

$$\partial_- C = 0, \quad [C(x), C(y)] = 0, \quad [B(x), C(y)] = -i\nabla_T{}^2 \delta^{(3)}(\boldsymbol{x}^+ - \boldsymbol{y}^+). \tag{2·15}$$

It is useful to point out here that the physical degrees of freedom are carried by T_μ, as is seen from (2·10), and the remaining degrees of freedom are carried by the pair of functions B and C. It should be noted in particular that solving the on-mass-shell condition

$$0 = p^2 = n_- p_-^2 + 2n_+ n_- p_+ p_- - n_- p_+^2 - p_\perp^2, \quad (p_\perp^2 = p_1^2 + p_2^2) \tag{2·16}$$

yields

$$p_- = \frac{\sqrt{p_+^2 + n_- p_\perp^2} - n_+ p_+}{n_-}, \quad p^- = \sqrt{p_+^2 + n_- p_\perp^2}, \quad p^+ = \frac{n_+\sqrt{p_+^2 + n_- p_\perp^2} - p_+}{n_-} \tag{2·17}$$

from which follows

$$\frac{1}{p_-} = \frac{n_+p_+ + \sqrt{p_+^2 + n_-p_\perp^2}}{p_\perp^2 + n_-p_+^2}. \tag{2·18}$$

Therefore we obtain the factors $(p_\perp^2 + n_-p_+^2)^{-1}$ from the physical operators. It turns out that they are canceled by those resulting from $\frac{1}{\nabla_T{}^2}$, which is appled to B and C so that we obtain the following ML form of the x^--ordered gauge field propagator

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle = \frac{1}{(2\pi)^4}\int d^4k D_{\mu\nu}(k)\mathrm{e}^{-ik\cdot(x-y)} \tag{2·19}$$

where

$$D_{\mu\nu}(k) = \frac{i}{k^2+i\varepsilon}\left\{-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{k_- + i\varepsilon\mathrm{sgn}(k_+)} - n^2\frac{k_\mu k_\nu}{(k_- + i\varepsilon\mathrm{sgn}(k_+))^2}\right\}. \tag{2·20}$$

We see from (2·20) that RG fields are indispensable to regularize singularities associated with the gauge fixing in such a way that causality is preserved in complex k_- coordinates. Because the factors $(p_\perp^2 + n_-p_+^2)^{-1}$, which becomes singular in the axial region due to $n_- < 0$, drop out completely from the gauge field propagator, we can formally extrapolate the gauge field propagator into the axial region. It is expected from this that we can develop a perturbation theory free from infrared divergences in the AG formulation. In next section we consider introducing RG fields so as to regularize the infrared divergences in the AG formulation.

§3. Constituent operators in the axial gauge formulation

3.1. *general solution of the gauge field equation (2·2)*

We begin by obtaining the general solution of (2·2) in a way which is independent of the evolution parameter. Let $a_\mu(x)$ be a field satisfying

$$\Box a_\mu = J_\mu. \tag{3·1}$$

Then it holds that

$$A_\mu - \frac{\partial_\mu}{\nabla_\perp{}^2}\partial_i A_i + \frac{n_\mu}{\nabla_T{}^2}B = a_\mu - \frac{\partial_\mu}{\nabla_\perp{}^2}\partial_i a_i, \quad (\nabla_\perp{}^2 = \partial_1^2 + \partial_2^2) \tag{3·2}$$

which is verified as follows: If we apply D'Alembert's operator, which is described in our formulation as

$$\Box = n_-\partial_-^2 + 2n_+\partial_+\partial_- - n_-\partial_+^2 - \nabla_\perp{}^2,$$

to both sides of (3·2), then we obtain the same result, $J_\mu - \frac{\partial_\mu}{\nabla_\perp{}^2}\partial_i J_i$. Therefore the difference between the left hand side and the right hand side is a solution of the free massless

D'Alembert's equation. The solutions which tend to zero at spatial infinity ($|\boldsymbol{x}_\perp|\to\infty$) vanish identically .

In addition we can show that a_μ satisfies

$$\partial^\mu a_\mu = 0. \tag{3·3}$$

In fact operating on (3·2) with ∂^μ yields

$$\begin{aligned} \partial^\mu A_\mu - \frac{\partial^\mu\partial_\mu}{\nabla_\perp{}^2}\partial_i A_i &= \partial^\mu a_\mu - \frac{1}{\nabla_\perp{}^2}\partial_i J_i \\ \therefore \quad \partial_i(\Box A_i - \partial_i\partial^\mu A_\mu - J_i) &= \partial_i(\partial^\mu F_{\mu i} - J_i) = -\nabla_\perp{}^2\partial^\mu a_\mu. \end{aligned}$$

The left hand side of the second line vanishes due to the spatial component of the gauge field equation and so does the right hand side.

¿From (3·2) A_μ is described as

$$A_\mu = a_\mu - \frac{\partial_\mu}{\nabla_\perp{}^2}\partial_i a_i + \frac{\partial_\mu}{\sqrt{-\nabla_\perp{}^2}}X - \frac{n_\mu}{\nabla_T{}^2}B \tag{3·4}$$

where we have denoted $-\frac{1}{\sqrt{-\nabla_\perp{}^2}}\partial_i A_i$ as X. Throughout this paper the operator $\sqrt{-\nabla_\perp{}^2}$ is understood as $|\boldsymbol{k}_\perp|$ in the Fourier transforms of the relevant operators. The minus component of (3·4) has to vanish identically due to the gauge fixing condition. Imposing $A_- = 0$ gives rise to the following constraint

$$\frac{\partial_-}{\sqrt{-\nabla_\perp{}^2}}X = -\{a_- - \frac{\partial_-}{\nabla_\perp{}^2}\partial_i a_i\} + \frac{n_-}{\nabla_T{}^2}B \tag{3·5}$$

the first term of which is rewritten, due to $\partial_i a_i = \partial^+ a_+ + \partial^- a_-$, as

$$\begin{aligned} a_- - \frac{\partial_-}{\nabla_\perp{}^2}\partial_i a_i &= -\frac{1}{\nabla_\perp{}^2}\{\Box a_+ - \partial^+(\partial_+ a_- - \partial_- a_+)\} \\ &= -\frac{1}{\nabla_\perp{}^2}\{J_- - \partial^+(\partial_+ a_- - \partial_- a_+)\}. \end{aligned} \tag{3·6}$$

Thus integrating (3·5) with respect to x^- yields X expressed as

$$\frac{1}{\sqrt{-\nabla_\perp{}^2}}X = -\frac{\partial_-^{-1}}{\nabla_\perp{}^2}\{\partial^+(\partial_+ a_- - \partial_- a_+) - J_-\} + \Lambda \tag{3·7}$$

where we have introduced integration constants involving C in the same manner as in (2·13). Substituting (3·7) into X in (3·4), we have A_μ expressed as

$$A_\mu = a_\mu - \frac{\partial_\mu}{\partial_-}a_- - \frac{n_\mu}{\nabla_T{}^2}B + \partial_\mu\Lambda. \tag{3·8}$$

It is straightforward to show that, because $\partial_- B = \partial_- C = 0$, X and A_μ satisfy the following equations:

$$\frac{\partial^\mu \partial_\mu}{\sqrt{-\nabla_\perp^2}} \partial_- X = -n_- B + \frac{\partial_-}{\nabla_\perp^2} \partial_i J_i - J_-, \tag{3·9}$$

$$\partial^\mu A_\mu = C - n_- x^- B + \frac{n_- n_+}{\nabla_T^2} \partial_+ B - (\partial_-)^{-1} J_-. \tag{3·10}$$

We see from (3·9) that X is nothing but the dipole ghost field.

3.2. *Commutation relations of the constituent operators*

In the axial region, where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ and x^+ is chosen as the evolution parameter, we obtain from (2·1) the canonical conjugate momenta

$$\pi^+ = \frac{\delta L}{\delta \partial_+ A_+} = 0, \quad \pi^- = \frac{\delta L}{\delta \partial_+ A_-} = F_{+-}, \quad \pi^i = \frac{\delta L}{\delta \partial_+ A_i} = F^+_{\ i},$$
$$\pi_B = \frac{\delta L}{\delta \partial_+ B} = 0, \quad \pi_\Psi = \frac{\delta L}{\delta \partial_+ \Psi} = i\bar{\Psi}\gamma^+. \tag{3·11}$$

If we impose the gauge fixing condition, $A_- = 0$, on the equation for π^-, then we obtain a constraint

$$\pi^- = -\partial_- A_+ \tag{3·12}$$

in addition to the Gauß law constraint

$$\partial_- \pi^- + \partial_i \pi^i = J^+. \tag{3·13}$$

As a consequence, A_+ becomes a dependent variable so that there remain only A_i, π^i ($i = 1, 2$) as independent bosonic canonical variables. Therefore, as quantization conditions we have only equal x^+-time commutation and/or anticommutation relations on the independent canonical variables; the nonvanishing ones are

$$[A_i(x), \pi^j(y)] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-), \quad \{\Psi(x), \bar{\Psi}(y)\gamma^+\} = \delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-). \tag{3·14}$$

The degrees of freedom of the system should not change when we move from the TG formulation to the AG one, because the field equations and the gauge fixing condition are the same. How can we resolve this paradox? The only answer is that B and C take the place of the canonical variables A_+, π^+ in the TG formulation but happen not to be canonical variables in the AG formulation. This reflects the fact that B and C are introduced as x^- independent static fields so that x^+ can not be the evolution parameter for them. Therefore, we cannot obtain their quantization conditions from the Dirac procedure.[16] To supplement the insufficient quantization conditions, we assume that the commutation relations of B given in (2·7) can be extrapolated into the axial region. This is a reasonable assumption, because we have introduced degrees of freedom for c-number residual gauge transformations and because B generates them. However it is not straightforward to make the extrapolation because the Laplace operator ${\nabla_T}^2$ becomes hyperbolic so that its inverse gives rise to singularities. Note that $n_- = \cos 2\theta < 0$ in the axial region. We regularize these singularities by the principal value method. Consequently we obtain vanishing contributions at equal x^+-time due to $({\nabla_T}^2)^{-1}$. For example, we obtain

$$\begin{aligned}
&[\frac{1}{{\nabla_T}^2}B(x), \frac{1}{\sqrt{-{\nabla_\perp}^2}}X(y)]_{x^+=y^+} = \frac{-i}{{\nabla_T}^2}\delta^{(3)}(\boldsymbol{x}^+ - \boldsymbol{y}^+)|_{x^+=y^+} \\
&= \frac{i}{(2\pi)^3}\int d^3\boldsymbol{k}_+ \frac{1}{|\boldsymbol{k}_\perp|^2 + n_- k_+^2} \mathrm{e}^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} = 0,
\end{aligned} \tag{3·15}$$

$$\begin{aligned}
&[\frac{1}{{\nabla_T}^2}B(x), \Psi(y)]_{x^+=y^+} = \frac{e}{{\nabla_T}^2}\delta^{(3)}(\boldsymbol{x}^+ - \boldsymbol{y}^+)\Psi(y)|_{x^+=y^+} \\
&= \frac{-e}{(2\pi)^3}\int d^3\boldsymbol{k}_+ \frac{1}{|\boldsymbol{k}_\perp|^2 + n_- k_+^2} \mathrm{e}^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)}\Psi(y) = 0
\end{aligned} \tag{3·16}$$

where we have made use of the fact that

$$\int_{-\infty}^{\infty} dk_+ \frac{1}{|\boldsymbol{k}_\perp|^2 + n_- k_+^2} = 0 \tag{3·17}$$

as a result of regularizing the singular integral by the principal value method. In this way B makes vanishing contributions to equal x^+-time commutation relations.

To express A_+ in terms of the independent canonical variables, we integrate (3·12) and (3·13) with respect to x^-. By integrating (3·13) we obtain

$$\pi^- = -\partial_- A_+ = \frac{1}{\partial_-}(J^+ - \partial_i \pi^i) + \text{constant}. \tag{3·18}$$

The integration constant is determined by comparing (3·18) with that given by A_+ in (3·8) and we get

$$\partial_+ a_- - \partial_- a_+ = \frac{1}{\partial_-}(J^+ - \partial_i \pi^i), \tag{3·19}$$

$$\text{constant} = -\frac{n_-}{{\nabla_T}^2}\partial_+ B. \tag{3·20}$$

Integrating (3·18) with respect to x^- and then comparing the result with A_+ in (3·8), we obtain

$$A_+ = -\frac{1}{\partial_-^2}(J^+ - \partial_i \pi^i) - \frac{n_+}{{\nabla_T}^2}B + \partial_+ \Lambda. \tag{3·21}$$

We see from (3·21) that A_+ consists of the conventional physical operator plus RG operators. Furthermore, we see from (3·7) and (3·19) that $\partial_+ a_- - \partial_- a_+$ plays fundamental roles in the axial gauge formulation. If we define $\tilde{\Sigma}$ by

$$\partial_+ a_- - \partial_- a_+ = \frac{1}{\partial_-}(J^+ - \partial_i \pi^i) = \sqrt{-\nabla_\perp{}^2}\tilde{\Sigma}, \tag{3·22}$$

then $\partial^+\tilde{\Sigma}$ is expressed as

$$\partial^+\tilde{\Sigma} = -\frac{1}{\sqrt{-\nabla_\perp{}^2}}\partial_-\partial_i A_i - \frac{n_-\sqrt{-\nabla_\perp{}^2}}{\nabla_T{}^2}B + \frac{1}{\sqrt{-\nabla_\perp{}^2}}J_-, \tag{3·23}$$

so that at $x^+ = y^+$ we obtain

$$[\tilde{\Sigma}(x), \partial^+\tilde{\Sigma}(y)] = i\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-), \quad [\tilde{\Sigma}(x), \tilde{\Sigma}(y)] = [\partial^+\tilde{\Sigma}(x), \partial^+\tilde{\Sigma}(y)] = 0. \tag{3·24}$$

Here we have assumed the equal x^+-time current commutation relations:

$$[J^+(x), J^+(y)] = [J^+, J_-(y)] = [J_-(x), J_-(y)] = 0. \tag{3·25}$$

In addition, both $\tilde{\Sigma}$ and $\partial^+\tilde{\Sigma}$ are gauge invariant so that due to (2·7) their four dimensional commutator with B vanishes:

$$[\tilde{\Sigma}(x), B(y)] = 0, \quad [\partial^+\tilde{\Sigma}(x), B(y)] = 0. \tag{3·26}$$

Note in particular that, when we calculate the commutation relations of $\partial^+\tilde{\Sigma}$ with X and Ψ, we do not obtain any $\delta^{(3)}(\boldsymbol{x}^+ - \boldsymbol{y}^+)$ contributions due to (3·15) and (3·16). Consequently we obtain the following equal x^+-time commutation relations:

$$[\tilde{\Sigma}(x), X(y)] = -\frac{i}{\partial_-}\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-), \quad [\partial^+\tilde{\Sigma}(x), X(y)] = 0, \tag{3·27}$$

$$[\tilde{\Sigma}(x), \pi^i(y)] = 0, \quad [\partial^+\tilde{\Sigma}(x), \pi^i(y)] = -\frac{i}{\sqrt{-\nabla_\perp{}^2}}\partial_-\partial_i\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-), \tag{3·28}$$

$$[\tilde{\Sigma}(x), \Psi(y)] = -e\frac{(\partial_-)^{-1}}{\sqrt{-\nabla_\perp{}^2}}\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-)\Psi(y), \quad [\tilde{\Sigma}(x), J_-(y)] = 0, \tag{3·29}$$

$$[\partial^+\tilde{\Sigma}(x), \Psi(y)] = \frac{en_-^{-1}}{\sqrt{-\nabla_\perp{}^2}}\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-)\gamma^+\gamma_-\Psi(y), \quad [\partial^+\tilde{\Sigma}(x), J_-(y)] = 0, \tag{3·30}$$

$$[\tilde{\Sigma}(x), J^+(y)] = [\partial^+\tilde{\Sigma}(x), J^+(y)] = 0. \tag{3·31}$$

Now we can calculate the commutation relations of C, which is introduced as the integration constant in (3·7). We rewrite (3·7) to express C in terms of operators whose commutation relations are known; we get

$$\frac{\sqrt{-\nabla_\perp{}^2}}{\nabla_T{}^2}C = \frac{\partial^+}{\partial_-}\tilde{\Sigma} - \frac{\partial_-^{-1}}{\sqrt{-\nabla_\perp{}^2}}J_- - X + \frac{\sqrt{-\nabla_\perp{}^2}}{\nabla_T{}^2}\left(n_- x^- B + \frac{n_- n_+}{\nabla_T{}^2}\partial_+ B\right). \tag{3·32}$$

Then using (3·15),(3·16) and (3·24) $\sim$ (3·30) we obtain at $x^+ = y^+$

$$[\frac{1}{\nabla_T^2}C(x), \tilde{\Sigma}(y)] = [\frac{1}{\nabla_T^2}C(x), \partial^+\tilde{\Sigma}(y)] = [\frac{1}{\nabla_T^2}C(x), \Psi(y)] = 0, \tag{3·33}$$

$$[\frac{1}{\nabla_T^2}C(x), X(y)] = [\frac{1}{\nabla_T^2}C(x), J^+(y)] = [\frac{1}{\nabla_T^2}C(x), J_-(y)] = 0. \tag{3·34}$$

If we use (2·7) and (3·26), then we obtain the following four dimensional commutation relation

$$[\frac{1}{\nabla_T^2}C(x), B(y)] = [\frac{-1}{\sqrt{-\nabla_\perp^2}}X(x), B(y)] = i\delta^{(3)}(\boldsymbol{x}^+ - \boldsymbol{y}^+). \tag{3·35}$$

It follows from (3·33) $\sim$ (3·35) that

$$[\frac{1}{\nabla_T^2}C(x), \frac{1}{\nabla_T^2}C(y)]|_{x^+=y^+} = 0. \tag{3·36}$$

It is clear that $A_i^\perp \equiv A_i - \frac{\partial_i\partial_j}{\nabla_\perp^2}A_j$ and $\pi_\perp^i \equiv \pi^i - \frac{\partial_i\partial_j}{\nabla_\perp^2}\pi^j$ are gauge invariant and thus commute with B and C at equal x^+-time. Therefore B and C do not give rise to any unwanted contributions to the equal x^+-time canonical commutation relations. Now that we have $A_i^\perp, \pi_\perp^i, \tilde{\Sigma}, \partial^+\tilde{\Sigma}, B$ and C as fundamental operators, we can reconstruct our formulation in terms of them. To do this, we divide A_μ into physical and RG parts as follows

$$A_\mu = T_\mu - \frac{n_\mu}{\nabla_T^2}B + \partial_\mu \Lambda \tag{3·37}$$

where

$$T_+ = -\frac{\sqrt{-\nabla_\perp^2}}{\partial_-}\tilde{\Sigma} = \frac{1}{\partial_-^2}(\partial_i\pi^i - J^+), \tag{3·38}$$

$$T_- = 0, \quad T_i = A_i^\perp + \frac{1}{\sqrt{-\nabla_\perp^2}}\frac{\partial_i}{\partial_-}\left(\partial^+\tilde{\Sigma} - \frac{1}{\sqrt{-\nabla_\perp^2}}J_-\right), \tag{3·39}$$

$$\pi^i = \partial^+ A_i^\perp + \frac{\partial_i}{\sqrt{-\nabla_T^2}}\left(\partial_-\tilde{\Sigma} - \frac{1}{\sqrt{-\nabla_T^2}}J^+\right). \tag{3·40}$$

It is easily seen that T_μ satisfies

$$\partial^\mu f_{\mu\nu} = J_\nu, \quad (f_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu), \tag{3·41}$$

$$\partial^\mu T_\mu = -\frac{1}{\partial_-}J_-, \tag{3·42}$$

and that T_i and π^i satisfy the equal x^+-time canonical quantization conditions

$$[T_i(x), T_j(y)] = [\pi^i(x), \pi^j(y)] = 0, \quad [T_i(x), \pi^j(y)] = i\delta_{ij}\delta^{(3)}(\boldsymbol{x}^- - \boldsymbol{y}^-),$$
$$[T_i(x), \Psi(y)] = [\pi^i(x), \Psi(y)] = 0. \tag{3·43}$$

§4. Translational generators in the axial gauge formulation

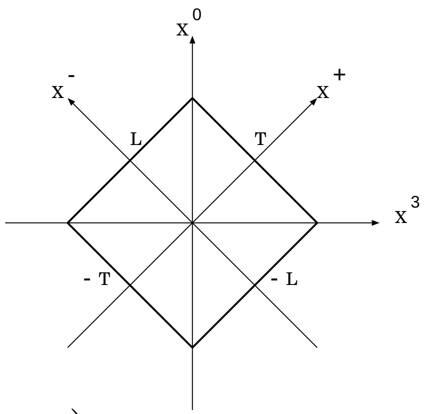
In this section we start with the canonical energy-momentum tensor

$$T^{\mu\nu} = -F^{\nu\sigma}\partial^\mu A_\sigma + \frac{g^{\mu\nu}}{4}F^{\rho\sigma}F_{\rho\sigma} + i\bar{\Psi}\gamma^\nu\partial^\mu\Psi \tag{4·1}$$

and obtain the conserved translation generators. Note that in our formulation we have to resort to a nonstandard way of deriving the conserved generators. This is because A_μ in (3·37) has the term depending explicitly on x^- as well as x^--independent terms. As a result, the traditional formula for the Hamiltonian contains divergences. This reflects the fact that the integral $\int_{-\infty}^{\infty}\partial_- T_\mu{}^- dx^-$ does not vanish, although x^- is one of space coordinates. From the divergence equation, $\partial_\nu T_\mu{}^\nu = 0$, we obtain

$$\oint T_\mu{}^\nu d\sigma_\nu = 0\,, \tag{4·2}$$

where the integral is taken over a closed surface. From this we shall derive the conserved generators. For the transverse directions we are justified in assuming that the integral $\int_{-\infty}^{\infty}\partial_i T_\mu{}^i dx^i$ vanishes. (Here, repeated indices do not imply sum over i.) Therefore, as the closed surface we use the one shown in Fig. 1, whose bounds T and L are taken to ∞ after the calculations are finished. A similar procedure was first used by one of the authors of the present paper.[13] It is straightforward to obtain



$$0 = \int d^2\boldsymbol{x}_\perp \left(\int_{-L}^{L} dx^- \left[T_\mu{}^+(x)\right]_{x^+=-T}^{x^+=T} + \int_{-T}^{T} dx^+ \left[T_\mu{}^-(x)\right]_{x^-=-L}^{x^-=L} \right) \tag{4·3}$$

where

$$\left[T_\mu{}^+(x)\right]_{x^+=-T}^{x^+=T} = T_\mu{}^+(x)|_{x^+=T} - T_\mu{}^+(x)|_{x^+=-T},$$
$$\left[T_\mu{}^-(x)\right]_{x^-=-L}^{x^-=L} = T_\mu{}^-(x)|_{x^-=L} - T_\mu{}^-(x)|_{x^-=-L}. \tag{4·4}$$

Let us illustrate the case of $\mu = +$ in detail. Because A_μ has the RG term, we divide $T_+{}^+$ and $T_+{}^-$ into terms containing only physical operators, terms containing products of physical operators and RG operators and terms consisting solely of RG operators. Then, we rewrite these terms, which are to be integrated by parts, in the form of derivatives minus terms resulting from the integration by parts. As a result we obtain

$$T_+{}^+ = F^+_{\ i}\partial_+ A_i - \frac{1}{2}\{(F_{-+})^2 + F^+_{\ i}F_{+i} + F^-_{\ i}F_{-i} - (F_{12})^2\} + i\bar{\Psi}\gamma^+\partial_+\Psi$$

$$
\begin{aligned}
&= \frac{1}{2}\{(f_{-+})^2 + f_{+i}f^{+}_{\ i} - f_{-i}f^{-}_{\ i} + (f_{12})^2\} + i\bar{\Psi}\gamma^+ D_+\Psi - \partial_i f^+_i \partial_+ \Lambda \\
&+ J^+(-\frac{n_+}{\nabla_T^2}B + \partial_+\Lambda) - \partial_-\left(f_{-+}T_+ + T_+\frac{n_-}{\nabla_T^2}\partial_+ B + T_i\frac{1}{\nabla_T^2}\partial_i B\right) \\
&+ \partial_i\left(f^+_i(T_+ + \partial_+\Lambda)\right) - \frac{1}{2}\left(\frac{n_-}{\nabla_T^2}\partial_+ B\frac{n_-}{\nabla_T^2}\partial_+ B + \frac{n_-}{\nabla_T^2}\partial_i B\frac{1}{\nabla_T^2}\partial_i B\right),
\end{aligned}
\tag{4·5}
$$

$$
\begin{aligned}
&T_+{}^- = F_{-+}\partial_+ A_+ + F^{-}_{\ i}\partial_+ A_i + i\bar{\Psi}\gamma^-\partial_+\Psi \\
&= f^{-}_{\ i}f_{+i} + i\bar{\Psi}\gamma^- D_+\Psi + \frac{1}{\partial_-}J^-\frac{\partial_+}{\nabla_T^2}B - J^-\frac{n_+}{\nabla_T^2}B + B\frac{1}{\nabla_T^2}\partial_+ C \\
&+ \partial_i\left(f^{-}_{\ i}(T_+ + \partial_+\Lambda) + \partial_+ T_i\frac{1}{\nabla_T^2}B + \frac{1}{\nabla_T^2}\partial_i B\frac{n_- n_+}{(\nabla_T^2)^2}\partial_i\partial_+^2 B - \frac{1}{\nabla_T^2}\partial_i B\frac{1}{\nabla_T^2}\partial_+ C\right) \\
&+ \partial_+\left(f_{-+}(T_+ + \partial_+\Lambda) - \partial_i T_i\frac{1}{\nabla_T^2}B + \frac{n_-}{\nabla_T^2}\partial_+ B\frac{n_- n_+}{(\nabla_T^2)^2}\partial_+^3 B - \frac{1}{\nabla_T^2}B\frac{n_- n_+}{\nabla_T^2}\partial_+ B\right) \\
&- \partial_+\left(\frac{n_-}{\nabla_T^2}\partial_+ B\frac{1}{\nabla_T^2}\partial_+ C\right) + \frac{x^-}{2}\partial_+\left(\frac{n_-}{\nabla_T^2}\partial_+ B\frac{n_-}{\nabla_T^2}\partial_+ B + \frac{n_-}{\nabla_T^2}\partial_i B\frac{1}{\nabla_T^2}\partial_i B\right).
\end{aligned}
\tag{4·6}
$$

When we substitute (4·5) and (4·6) into (4·3) with $\mu = +$, the derivative terms with respect to x^i give vanishing contributions, whereas the last x^- independent term of (4·5) is cancelled out by the last linear term of x^- in (4·6). We also see that B and C contained in the derivative terms with respect to x^+ in the fourth and fifth lines of (4·6) give vanishing contributions in the limit $T\to\infty$ because x^+ is one of spatial coordinates for B and C. As to $B\frac{1}{\nabla_T^2}\partial_+ C$ in the second line, the contribution from the upper bound L cancels out that from the lower bound $-L$. However we keep it, because it is the Hamiltonian density for the RG fields. We see furthermore, that the derivative term with respect to x^- in the third line of (4·5) gives rise to a vanishing contribution, partly because its first term is cancelled out by the corresponding one in the derivative term with respect to x^+ in the fourth line of (4·6) and partly because the physical T_+ and T_i vanish in the limit $L\to\infty$.

With the redundant terms eliminated, we turn to consideration of the regularization term. The first term in the third line of (4·5) is the relevant one. We see that for this to work as the regularization term we have to get rid of $-\partial_i f^+_i\partial_+\Lambda$, the remainder of the term $F^{+}_{\ i}\partial_+ A_i$, in the second line of (4·5). This is because if we combine this with $J^+\partial_+\Lambda$, then we get

$$
(J^+ - \partial_i f^+_i)\partial_+\Lambda = -\partial_- f_{-+}\partial_+\Lambda = \partial_-(T_+\partial_-\partial_+\Lambda - f_{-+}\partial_+\Lambda) \tag{4·7}
$$

so that all the derivative terms with respect to x^- in (4·5) are cancelled out by corresponding derivative terms with respect to x^+ in (4·6). This results in losing the necessary regularizing term. We remark that Robertson and McCartor[8)] overlooked this fact so that they did not obtain the regularization term in the light front formulation of QED. We have to discard the unnecessary term in such a way that the divergence equation is preserved. It turns out that if we supplement ${T_+}^-$ with

$$\partial_+ \left(\frac{1}{\partial_-^2} J^+ \partial_+ \partial_- \Lambda - \frac{1}{\partial_-} J^+ \partial_+ \Lambda \right) \tag{4·8}$$

and if we denote ${T_+}^+$ with $-\partial_i f_i^+ \partial_+ \Lambda$ subtracted as ${\theta_+}^+$, and ${T_+}^-$ with (4·8) added as ${\theta_+}^-$, then we obtain the following divergence equation:

$$\partial_+ {\theta_+}^+ + \partial_- {\theta_+}^- = \partial_j (f_{+i} f_{ji}) - \partial_i (f_{-+} f^-{}_i) - i\partial_i (\bar{\Psi} \gamma^i \partial_+ \Psi) - \partial_i (J_i (T_+ + \partial_+ \Lambda)) \tag{4·9}$$

where

$$\begin{aligned} {\theta_+}^+ &= \frac{1}{2} \{ (f_{-+})^2 + f_{+i} f^+{}_i - f_{-i} f^-{}_i + (f_{12})^2 \} \\ &\quad + \bar{\Psi} (m - i\gamma^- \partial_- - i\gamma^i D_i) \Psi + J^+ (-\frac{n_+}{\nabla_T^2} B + \partial_+ \Lambda), \\ {\theta_+}^- &= f^-{}_i f_{+i} - n_-^{-1} \bar{\Psi} \gamma^- \gamma^+ (m - i\gamma^- \partial_- - i\gamma^i D_i) \Psi \\ &\quad + \frac{1}{\partial_-} J_- \frac{\partial_+}{\nabla_T^2} B - J^- \frac{n_+}{\nabla_T^2} B + \partial_+ \left(\frac{1}{\partial_-^2} J^+ \partial_+ \partial_- \Lambda - \frac{1}{\partial_-} J^+ \partial_+ \Lambda \right). \end{aligned} \tag{4·10}$$

Eq.(4·9) indicates that it is enough to consider only ${\theta_+}^+$ and ${\theta_+}^-$.

Now taking the limits $L \to \infty$ and $T \to \infty$ in (4·3) gives us a conservation law which contains RG fields:

$$0 = \int d^3\boldsymbol{x}^- \left[{\theta_+}^+ \right]_{x^+=-\infty}^{x^+=\infty} + \int d^3\boldsymbol{x}^+ \left[{\theta_+}^- + B \frac{1}{\nabla_T^2} \partial_+ C \right]_{x^-=-\infty}^{x^-=\infty}. \tag{4·11}$$

The derivative term in (4·8) gives a vanishing contribution to (4·11) and any other terms of ${\theta_+}^-$ are assumed to vanish in the limits $x^- \to \pm\infty$ in accordance with the traditions of the AG theories. Then we obtain that the first term is conserved by itself. We can add to it the constant Hamiltonian for the RG fields and thus

we have

$$P_+ = \int d^3\boldsymbol{x}^- {\theta_+}^+(x) + \int d^3\boldsymbol{x}^+ B(x) \frac{1}{\nabla_T^2} \partial_+ C(x). \tag{4·12}$$

We can similarly derive conservation laws for the $-$ and transverse directions as follows:

$$0 = \int d^3\boldsymbol{x}^- \left[{\theta_-}^+ \right]_{x^+=-\infty}^{x^+=\infty} + \int d^3\boldsymbol{x}^+ \left[{\theta_-}^- - \frac{1}{2} B \frac{n_-}{\nabla_T^2} B \right]_{x^-=-\infty}^{x^-=\infty}, \tag{4·13}$$

$$0 = \int d^3\boldsymbol{x}^- \left[{\theta_i}^+ \right]_{x^+=-\infty}^{x^+=\infty} + \int d^3\boldsymbol{x}^+ \left[{\theta_i}^- + B \frac{1}{\nabla_T^2} \partial_i C \right]_{x^-=-\infty}^{x^-=\infty} \tag{4·14}$$

where

$${\theta_-}^+ = f^+{}_i f_{-i} + i\bar{\Psi} \gamma^+ \partial_- \Psi, \tag{4·15}$$

$$\theta_-{}^- = \frac{1}{2}\{(f_{-+})^2 - f^+{}_i f_{+i} + f^-{}_i f_{-i} + (f_{12})^2\} + i\bar{\Psi}\gamma^-\partial_-\Psi$$
$$- J^- \frac{n_-}{\nabla_T^2} B - \partial_+ (\frac{1}{\partial_-} J^+ \frac{n_-}{\nabla_T^2} B), \tag{4·16}$$
$$\theta_i{}^+ = f^+{}_j \partial_i T_j + i\bar{\Psi}\gamma^+\partial_i\Psi, \tag{4·17}$$
$$\theta_i{}^- = f_{-+}\partial_i + f^-{}_j\partial_i T_j + i\bar{\Psi}\gamma^-\partial_i\Psi$$
$$- \frac{\partial_i}{\partial_-} J_- \frac{1}{\nabla_T^2} B - J^-\partial_i\Lambda + \partial_+ \left(\frac{1}{\partial_-^2} J^+ \partial_i \partial_- \Lambda - \frac{1}{\partial_-} J^+ \partial_i \Lambda \right). \tag{4·18}$$

From (4·13) and (4·14) we obtain the generators of translations for the $-$ and transverse directions:

$$P_- = \int d^3\boldsymbol{x}^- \theta_-{}^+(x) - \frac{1}{2}\int d^3\boldsymbol{x}^+ B(x) \frac{n_-}{\nabla_T^2} B(x), \tag{4·19}$$
$$P_i = \int d^3\boldsymbol{x}^- \theta_i{}^+(x) + \int d^3\boldsymbol{x}^+ B(x) \frac{1}{\nabla_T^2} \partial_i C(x). \tag{4·20}$$

We end this section by showing that the Heisenberg equations for the constituent fields hold. For that purpose we use the expressions

$$f_{-+} = \frac{1}{\partial_-}(\partial_i \pi^i - J^+), \quad f^+_i = \pi^i, \quad f_{+i} = \frac{\pi^i - n_+ f_{-i}}{(-n_-)}, \quad f^-_i = \frac{n_+\pi^i - f_{-i}}{(-n_-)}, \tag{4·21}$$

to express the Hamiltonian in terms of T_i and π^i as follows

$$P_+ = \int d^3\boldsymbol{x}^- \theta_+{}^+(x) + \int d^3\boldsymbol{x}^+ B(x) \frac{1}{\nabla_T^2} \partial_+ C(x)$$
$$= \frac{1}{2}\int d^3\boldsymbol{x}^- \{(\partial_-^{-1}(\partial_i\pi^i - J^+))^2 + (-n_-)^{-1}(\pi^i - n_+ f_{-i})^2 + (-n_-)(f_{-i})^2 + (f_{12})^2\}$$
$$+ \int d^3\boldsymbol{x}^- \{\bar{\Psi}(m - i\gamma^-\partial_- - i\gamma^i D_i)\Psi + J^+(\partial_+\Lambda - \frac{n_+}{\nabla_T^2} B)\} + \int d^3\boldsymbol{x}^+ B \frac{\partial_+}{\nabla_T^2} C. \tag{4·22}$$

Note here that we can use the equal x^+-time commutation relations to calculate the commutation relations with the operators contained in $\theta_+{}^+(x)$, while B and C are regarded as independent of other constituent operators so that they commute with the others at all times. As a consequence we obtain

$$[T_i(x), P_+] = i\frac{\partial_i}{\partial_-^2}(\partial_j\pi^j - J^+) + i\frac{\pi^i - n_+ f_{-i}}{(-n_-)} = i(\partial_i T_+ + f_{+i}) = i\partial_+ T_i, \tag{4·23}$$
$$[\Psi(x), P_+] = e(T_+ - \frac{n_+}{\nabla_T^2} B + \partial_+\Lambda)\Psi - \frac{\gamma^+}{n_-}(m - i\gamma^-\partial_- - i\gamma^i D_i)\Psi(x) \tag{4·24}$$

where we have used the equality $(\gamma^+)^2 = -n_-$ as well as the fact that Ψ commutes with B and C contained in $\theta_+{}^+$ at equal x^+-time because they are multiplied by $\frac{1}{\nabla_T^2}$. Equating

(4·24) with $\partial_+\Psi$ and then multiplying both sides by γ^+ provides us with the field equation of Ψ.

In quite the same manner we obtain

$$[T_i(x), P_r] = i\partial_r T_i, \quad [\Psi(x), P_r] = i\partial_r \Psi. \quad (r = -, 1, 2) \tag{4·25}$$

Equations for B and C are calculated by the same rules and we get

$$[\frac{1}{\nabla_T^2}B(x), P_r] = i\frac{\partial_r}{\nabla_T^2}B, \quad [\frac{1}{\nabla_T^2}C(x), P_r] = i\frac{\partial_r}{\nabla_T^2}C, \quad (r = +, 1, 2) \tag{4·26}$$

$$[\frac{1}{\nabla_T^2}B(x), P_-] = 0, \quad [\frac{1}{\nabla_T^2}C(x), P_-] = -i\frac{n_-}{\nabla_T^2}B. \tag{4·27}$$

It follows that

$$[A_i(x), P_\mu] = i\partial_\mu A_i. \tag{4·28}$$

§5. A perturbation formulation free from infrared divergences

5.1. *calculation of the propagator*

Now it is straightforward to develop an x^+-time ordered perturbation theory by employing the following free and interaction Hamiltonians:

$$\begin{aligned} H_0 = &\frac{1}{2}\int d^3\boldsymbol{x}^- \{(\partial_-^{-1}\partial_i\pi^i)^2 - n_-^{-1}(\pi^i - n_+ f_{-i})^2 - n_-(f_{-i})^2 + (f_{12})^2\} \\ &+ \int d^3\boldsymbol{x}^- \bar{\Psi}(m - i\gamma^-\partial_- - i\gamma^i\partial_i)\Psi + \int d^3\boldsymbol{x}^+ B\frac{\partial_+}{\nabla_T^2}C, \end{aligned} \tag{5·1}$$

$$H_I = \int d^3\boldsymbol{x}^- \{J^\mu A_\mu - \frac{1}{2}J^+\frac{1}{\partial_-^2}J^+\} \tag{5·2}$$

where we have denoted the free fields in the interaction representation with the same notation as those in the Heisenberg representation. Thus

$$A_\mu = T_\mu + \Gamma_\mu, \quad \Gamma_\mu = -\frac{n_\mu}{\nabla_T^2}B + \partial_\mu\Lambda \tag{5·3}$$

is the free gauge field. The physical part, $T_\mu = a_\mu - \frac{\partial_\mu}{\partial_-}a_-$, is described in terms of free fields $\partial_+a_- - \partial_-a_+ = \sqrt{-\nabla_\perp{}^2}\tilde{\Sigma}$ and $a_i^\perp = a_i - \frac{\partial_i\partial_j}{\nabla_\perp{}^2}a_j$ as

$$T_+ = a_+ - \frac{\partial_+}{\partial_-}a_- = -\frac{\sqrt{-\nabla_\perp{}^2}}{\partial_-}\tilde{\Sigma}, \tag{5·4}$$

$$T_- = 0, \quad T_i = a_i - \frac{\partial_i}{\partial_-}a_- = a_i^\perp + \frac{\partial_i}{\sqrt{-\nabla_\perp{}^2}}\frac{\partial^+}{\partial_-}\tilde{\Sigma}. \tag{5·5}$$

If we consider the fact that the conjugate momentum of T_i is given by

$$\pi^i = f^+_{\ i} = \partial^+ a_i^\perp + \frac{\partial_i}{\sqrt{-\nabla_\perp^{\ 2}}}\partial_- \tilde{\Sigma}, \tag{5·6}$$

then we see that $\tilde{\Sigma},\ \partial^+\tilde{\Sigma}$ and $a_i^\perp,\ \partial^+ a_i^\perp$ are two pairs of canonical variables. As a consequence we can express T_μ in terms of creation and annihilation operators as

$$T_\mu(x) = \frac{1}{\sqrt{2(2\pi)^3}}\int \frac{d^3\boldsymbol{k}_-}{\sqrt{k^+}}\sum_{\lambda=1}^{2}\epsilon_\mu^{(\lambda)}(k)\{a_\lambda(\boldsymbol{k}_-)e^{-ik\cdot x} + \text{h.c.}\} \tag{5·7}$$

where

$$k^+ = \sqrt{k_-^2 - n_- k_\perp^2},\quad k^- = \frac{k_- - n_+ k^+}{n_-},\quad k_+ = \frac{n_+ k_- - k^+}{n_-},\quad k_\perp = \sqrt{k_1^2 + k_2^2} \tag{5·8}$$

and the operators $a_\lambda(\boldsymbol{k}_-)$ and $a_\lambda^\dagger(\boldsymbol{k}_-)$ $(\lambda = 1, 2)$ are normalized so as to satisfy the usual commutation relations,

$$[a_\lambda(\boldsymbol{k}_-),\ a_{\lambda'}(\boldsymbol{q}_-)] = 0,\quad [a_\lambda(\boldsymbol{k}_-),\ a_{\lambda'}^\dagger(\boldsymbol{q}_-)] = \delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}_- - \boldsymbol{q}_-). \tag{5·9}$$

Note that

$$\epsilon_\mu^{(1)}(k) = \left(-\frac{k_\perp}{k_-},\ 0,\ -\frac{k^+ k_1}{k_- k_\perp},\ -\frac{k^+ k_2}{k_- k_\perp}\right), \tag{5·10}$$

$$\epsilon_\mu^{(2)}(k) = (0,\ 0,\ -\frac{k_2}{k_\perp},\ \frac{k_1}{k_\perp}) \tag{5·11}$$

are the polarization vectors satisfying

$$k^\mu \epsilon_\mu^{(\lambda)}(k) = 0,\quad n^\mu \epsilon_\mu^{(\lambda)}(k) = 0,\quad (\lambda = 1, 2) \tag{5·12}$$

$$\sum_{\lambda=1}^{2}\epsilon_\mu^{(\lambda)}(k)\epsilon_\nu^{(\lambda)}(k) = -g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{k_-} - n^2\frac{k_\mu k_\nu}{k_-^2}. \tag{5·13}$$

We expand B and C in terms of zero-norm creation and annihilation operators as follows:

$$-\frac{1}{\nabla_T^{\ 2}}B(x) = \frac{1}{\sqrt{(2\pi)^3}}\int \frac{d^3\boldsymbol{k}_+}{\sqrt{k_+}}\theta(k_+)\{B(\boldsymbol{k}_+)e^{-ik\cdot x} + B^\dagger(\boldsymbol{k}_+)e^{ik\cdot x}\}|_{x^-=0}, \tag{5·14}$$

$$C(x) = \frac{i}{\sqrt{(2\pi)^3}}\int d^3\boldsymbol{k}_+\sqrt{k_+}\theta(k_+)\{C(\boldsymbol{k}_+)e^{-ik\cdot x} - C^\dagger(\boldsymbol{k}_+)e^{ik\cdot x}\}|_{x^-=0}, \tag{5·15}$$

where

$$[B(\boldsymbol{k}_+),\ C^\dagger(\boldsymbol{q}_+)] = [C(\boldsymbol{k}_+),\ B^\dagger(\boldsymbol{q}_+)] = -\delta^{(3)}(\boldsymbol{k}_+ - \boldsymbol{q}_+), \tag{5·16}$$

and all other commutators are zero. We note here that limiting the k_+- integration region to be $(0, \infty)$ is indispensable to obtain the ML form of gauge field propagator. We define the vacuum state and physical space V_P, respectively by

$$B(\boldsymbol{k}_+)|\Omega\rangle = C(\boldsymbol{k}_+)|\Omega\rangle = 0, \tag{5·17}$$

$$V_P = \{\ |\mathrm{phys}\rangle \mid B(\boldsymbol{k}_+)|\mathrm{phys}\rangle = 0\ \}. \tag{5·18}$$

Now we can calculate the x^+-ordered gauge field propagator

$$\begin{aligned} D_{\mu\nu}(x-y) &= \langle\Omega|\{\theta(x^+ - y^+)A_\mu(x)A_\nu(y) + \theta(y^+ - x^+)A_\nu(y)A_\mu(x)\}|\Omega\rangle \\ &= \frac{1}{(2\pi)^4}\int d^4q D_{\mu\nu}(q)e^{-iq\cdot(x-y)}. \end{aligned} \tag{5·19}$$

It is straightforward to show that its physical part is given by

$$D^p_{\mu\nu}(q) = \frac{i}{q^2+i\epsilon}\left(-g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{q_-} - n^2\frac{q_\mu q_\nu}{q_-^2}\right) - \delta_{\mu+}\delta_{\nu+}\frac{i}{q_-^2}, \tag{5·20}$$

where $q^2 = n_- q_-^2 + 2n_+ q_+ q_- - n_- q_+^2 - q_\perp^2$. We investigate in detail how the RG fields play roles as regulators. In the case that $\mu = i$ and $\nu = j$ we obtain the RG contribution

$$\langle\Omega|T\left(\mathit{\Gamma}_i(x)\mathit{\Gamma}_j(y)\right)|\Omega\rangle = \frac{1}{(2\pi)^4}\int d^4q D^g_{ij}(q)e^{-iq\cdot(x-y)} \tag{5·21}$$

where

$$\begin{aligned} D^g_{ij}(q) = q_i q_j \int_0^\infty dk_+ \Big[&\delta'(q_-)\frac{n_-}{n_- k_+^2 + q_\perp^2}\left(\frac{i}{k_+ - q_+ - i\epsilon} - \frac{i}{k_+ + q_+ - i\epsilon}\right) \\ &- \delta(q_-)\frac{2n_- n_+ k_+}{(n_- k_+^2 + q_\perp^2)^2}\left(\frac{i}{k_+ - q_+ - i\epsilon} + \frac{i}{k_+ + q_+ - i\epsilon}\right)\Big]. \end{aligned} \tag{5·22}$$

Note that the explicit x^- dependence gives rise to the factor $\delta'(q_-)$. Note also that there is no on-mass-shell condition among the RG field's momenta k_+, k_1, k_2 so that there remains a k_+-integration. As a consequence there arise singularities resulting from the inverse of the hyperbolic Laplace operator. Nevertheless, when we regularize the singularities as the principal values, the integral on the first line of (5·22) turns out to be well-defined. In fact we can rewrite its integrand as a sum of simple poles and make use of the integral formulas

$$\int_0^\infty dk_+\left(\frac{i}{k_+ - q_+ - i\epsilon} - \frac{i}{k_+ + q_+ - i\epsilon}\right) = -\pi \mathrm{sgn}(q_+), \tag{5·23}$$

$$\mathrm{P}\int_0^\infty dk_+\left(\frac{1}{k_+ - a} - \frac{1}{k_+ + a}\right) = 0, \tag{5·24}$$

where $a=\frac{q_\perp}{\sqrt{-n_-}}$. As a result we obtain

$$\int_0^\infty dk_+ \frac{n_-}{n_-k_+^2+q_\perp^2}\left(\frac{i}{k_+-q_+-i\epsilon}-\frac{i}{k_++q_+-i\epsilon}\right)=-\frac{n_-\pi\mathrm{sgn}(q_+)}{n_-q_+^2+q_\perp^2}. \qquad (5\cdot25)$$

On the other hand, the integral on the second line of (5·22) yields a linear divergence, as is seen by rewriting its integrand as a sum of simple and double poles:

$$\begin{aligned}&\frac{2n_+n_-k_+}{(n_-k_+^2+q_\perp^2)^2}\left(\frac{i}{k_+-q_+-i\epsilon}+\frac{i}{k_++q_+-i\epsilon}\right)\\ &=\frac{2n_+n_-q_+}{(n_-q_+^2+q_\perp^2)^2}\left(\frac{i}{k_+-q_+-i\epsilon}-\frac{i}{k_++q_+-i\epsilon}\right)-\frac{n_+}{a}\frac{n_-q_+^2-q_\perp^2}{(n_-q_+^2+q_\perp^2)^2}\left(\frac{i}{k_+-a}\right.\\ &\left.-\frac{i}{k_++a}\right)-\frac{n_+}{n_-q_+^2+q_\perp^2}\left(\frac{i}{(k_+-a)^2}+\frac{i}{(k_++a)^2}\right).\end{aligned} \qquad (5\cdot26)$$

The integration of the first and second terms on the right hand side can be carried out with the help of (5·23) and (5·24). However we cannot regularize a linear divergence resulting from a double pole by the principal value prescription.[17)] We show below that this linear divergence is necessary to cancel a corresponding one that occurs in the physical part. For later convenience we rewrite the linearly diverging integration in the form

$$\mathrm{P}\int_0^\infty dk_+\left(\frac{1}{(k_+-a)^2}+\frac{1}{(k_++a)^2}\right)=\mathrm{P}\int_0^\infty dk_+\frac{2n_-(n_-k_+^2-q_\perp^2)}{(n_-k_+^2+q_\perp^2)^2}. \qquad (5\cdot27)$$

Substituting these results and (5·27) into (5·22) yields

$$D^g_{ij}(q)=\frac{n_-q_iq_j}{q^2+i\epsilon}\delta'(q_-)\pi\mathrm{sgn}(q_+)-i\frac{n_+q_iq_j}{q^2+i\epsilon}\delta(q_-)\int_0^\infty dk_+\frac{2n_-(n_-k_+^2-q_\perp^2)}{(n_-k_+^2+q_\perp^2)^2}, \qquad (5\cdot28)$$

where we have made use of the identity

$$\frac{1}{q^2+i\epsilon}\delta'(q_-)=-\frac{1}{n^2q_+^2+q_\perp^2}\delta'(q_-)+\frac{2n_+q_+}{(n^2q_+^2+q_\perp^2)^2}\delta(q_-). \qquad (5\cdot29)$$

Thus, for the sum of (5·20) and (5·28) we obtain

$$\begin{aligned}D_{ij}(q)=\frac{i}{q^2+i\epsilon}\Big(&-g_{ij}-n^2\frac{q_iq_j}{q_-^2}-in_-q_iq_j\pi\mathrm{sgn}(q_+)\delta'(q_-)\\ &-n_+q_iq_j\delta(q_-)\int_0^\infty dk_+\frac{2n_-(n_-k_+^2-q_\perp^2)}{(n_-k_+^2+q_\perp^2)^2}\Big).\end{aligned} \qquad (5\cdot30)$$

Now we can demonstrate that the linear divergence resulting from $\frac{1}{q_-^2}$ is canceled by the final term of (5·30) when we restore $D_{ij}(x)$ by substituting (5·30) into (5·19). It should be noted here that q_- is conjugate to the spatial variable x^-, while k_+ is conjugate to the temporal variable x^+. To make the infrared divergence cancellation mechanism work, both integration variables have to be either spatial or temporal. We show in Appendix A that if we change the integration variable from the spatial q_- to the temporal $k_+ = \frac{n_+q_- - \sqrt{q_-^2 - n_- q_\perp^2}}{n_-}$, then the following integral formula holds:

$$\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} + \int_0^{\infty} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} = 0. \tag{5·31}$$

From (5·31) we see that the linearly diverging terms of the inverse Fourier transform of (5·30) are given by (we omit $-in^2q_iq_j$ for the moment)

$$\begin{aligned} &\int_{-\infty}^{\infty} dq_- \frac{1}{q^2+i\epsilon}\left(\frac{1}{q_-^2} - \delta(q_-)\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2}\right) e^{-iq_-x^-} \\ =& \int_{-\infty}^{\infty} dq_- \frac{e^{-iq_-x^-}}{q_-^2(q^2+i\epsilon)} + \frac{1}{n^2q_+^2 + q_\perp^2}\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2}. \\ =& \int_{-\infty}^{\infty} dq_- \left(\frac{e^{-iq_-x^-}-1}{q_-^2}\frac{1}{q^2+i\epsilon} + \frac{2n_+q_+ + n^2q_-}{q_-(n^2q_+^2+q_\perp^2)(q^2+i\epsilon)}\right). \end{aligned} \tag{5·32}$$

We see from this that the last integrals diverge at most logarithmically, but logarithmic divergences can be regularized by the principal value prescription so that there arise no divergences from (5·32). This verifies that the following identity holds:

$$\begin{aligned} &\frac{1}{q_-^2} + i\pi\mathrm{sgn}(q_+)\delta'(q_-) - \delta(q_-)\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \\ =& \mathrm{Pf}\frac{1}{q_-^2} + i\pi\mathrm{sgn}(q_+)\delta'(q_-) = \frac{1}{(q_- + i\epsilon\mathrm{sgn}(q_+))^2} \end{aligned} \tag{5·33}$$

where Pf denotes Hadamard's finite part. Now substituting (5·33) into (5·30) yields the ML form of gauge field propagator:

$$D_{ij}(q) = \frac{i}{q^2+i\epsilon}\left(-g_{ij} - \frac{n^2q_iq_j}{(q_- + i\epsilon\mathrm{sgn}(q_+))^2}\right). \tag{5·34}$$

For other cases we omit detailed demonstrations because the calculations are similar. In the case that $\mu = +$ and $\nu = i$, we obtain the following RG contribution

$$\begin{aligned} D^g_{+i}(q) = q_i\int_0^{\infty} dk_+ \Big[&\delta(q_-)\frac{n_+}{n_-k_+^2+q_\perp^2}\left(\frac{i}{k_+ - q_+ - i\epsilon} - \frac{i}{k_+ + q_+ - i\epsilon}\right) \\ &+ \delta'(q_-)\frac{n_-k_+}{n_-k_+^2+q_\perp^2}\left(\frac{i}{k_+ - q_+ - i\epsilon} + \frac{i}{k_+ + q_+ - i\epsilon}\right) \\ &- \delta(q_-)\frac{2n_-n_+k_+^2}{(n_-k_+^2+q_\perp^2)^2}\left(\frac{i}{k_+ - q_+ - i\epsilon} - \frac{i}{k_+ + q_+ - i\epsilon}\right)\Big]. \end{aligned} \tag{5·35}$$

Because the integral on the first line gives rise to the imaginary part of $\frac{1}{q_-+i\epsilon\mathrm{sgn}(q_+)}$, we obtain

$$D_{+i}(q)=\frac{i}{q^2+i\epsilon}\left(\frac{n_+q_i}{q_-+i\epsilon\mathrm{sgn}(q_+)}-\frac{n^2q_+q_i}{(q_-+i\epsilon\mathrm{sgn}(q_+))^2}\right). \tag{5·36}$$

For the case that $\mu=\nu=+$ the RG contribution turns out to be

$$\begin{aligned}D^g_{++}(q)=&\int_0^\infty dk_+\Big[\delta'(q_-)\frac{n_-k_+^2}{n_-k_+^2+q_\perp^2}\Big(\frac{i}{k_+-q_+-i\epsilon}-\frac{i}{k_++q_+-i\epsilon}\Big)\\&+\delta(q_-)\frac{2n_+k_+q_\perp^2}{(n_-k_+^2+q_\perp^2)^2}\left(\frac{i}{k_+-q_+-i\epsilon}+\frac{i}{k_++q_+-i\epsilon}\right)\Big],\end{aligned} \tag{5·37}$$

so that we obtain

$$\begin{aligned}D^g_{++}(q)=&\frac{1}{q^2+i\epsilon}\Big(2n_+q_+\delta(q_-)\pi\mathrm{sgn}(q_+)\\&+n^2q_+^2\delta'(q_-)\pi\mathrm{sgn}(q_+)-q_\perp^2\delta(q_-)\int_{-\infty}^\infty dq_-\frac{i}{q_-^2}\Big).\end{aligned} \tag{5·38}$$

Here it is noteworthy that the last term in (5·38) also cancels the linear divergence resulting from the contact term in (5·20). In fact it also yields a contact term, as is seen from $-\delta(q_-)\frac{q_\perp^2}{q^2+i\epsilon}=\delta(q_-)\left(1+\frac{n^2q_+^2}{q^2+i\epsilon}\right)$. Therefore, combining it with the corresponding one in (5·20) and carrying out the inverse Fourier transform, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^\infty dq_-\left(\delta(q_-)\int_{-\infty}^\infty dq_-\frac{1}{q_-^2}-\frac{1}{q_-^2}\right)e^{-q_-x^-}=\frac{1}{\pi}\int_0^\infty dq_-\frac{1-\mathrm{cos}q_-x^-}{q_-^2}=\frac{|x^-|}{2}. \tag{5·39}$$

Consequently, taking into account the fact that the Fourier transform of $\frac{|x^-|}{2}$ is $-\frac{1}{2}(\frac{1}{(q_-+i\epsilon)^2}+\frac{1}{(q_--i\epsilon)^2})$, we obtain

$$\begin{aligned}D_{++}(q)=&\frac{i}{q^2+i\epsilon}\left(-g_{++}+\frac{2n_+q_+}{q_-+i\epsilon\mathrm{sgn}(q_+)}-\frac{n^2q_+^2}{(q_-+i\epsilon\mathrm{sgn}(q_+))^2}\right)\\&-\delta_{\mu+}\delta_{\nu+}\frac{i}{2}\left(\frac{1}{(q_-+i\epsilon)^2}+\frac{1}{(q_--i\epsilon)^2}\right).\end{aligned} \tag{5·40}$$

This finishes the demonstration that, due to the RG fields, the linear divergences are eliminated even in the most singular component of x^+-ordered gauge field propagator.

We end this subsection by investigating the light-front limit $\theta\to\frac{\pi}{4}+0$. On the mass-shell, $k^2=0$, k_+ is given by $k_+=\frac{n_+k_--\sqrt{k_-^2-n_-k_\perp^2}}{n_-}$ so we get

$$\lim_{\theta\to\frac{\pi}{4}+0}k_+=\begin{cases}\frac{k_\perp^2}{2k_-} & (k_->0)\\ \infty & (k_-<0)\end{cases} \tag{5·41}$$

and contributions to T_μ in (5·7) from the integration region in which $k_-<0$ vanish due to the Riemann-Lebesgue lemma. As a result, the integration region k_- in T_μ is limited to

$(0,\infty)$. It is worthwhile noting that in the limit, $n_- = n^2 \to 0$, so we do not have the linear divergences except for the contact term in the most singular component of the gauge field propagator:

$$D_{++}(q) = \frac{2q_+}{q_-}\frac{i}{q^2+i\epsilon} - \frac{i}{q_-^2} - i\frac{2q_+}{q_\perp^2}\delta(q_-)(-i\pi\mathrm{sgn}(q_+)) + \frac{4i}{q_\perp^2}\delta(q_-)\int_0^\infty dk_+. \qquad (5\cdot42)$$

However, the linear divergence resulting from the contact term is also cancelled out due to an equality

$$\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} - \int_0^\infty dk_+ \frac{4}{q_\perp^2} = 0, \qquad (5\cdot43)$$

which is verified by changing the integration variable from q_- to $k_+ = \frac{q_\perp^2}{2q_-}$.[15)]

In the light-front formulation $\psi_- = \frac{1}{\sqrt{2}}\gamma^0\gamma^-\psi$ is a dependent field and has to be expressed in terms of the independent field $\psi_+ = \frac{1}{\sqrt{2}}\gamma^0\gamma^+\psi$. We follow Morara and Soldati[7)] to solve this problem and obtain the following free and interaction Hamiltonians

$$H_0 = \int d^3\boldsymbol{x}^- \{\frac{1}{2}(f_{-+})^2 + \frac{1}{2}(f_{12})^2 + i\bar{\psi}\gamma^-\partial_-\psi\} + \int d^3\boldsymbol{x}^+ B\frac{\partial_+}{\nabla_\perp^2}C, \qquad (5\cdot44)$$

$$H_I = \int d^3\boldsymbol{x}^- \{J^\mu A_\mu + e^2\bar{\psi}\gamma^\mu A_\mu \frac{\gamma^+}{2i\partial_-}\gamma^\nu A_\nu\psi - \frac{1}{2}J^+\frac{1}{\partial_-^2}J^+\}. \qquad (5\cdot45)$$

The x^+-ordered electron propagator is given by Chang and Yan[18)] to be

$$S_F = \frac{i}{(2\pi)^4}\int d^4p\ e^{-ip\cdot x}\left[\frac{(\not p + m)}{p^2 - m^2 + i\varepsilon} - \frac{1}{2}\frac{\gamma^+}{p^+}\right]. \qquad (5\cdot46)$$

We see here that there appears a noncovariant instant term, but its contributions to S matrices are cancelled by those resulting from the noncovariant fourpoint interaction term.

5.2. *calculation of the electron self energy*

In order to make comparison with previous calculations we shall regulate our calculation with the inclusion of Pauli-Villars fields. If we perform the calculation in the usual order for equal-time quantization, we should expect to need only one Pauli-Villars photon. However, it is common experience that more Pauli-Villars fields are needed when one performs the x^+ integral first. In Feynman gauge three Pauli-Villars photons[11)] or one Pauli-Villars electron and one Pauli-Villars photon are needed.[12)] In light-front gauge with higher derivatives so that A_+ is a degree of freedom, a much more complicated regulation is required including (in addition to the higher derivatives): three Pauli-Villars electrons, a Pauli-Villars photon and two carefully treated cutoffs.[12)] In all these cases the Pauli-Villars electrons are included with flavor changing currents that break gauge invariance. The breaking of gauge invariance is removed by taking the masses of the Pauli-Villars electrons to infinity after the calculation is complete. It is not obvious that taking such a limit really restores gauge invariance but it can be shown to be true.[12)]

In the present calculation we keep one Pauli-Villars photon and investigate how many Pauli-Villars electrons are required to successfully regularize the one loop electron self energy. We find that we need two Pauli-Villars electrons and so we use the Lagrangian

$$\sum_{i=0}^{1}(-1)^{i+1}\frac{1}{4}F^{i\mu\nu}F^{i}{}_{\mu\nu} - BA_{-} + \sum_{i}\frac{1}{\nu_i}\bar{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - e\bar{\psi}\gamma^\mu\psi A_\mu, \tag{5·47}$$

where

$$\psi = \sum_{i=0}^{2}\psi_i, \quad \sum_{i=0}^{2}\nu_i = 0, \quad A_\mu = \sum_{i=0}^{1}A^i_\mu. \tag{5·48}$$

We shall have m_0 equal to the physical electron mass and $\nu_0 = 1$. For the other PV condition we shall require that $\{\nu_i\}$ and $\{m_i\}$ be chosen such that the requirement from chiral symmetry — that the renormalized mass be zero if the bare mass is zero — be satisfied. That is, we shall require that

$$\delta m|_{m_0=0} = 0. \tag{5·49}$$

Chiral transformations are dynamic in light-cone quantization and it is common that that requirement from chiral symmetry has to be imposed with an extra Pauli-Villars field.[18)] As is usual, we shall regulate the infrared singularities with photon masses and shall add these to regulate the propagator rather than add a mass term to regulate the Lagrangian.

Using eqn's (5·34), (5·36), (5·40) and (5·46) we find that the one loop electron self energy is given by

$$\frac{e^2}{(2\pi)^4}\int d^4q\sum_{i,j}(-1)^j\nu_i D_{\mu\nu}(q,\mu_j)\langle p|\gamma^0\gamma^\mu\{\frac{(p-q)_\rho\gamma^\rho + m_i}{(p-q)^2 - m_i^2 + i\varepsilon} - \frac{1}{2}\frac{\gamma^+}{(p-q)^+}\}\gamma^\nu|p\rangle. \tag{5·50}$$

We shall take $p_\perp = 0$. Due to the fact that they do not depend on masses, the contributions from the noncovariant term in the fermion propagator (the second term in (5·46)), the contributions from the contact term in the gauge propagator (the last term in (5·40)) and the contributions from the second and third terms in (5·45) all go to zero in the sum over i and j. We can write the gauge propagator as

$$D_{\mu\nu}(q,\mu) = i\frac{-g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{q_- + i\varepsilon\mathrm{sgn}(q_+)}}{q^2 - \mu^2 + i\varepsilon} \tag{5·51}$$

We then get

$$\Sigma(p) = \frac{ie^2}{(2\pi)^4}\int d^4q\sum_{i,j}\nu_i(-1)^j\,\frac{\gamma^\mu\{\gamma\cdot(p-q) + m_i\}\gamma^\nu}{(p-q)^2 - m_i^2 + i\varepsilon}\cdot\frac{-g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{q_- + i\varepsilon\mathrm{sgn}(q_+)}}{q^2 - \mu_j^2 + i\varepsilon}. \tag{5·52}$$

We divide this expression into two pieces, $\Sigma^{(1)}(p)$ and $\Sigma^{(2)}(p)$, according to

$$\Sigma^{(1)}(p) = \frac{-ie^2}{(2\pi)^4}\int d^4q \sum_{i,j} \nu_i(-1)^j \; \frac{\gamma^\mu\{\gamma\cdot(p-q)+m_i\}\gamma_\mu}{[(p-q)^2-m_i^2+i\varepsilon][q^2-\mu_j^2+i\varepsilon]} \tag{5·53}$$

$$\Sigma^{(2)}(p) = \frac{ie^2}{(2\pi)^4}\int d^4q \sum_{i,j} \nu_i(-1)^j \; \frac{\gamma^\mu\{\gamma\cdot(p-q)+m_i\}\gamma^\nu}{(p-q)^2-m_i^2+i\varepsilon}\frac{\frac{n_\mu q_\nu + n_\nu q_\mu}{q_-+i\varepsilon \mathrm{sgn}(q_+)}}{q^2-\mu_j^2+i\varepsilon}. \tag{5·54}$$

We now perform the γ algebra and take the inner product $\langle p|\gamma^0\Sigma^{(1)}(p)|p\rangle$ to get

$$\delta m^{(1)} = \frac{ie^2}{(2\pi)^4}\int d^4q \sum_{i,j} \nu_i(-1)^j \frac{2\frac{p^+}{m_0}(p_+-q_+)+2\frac{p^-}{m_0}(p_--q_-)-4m_i}{[(p-q)^2-m_i^2+i\varepsilon][q^2-\mu_j^2+i\varepsilon]}. \tag{5·55}$$

Here it is important to note that when we perform the q_+ integration, we obtain an extra contribution from the semicircle at infinity to the term propotional to $(p_+ - q_+)$ as in the following

$$\int_{-\infty}^{\infty} dq_+ \frac{(p-q)_+}{[(p-q)^2-m_i^2+i\varepsilon][q^2-\mu_j^2+i\varepsilon]}$$
$$= \frac{-i\pi}{4q_-(p-q)_-}\left[\frac{\{\mathrm{sgn}(q_-)+\mathrm{sgn}((p-q)_-)\}\frac{(p-q)_\perp^2+m_i^2}{2(p-q)_-}}{p_+ - \frac{(p-q)_\perp^2+m_i^2-i\varepsilon}{2(p-q)_-} - \frac{q_\perp^2+\mu_j^2-i\varepsilon}{2q_-}} + \mathrm{sgn}(q_-)\right].$$

and that it gives rise to a logarithmic divergence

$$\int_{-\infty}^{\infty} dq_- \frac{\mathrm{sgn}(q_-)}{4q_-(p-q)_-} = \frac{1}{2p_-}\int_0^1 dx \frac{1}{1-x}$$

so that a similar divergence resulting from $\frac{(p-q)_\perp^2+m_i^2}{2(p-q)_-}$ is cancelled. As a consequence, changing the integration variable from q_- to $x=\frac{q_-}{p_-}$ yields

$$\delta m^{(1)} = \frac{e^2}{(2\pi)^3}\int d^2q_\perp \int_0^1 dx \sum_{i,j}\nu_i(-1)^j \frac{m_0(1-x)-2m_i}{m_0^2x(1-x)-m_i^2x-\mu_j^2(1-x)-q_\perp^2}$$
$$+ \frac{e^2}{2(2\pi)^3 m_0}\int d^2q_\perp \int_0^1 dx \sum_{i,j}\nu_i(-1)^j \frac{m_0(2x-1)+m_i^2-\mu_j^2}{m_0^2x(1-x)-m_i^2x-\mu_j^2(1-x)-q_\perp^2}.$$

By performing the x integration of the term in the second line we obtain

$$\int d^2q_\perp \int_0^1 dx \sum_{i,j}\nu_i(-1)^j \frac{m_0(2x-1)+m_i^2-\mu_j^2}{m_0^2x(1-x)-m_i^2x-\mu_j^2(1-x)-q_\perp^2}$$
$$= -\int d^2q_\perp \sum_{i,j}\nu_i(-1)^j \log\left(\frac{q_\perp^2+m_i^2}{q_\perp^2+\mu_j^2}\right) = -\int d^2q_\perp \sum_i \nu_i \log\left(\frac{q_\perp^2+\mu_1^2}{q_\perp^2+\mu_0^2}\right).$$

We see here that, due to the Pauli-Villars photon, this term is independent of Pauli-Villars electron masses so that if we impose the Pauli-Villars condition $\sum_i \nu_i = 0$, it is trivially zero. Consequently we obtain

$$\delta m^{(1)} = \frac{e^2}{(2\pi)^3} \int d^2 q_\perp \int_0^1 dx \sum_{i,j} \nu_i (-1)^j \frac{m_0(1-x) - 2m_i}{m_0^2 x(1-x) - m_i^2 x - \mu_j^2 (1-x) - q_\perp^2}. \tag{5·56}$$

For terms involving only the physical electron mass we have

$$\delta m^{(1)}_{physical} = \frac{e^2}{(2\pi)^3} \int d^2 q_\perp \int_0^1 dx \sum_j (-1)^j \frac{m_0(1+x)}{m_0^2 x^2 + \mu_j^2 (1-x) + q_\perp^2}. \tag{5·57}$$

The remaining contributions from $\Sigma^{(1)}$ are given by

$$\delta m^{(1)}_{PV} = \frac{e^2}{(2\pi)^3} \int d^2 q_\perp \int_0^1 dx \sum_{i=1}^{2} \sum_j \nu_i (-1)^j \frac{m_0(1-x) - 2m_i}{m_0^2 x(1-x) - m_i^2 x - \mu_j^2 (1-x) - q_\perp^2}. \tag{5·58}$$

It is not difficult to show that this quantity goes to zero as the masses of the Pauli-Villars electrons go to infinity. In this way we obtain the previous result that one Pauli-Villars electron and one Pauli-Villars photon are enough to regularize $\delta m^{(1)}$.

We now consider the contributions from $\Sigma^{(2)}$. Taking the matrix element, we find that

$$\begin{aligned}\delta m^{(2)} = &- \frac{ie^2}{(2\pi)^4} \frac{2p^+}{m_0} \int d^4 q \sum_{i,j} \nu_i (-1)^j \frac{1}{[q^2 - \mu_j^2 + i\varepsilon][q_- + i\varepsilon \mathrm{sgn}(q_+)]} \\ &- \frac{ie^2}{(2\pi)^4} \int d^4 q \sum_{i=1}^{2} \sum_j \nu_i (-1)^j \Bigg(\frac{2(m_0 - m_i)}{[(p-q)^2 - m_i^2 + i\varepsilon][q^2 - \mu_j^2 + i\varepsilon]} \\ &- \frac{2\frac{m_0^2 - m_i^2}{m_0} p^+}{[(p-q)^2 - m_i^2 + i\varepsilon][q^2 - \mu_j^2 + i\varepsilon][q_- + i\varepsilon \mathrm{sgn}(q_+)]} \Bigg). \end{aligned} \tag{5·59}$$

The first term is trivially zero and the second term goes to zero as the masses of the Pauli-Villars electrons go to infinity. Thus we can discard them and concentrate on the third term, which is given by

$$\delta m^{(2)} = \delta m_1^{(2)} + \delta m_2^{(2)} \tag{5·60}$$

where

$$\begin{aligned}\delta m_1^{(2)} = &\frac{-ie^2}{(2\pi)^4} \int d^2 q_\perp \int \frac{dq_-}{4q_-(p-q)_-} \sum_{i,j} \frac{2\nu_i (-1)^j \frac{m_0^2 - m_i^2}{m_0} p^+}{[p_+ - -\frac{(p-q)_\perp^2 + m^2 - i\varepsilon}{2(p-q)_-} - \frac{q_\perp^2 + \mu^2 - i\varepsilon}{2q_-}]} \\ &\times \frac{q_-}{q_-^2 + \varepsilon^2} \int_{-\infty}^{\infty} dq_+ \left(\frac{1}{q_+ - p_+ + \frac{(p-q)_\perp^2 + m^2 - i\varepsilon}{2(p-q)_-}} - \frac{1}{q_+ - \frac{q_\perp^2 + \mu_j^2 - i\varepsilon}{2q_-}} \right) \end{aligned} \tag{5·61}$$

$$\delta m_2^{(2)} = \frac{ie^2}{(2\pi)^4}\int d^2q_\perp \int \frac{dq_-}{4q_-(p-q)_-}\sum_{i,j}\frac{2\nu_i(-1)^j\frac{m_0^2-m_i^2}{m_0}p^+}{[p_+ - -\frac{(p-q)_\perp^2+m^2-i\varepsilon}{2(p-q)_-} - \frac{q_\perp^2+\mu^2-i\varepsilon}{2q_-}]}$$
$$\times\frac{i\varepsilon}{q_-^2+\varepsilon^2}\int_{-\infty}^{\infty} dq_+\left(\frac{1}{q_+-p_+ + \frac{(p-q)_\perp^2+m^2-i\varepsilon}{2(p-q)_-}} - \frac{1}{q_+ - \frac{q_\perp^2+\mu_j^2-i\varepsilon}{2q_-}}\right). \tag{5·62}$$

By performing the q_+ integration $\delta m_1^{(2)}$ and $\delta m_2^{(2)}$ can be described, respectively, as

$$\delta m_1^{(2)} = \frac{-e^2}{(2\pi)^3}\int d^2q_\perp \int_0^1 dx \sum_{i,j}\frac{\nu_i(-1)^j\frac{m_0^2-m_i^2}{m_0}\frac{x}{x^2+\varepsilon^2}}{q_\perp^2+m_i^2x+\mu_j^2(1-x)-m_0^2x(1-x)} \tag{5·63}$$

$$\delta m_2^{(2)} = \frac{e^2}{(2\pi)^4}\int d^2q_\perp \int dq_- \frac{\varepsilon}{q_-^2+\varepsilon^2}\sum_{i,j}$$
$$\times\frac{\nu_i(-1)^j\frac{m_0^2-m_i^2}{m_0}p^+(L+2iT)}{2p_+q_-(p_--q_-)-q_-(q_\perp^2+m_i^2-i\varepsilon)-(p_--q_-)(q_\perp^2+\mu_j^2-i\varepsilon)} \tag{5·64}$$

where

$$L = \log\left(\frac{q_-^2(q_\perp^2+m_i^2-2p_+(p_--q_-))^2}{(p_--q_-)^2(q_\perp^2+\mu_j 2)^2}\right), \tag{5·65}$$

$$T = \tan^{-1}\left(\frac{\varepsilon(m_i^2-\mu_j^2-2p_+(p_--q_-))}{(q_\perp^2+\mu_j^2)(q_\perp^2+m_i^2-2p_+(p_--q_-))}\right). \tag{5·66}$$

Here we see that the term proportional to T is well-defined so that the factor $\frac{\varepsilon}{q_-^2+\varepsilon^2}$ behaves as $\pi\delta(q_-)$. As a result, it vanishes as ε tends to zero. Therefore we can discard it. The remaining integral violates the chiral symmetry condition. That is why we need the second Pauli-Villars electron. We impose the last Pauli-Villars condition — $\delta m|_{m_0=0}=0$. Actually, we see that $\delta m^{(2)}$ diverges as $m_0\to 0$; so we set

$$\frac{e^2}{(2\pi)^3}\int d^2q_\perp\sum_{i,j}\nu_i(-1)^j\left[\int_0^1 dx\frac{x}{x^2+\varepsilon^2}\frac{\frac{m_0^2-m_i^2}{m_0}}{q_\perp^2+m_i^2x+\mu_j^2(1-x)}\right.$$
$$\left.+\frac{1}{\pi}\int dq_-\frac{\varepsilon}{q_-^2+\varepsilon^2}\frac{\frac{m_0^2-m_i^2}{m_0}p^+\log\left|\frac{q_-(q_\perp^2+m_i^2)}{(p_--q_-)(q_\perp^2+\mu_j^2)}\right|}{q_-(q_\perp^2+m_i^2-i\epsilon)+(p-q)_-(q_\perp^2+\mu_j^2-i\epsilon)}\right]=0. \tag{5·67}$$

We can use this relation to write $\delta m^{(2)}$ as

$$\delta m^{(2)} = \delta m_{c1}^{(2)} + \delta m_{c2}^{(2)} \tag{5·68}$$

where

$$\delta m_{c1}^{(2)} = -\frac{e^2}{(2\pi)^3}\int d^2q_\perp\sum_{i,j}\nu_i(-1)^j\int_0^1 dx\frac{x}{x^2+\varepsilon^2}\frac{m_0^2-m_i^2}{m_0}$$

$$\times \left(\frac{1}{q_\perp^2 + m_i^2 x + \mu_j^2 (1-x) - m_0^2 x(1-x)} - \frac{1}{q_\perp^2 + m_i^2 x + \mu_j^2(1-x)} \right)$$

$$= -\frac{e^2}{8\pi^2} \sum_{i,j} \nu_i (-1)^j \int_0^1 dx \frac{x}{x^2+\varepsilon^2} \frac{m_i^2 - m_0^2}{m_0} \log\left(1 - \frac{m_0^2 x(1-x)}{m_i^2 x + \mu_j^2 (1-x)}\right), \tag{5·69}$$

$$\delta m_{c2}^{(2)} = \frac{e^2}{(2\pi)^3} \int d^2 q_\perp \frac{1}{\pi} \int dq_- \frac{\varepsilon}{q_-^2 + \varepsilon^2} \sum_{i,j} \nu_i (-1)^j$$
$$\times \left[\frac{\frac{m_0^2 - m_i^2}{m_0} p^+ \log\left(1 - \frac{2p_+(p_- - q_-)}{q_\perp^2 + m_i^2}\right)}{2p_+ q_-(p_- - q_-) - q_-(q_\perp^2 + m_i^2 - i\varepsilon) - (p_- - q_-)(q_\perp^2 + \mu_j^2 - i\varepsilon)} \right.$$
$$+ 2m_0(m_0^2 - m_i^2) q_-(p_- - q_-) \log \left| \frac{q_-(q_\perp^2 + m_i^2)}{(p_- - q_-)(q_\perp^2 + \mu_j^2)} \right|$$
$$\left. \times \frac{\{q_-(q_\perp^2 + m_i^2) + (p_- - q_-)(q_\perp^2 + \mu_j^2)\}^{-1}}{2p_+ q_-(p_- - q_-) - q_-(q_\perp^2 + m_i^2) - (p_- - q_-)(q_\perp^2 + \mu_j^2)} \right]. \tag{5·70}$$

Now the integrand of $\delta m_{c2}^{(2)}$ can be regarded as a continuous function of q_-. Consequently the factor $\frac{1}{\pi} \frac{\varepsilon}{q_-^2 + \varepsilon 2}$ behaves as $\delta(q_-)$ so that we obtain

$$\lim_{\varepsilon \to 0} \delta m_{c2}^{(2)} = -\frac{e^2}{(2\pi)^3} \int d^2 q_\perp \sum_{i,j} \nu_i (-1)^j \frac{\frac{m_0^2 - m_i^2}{m_0} \log\left(1 - \frac{m_0^2}{q_\perp^2 + m_i^2}\right)}{q_\perp^2 + \mu_j^2}. \tag{5·71}$$

It follows from (5·69) and (5·71)) that

$$\lim_{m_i \to \infty} \left(\lim_{\varepsilon \to 0} \delta m^{(2)} \right) = 0. \tag{5·72}$$

So the final answer for the electron self energy is regulated by one Pauli-Villars photon and is given by

$$\delta m = \frac{e^2}{(2\pi)^3 m_0} \int d^2 q_\perp \int_0^1 dx \sum_j (-1)^j \frac{m_0^2 (1+x)}{m_0^2 x^2 + \mu_j^2 (1-x) + q_\perp^2}. \tag{5·73}$$

That is the same answer as the one obtained by Feynman methods.[12),20)]

We end this section by making some remarks. If we regulate the spurious singularity instead by

$$\frac{1}{q_- + iq_+ \varepsilon}, \tag{5·74}$$

which has been often used as the ML prescription, and if we take ε to zero at the end of the calculation, then it improves properties of the q_+ integration so that we need not introduce extra Pauli-Villars electrons. It is also possible to include a regulator in the spurious pole of the form $\frac{1}{q_-+\epsilon}$; in that case ε can be taken to zero immediately after the q_+ integration is performed and, if we keep ϵ finite until after the Pauli-Villars electron masses are taken to infinity, we again obtain the Feynman answer. The main point is that the cancellation of the strongest singularity in the gauge propagator, shown in (5·39), allows a successful calculation of the electron self energy using standard techniques.

§6. Concluding remarks

In this paper we have constructed the quantization of QED in gauges fixed by specifying a constant, space-like vector, n, and the gauge condition, $n^\mu A_\mu = 0$. We have then constructed the axial gauge formulation of QED in the $\pm$-coordinates. Our framework has allowed us to consider axial gauges generally. The temporal and axial gauges in ordinary coordinates correspond, respectively, to $\theta = 0$ and $\theta = \frac{\pi}{2}$, while the light-cone formulation corresponds to $\theta = \frac{\pi}{4}$. The most important aspect of this framework is that it has enabled us to use the temporal gauge formulation to obtain the algebra of the RG fields, which are not canonical variables in the pure space-like case; they are nevertheless, necessary ingredients in that case.

We have obtained the commutation relations of B in the temporal region and extrapolated them into the axial region. We have also obtained the formal solution (3·8) of the gauge field equations and specified the relevant integration constants by comparing with the free gauge fields in the temporal formulation. We have made use of that solution to specify the integration constants that appear when we solve the constraint equations (3·12) and (3·13) in favor of the independent canonical variables. We have specified the equal x^+-time commutation relations of the constituent fields as being canonical and found eventually that the RG fields make vanishing contributions to the equal x^+-time commutation relations because they are multiplied by the operator $\frac{1}{\nabla_T{}^2}$. We have furthermore obtained the conserved Hamiltonian to which Hamiltonian for the RG fields is added. In this way we have succeeded in constructing a perturbative formulation of pure space-like axial gauge QED in which the RG fields regularize infrared divergences inherent in the traditional quantizations of those gauges. The resulting gauge field propagator has the ML form.

We have illustrated the effect of the RG fields by performing a calculation of the one loop self energy of the electron. In the usual light-cone gauge without higher derivatives so that A_+ is a constrained field, that calculation has not previously been performed successfully. The severe infrared divergences which result from solving the constraint equation for A_+ (without inclusion of the integration constants which we label B and C) have prevented a successful calculation. What we have found here is that including the integration constants (the RG fields) softens these severe infrared singularities and allows a successful calculation of the self energy using standard regulation techniques. The necessary regulation is slightly more complicated than in Feynman gauge but is considerably simpler than in the higher derivative regulated version of light-cone gauge where A_+ is a degree of freedom. Indeed we expect that the very complicated regulation procedures necessary in light-cone gauge without the RG fields[12)] are somehow mimicking the effects of the RG fields, which should be included, to a sufficient degree that an effective renormalization can be performed. We should also remark that if another order of performing the integral is used (such as performing the q_- integral first) then a smaller number of regulator fields is needed. We have studied the case of performing the q_+ integral first since that is the order that makes closest contact with nonperturbative light-cone calculations.

While we have only considered the case of one loop (with the q^+ integral performed first), we expect that it may be possible to use the techniques of Paston and Franke[18)] to show that the calculations with the relatively simple regulation are equivalent to Feynman methods to all orders. Since the quantization of QCD in Feynman gauge encounters difficulties which have not, so far, been solved, it may be that if the RG fields can be included in QCD, as they were here, the resulting formulation would have practical advantages over other formulations.

Appendix A

—— *Verification of* Eq. (5·31) ——

We change the integration variable from q_- to $k_+=\frac{n_+q_--\sqrt{q_-^2-n_-q_\perp^2}}{n_-}$. The quantity k_+ is a two-valued function of q_-, and it takes its minimum value, $\sqrt{-n_-}q_\perp\equiv m_0$, at $q_-=\frac{n_+}{-n_-}m_0$. Therefore, when we change the integration variable from q_- to k_+, we use $q_-=\frac{n_+k_+-\sqrt{k_+^2-m_0^2}}{-n_-}$ so that the region $m_0\leqq k_+<\infty$ corresponds to $-\infty<q_-\leqq\frac{n_+}{-n_-}m_0$, whereas we use $q_-=\frac{n_+k_++\sqrt{k_+^2-m_0^2}}{-n_-}$ so that the region $m_0\leqq k_+<\infty$ corresponds to $\frac{n_+}{-n_-}m_0\leqq q_-<\infty$. Hence we have

$$\int_{-\infty}^{\infty}\frac{dq_-}{q_-^2}=\int_{m_0}^{\infty}dk_+\frac{k_+-n_+\sqrt{k_+^2-m_0^2}}{(-n_-)\sqrt{k_+^2-m_0^2}}\left(\frac{-n_-}{n_+k_+-\sqrt{k_+^2-m_0^2}}\right)^2$$
$$+\int_{m_0}^{\infty}dk_+\frac{k_++n_+\sqrt{k_+^2-m_0^2}}{(-n_-)\sqrt{k_+^2-m_0^2}}\left(\frac{-n_-}{n_+k_++\sqrt{k_+^2-m_0^2}}\right)^2, \tag{A·1}$$

where the first term diverges, but the second term is finite. Then, combining the first integral with the second one of (5·31) and making use of an equality

$$(k_+-n_+\sqrt{k_+^2-m_0^2})(n_+k_++\sqrt{k_+^2-m_0^2})^2-2n_+n_-\sqrt{k_+^2-m_0^2}(n_-k_+^2-q_\perp^2)$$
$$=(k_++n_+\sqrt{k_+^2-m_0^2})(n_+k_+-\sqrt{k_+^2-m_0^2})^2 \tag{A·2}$$

yields

$$
\begin{aligned}
&\int_{m_0}^{\infty} dk_+ \frac{k_+ - n_+\sqrt{k_+^2 - m_0^2}}{(-n_-)\sqrt{k_+^2 - m_0^2}} \left(\frac{-n_-}{n_+k_+ - \sqrt{k_+^2 - m_0^2}} \right)^2 + \int_0^{\infty} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} \\
&= \int_{m_0}^{\infty} dk_+ \frac{k_+ - n_+\sqrt{k_+^2 - m_0^2}}{(-n_-)\sqrt{k_+^2 - m_0^2}} \left(\frac{n_+k_+ + \sqrt{k_+^2 - m_0^2}}{n_-k_+^2 + q_\perp^2} \right)^2 + \int_0^{\infty} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} \\
&= \int_{m_0}^{\infty} dk_+ \frac{k_+ + n_+\sqrt{k_+^2 - m_0^2}}{(-n_-)\sqrt{k_+^2 - m_0^2}} \left(\frac{n_+k_+ - \sqrt{k_+^2 - m_0^2}}{n_-k_+^2 + q_\perp^2} \right)^2 + \int_0^{m_0} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} .
\end{aligned}
\tag{A·3}
$$

We see that the first integral in the last line is identical with the second one in (A·1) and thus is finite. Carrying out the integrations, we obtain

$$
2\int_{m_0}^{\infty} dk_+ \frac{k_+ + n_+\sqrt{k_+^2 - m_0^2}}{(-n_-)\sqrt{k_+^2 - m_0^2}} \left(\frac{-n_-}{n_+k_+ + \sqrt{k_+^2 - m_0^2}} \right)^2 = \frac{\sqrt{-n_-}}{n_+}\frac{2}{q_\perp}, \tag{A·4}
$$

$$
\int_0^{m_0} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} = -\frac{\sqrt{-n_-}}{n_+}\frac{2}{q_\perp}. \tag{A·5}
$$

It follows that

$$
\int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} + \int_0^{\infty} dk_+ \frac{2n_+(n_-k_+^2 - q_\perp^2)}{(n_-k_+^2 + q_\perp^2)^2} = (\text{A·4}) + (\text{A·5}) = 0. \tag{A·6}
$$

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