Further nonperturbative calculations of the electron's magnetic moment

S. Chabysheva[a]

[a]Department of Physics, Southern Methodist University, Dallas, TX 75275

We continue to test our method of Pauli-Villars regularization on QED. Ultimately we hope to apply the method to the problem of deriving nonperturbative solutions for QCD. Developing it pursues the goal of working out a renormalization scheme which preserves Lorentz symmetries and gauge invariance in the framework of light-cone Hamiltonian approach. In a previous work a nonperturbative solution, developed as a Fock-state expansion and truncated up to one electron - one photon states, gave a good agreement with the Schwinger term of the usual Feynman series. In this paper we add one electron and two photon terms to the eigenstate. We expect to obtain good agreement with the next order contribution to the electron's anomalous magnetic moment — the Sommerfield-Peterman term. In addition, these calculations more strictly test the proposed prescripton for new singularities which arise due to the indefinite metric.

1. INTRODUCTION

Our ultimate objective is to develop nonperturbative methods which could be applied to the calculation of hadron wave function in QCD. In order to do this we have to regularize and renormalize the theory. Moreover this regularization should be carried out such way as to preserve symmetries — Lorentz and gauge invariance.

Our group has recently performed nonpreturbative calculations using the generalized Pauli-Villars (PV) method as an ultraviolet regulator of $(3+1)$-dimensional quantum field theories. In particular, Yukawa-like theories [1–5] where there are no infrared divergences and no need to protect gauge symmetry. This method explicitly preserves Lorentz invariance and sometimes, as in the case of QED, it effectively preserves gauge invariance. But for non-abelian theories or, for example, for calculations with an electron-positron loop, adding only negative metric fields is not enough — we have to include counterterms also.

To start, we introduce a sufficient number of PV fields in the Lagrangian to ensure that perturbation theory is finite. But we must also make sure that our nonpreturbative result, if expanded in a power series in the coupling constant, would agree with the usual Feynman series for processes that could be calculated perturbatively. The St.Petersburg group, Franke, Paston and Prokhvatilov, has derived what combination of PV fields and what kind of counterterms are needed to guarantee perturbative equivalence with Feynman methods for Yukawa and QCD [6,7].

In order to perform numerical calculations we also need to truncate our eigenstate. This truncation of the Fock space will break all symmetries. But we assume that the exact solution exists (and preserves all symmetries including gauge invariance) and if our approximate eigenstate is close to the exact one, even if the small difference is in the direction of maximum symmetry's breaking, it is still close to the correct answer.

To put our method to a test, and to learn more how to use it, we have to attempt a problem to which we know answer. The most recent and most severe test was a calculation of the electron's magnetic moment [8]. Here one uses the light-cone representation and a basis truncated to include only states with one electron and one electron - one photon. Once the one-electron eigenstate in the truncated subspace is found, the anomalous moment can be evaluated, without approximation, from the overlap of light-cone Fock-state wave functions [9].

Now we extend these calculations and include one electron - two photons states, and later will

add two electrons - one positron states for computation of electron-positron loop. Of course we do not expect to do better, or even as well as perturbation theory. But it is not our intent, we just wish to verify that our approximate noperturbative solution is a useful approximation to the same physics problem as that is solved so successfully by perturbation theory in QED.

2. NEW CALCULATIONS

We are working in Feynman gauge in light-front representation in 3+1 dimensions and start from the Lagrangian:

$$\sum_{i=0}^{1}\Bigg(-\frac{1}{4}(-1)^i F_i^{\mu\nu}\,F_{i,\mu\nu} + (-1)^i\overline{\psi}_i(i\gamma^\mu\partial_\mu - m_i)\psi_i + B_i\partial_\mu A_i^\mu + \frac{1}{2}B_iB_i\Bigg) - e\,\overline{\psi}\,\gamma^\mu\,\psi\,A_\mu, \quad (1)$$

where
$A^\mu = \sum_{i=0}^{1} A_i^\mu, \quad \psi = \sum_{i=0}^{1}\psi_i,$
$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial_\nu A_i^\mu.$

Here, $i = 0$ indicates the physical fields, and $i = 1$, the PV (negative-metric) fields.

The light-cone Hamiltonian is given by [8]:

$$P^- = \sum_{i,s}\int d\underline{p}\frac{m_i^2+p_\perp^2}{p^+}\,(-1)^i\,b_{i,s}^\dagger(\underline{p})\,b_{i,s}(\underline{p}) + \sum_{l,\mu}\int d\underline{k}\frac{m_l^2+k_\perp^2}{k^+}\,(-1)^l\epsilon^\mu a_l^{\mu\dagger}(\underline{k})\,a_l^\mu(\underline{k}) + \sum_{i,j,l,s,\mu}\int d\underline{p}\,d\underline{q}\Bigg\{b_{i,s}^\dagger(\underline{p})\Big[b_{j,s}(\underline{q})Q_{ij,2s}^\mu(\underline{p},\underline{q}) + b_{j,-s}(\underline{q})R_{ij,-2s}^\mu(\underline{p},\underline{q})\Big]a_{l\mu}^\dagger(\underline{q}-\underline{p}) + h.c.\Bigg\} \quad (2)$$

This operator we apply to our truncated light-cone Fock state, where for now we do not include an electron-positron pair.

$$\Phi_+(\underline{P}) = \sum_i z_i b_{i+}^\dagger(\underline{P})|0\rangle + \sum_{ijs}\int d\underline{q}\,G_{ijs}^\lambda(\underline{q})\,b_{is}^\dagger(\underline{P}-\underline{q})\,a_j^{\lambda\dagger}(\underline{q})|0\rangle + \sum_{ijks}\int d\underline{q_1}d\underline{q_2}\,D_{ijks}^{\lambda_1\lambda_2}(\underline{q_1},\underline{q_2})\frac{1}{\sqrt{1+\delta_{jk}}}\,b_{is}^\dagger \times(\underline{P}-\underline{q_1}-\underline{q_2})\,a_j^{\lambda_1\dagger}(\underline{q_1})\,a_k^{\lambda_2\dagger}(\underline{q_2})|0\rangle + \ldots \quad (3)$$

The vertices in the Hamiltonian are:

$$V_{ij\pm}^0(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \times\frac{\vec{p}_\perp\cdot\vec{q}_\perp \pm i\vec{p}_\perp\times\vec{q}_\perp + m_im_j + p^+q^+}{p^+q^+\sqrt{q^+-p^+}} \quad (4)$$

$$V_{ij\pm}^3(\underline{p},\underline{q}) = \frac{-e}{\sqrt{16\pi^3}} \times\frac{\vec{p}_\perp\cdot\vec{q}_\perp \pm i\vec{p}_\perp\times\vec{q}_\perp + m_im_j - p^+q^+}{p^+q^+\sqrt{q^+-p^+}} \quad (5)$$

$$V_{ij\pm}^1(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \times\frac{p^+\left(q^1\pm iq^2\right)+q^+\left(p^1\mp ip^2\right)}{p^+q^+\sqrt{q^+-p^+}} \quad (6)$$

$$V_{ij\pm}^2(\underline{p},\underline{q}) = \frac{e}{\sqrt{16\pi^3}} \times\frac{p^+\left(q^2\mp iq^1\right)+q^+\left(p^2\pm ip^1\right)}{p^+q^+\sqrt{q^+-p^+}} \quad (7)$$

$$U_{ij\pm}^0(\underline{p},\underline{q}) = \frac{\mp e}{\sqrt{16\pi^3}} \times\frac{m_j\left(p^1\pm ip^2\right)-m_i\left(q^1\pm iq^2\right)}{p^+q^+\sqrt{q^+-p^+}} \quad (8)$$

$$U_{ij\pm}^3(\underline{p},\underline{q}) = \frac{\pm e}{\sqrt{16\pi^3}} \times\frac{m_j\left(p^1\pm ip^2\right)-m_i\left(q^1\pm iq^2\right)}{p^+q^+\sqrt{q^+-p^+}} \quad (9)$$

$$U^1_{ij\pm}(\underline{p},\underline{q}) = \frac{\pm e}{\sqrt{16\pi^3}}\,\frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}} \tag{10}$$

$$U^2_{ij\pm}(\underline{p},\underline{q}) = \frac{ie}{\sqrt{16\pi^3}}\,\frac{m_i q^+ - m_j p^+}{p^+ q^+ \sqrt{q^+ - p^+}} \tag{11}$$

The wave functions that define our eigenstate must satisfy the coupled system of equations that results from mass-squared eigenvalue problem: $P^+P^-\Phi_+ = M^2\Phi_+$.

The first three coupled equations are:

$$\begin{aligned}
&m_i^2 z_i + \sum_{i',j,\lambda'} (-1)^{i'+j} P^+ \int^{P+} d\underline{q}\Big\{ G^{\lambda'}_{i'j-}(\underline{q}) \\
&\times \big[V^{\lambda'}_{+} + (\underline{P}-\underline{q},\underline{q}) + V^{\lambda'*}_{-}(\underline{P},\underline{q})\big] + G^{\lambda'}_{i'j+}(\underline{q}) \\
&\quad \times \big[U^{\lambda'}_{i'}(\underline{P}-\underline{q},\underline{q}) + U^{\lambda'}_{i}(\underline{P},\underline{q})\big]\Big\} = M^2 z_i, \quad (12)
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{m_i^2+q_\perp^2}{1-y} + \frac{\mu_j^2+q_\perp^2}{y}\right] G^{\lambda}_{ijs}(\underline{q}) + \sum_{i'} (-1)^{i'} \\
&\times \Big\{ z_{i'}\delta_{s,-}\big[V^{\lambda *}_{+}(\underline{P}-\underline{q},\underline{q}) + V^{\lambda}_{-}(\underline{P},\underline{q})\big] \\
&+ z_{i'}\delta_{s,+}\big[U^{\lambda}_{i}(\underline{P}-\underline{q},\underline{q}) + U^{\lambda}_{i'}(\underline{P},\underline{q})\big]\Big\} \\
&+ 2\sum_{i',k,\lambda'} \frac{(-1)^{i'+k}}{\sqrt{1+\delta_{jk}}} P^+ \int^{P^+-q^+} d\underline{q}'\Big\{ D^{\lambda\lambda'}_{i'jk,-s}(\underline{q},\underline{q}') \\
&\times \big[V^{\lambda'}_{2s}(\underline{P}-\underline{q}-\underline{q}',\underline{q}') + V^{\lambda'*}_{-2s}(\underline{P}-\underline{q},\underline{q}')\big] \\
&+ D^{\lambda\lambda'}_{i'jks}(\underline{q},\underline{q}')\big[U^{\lambda'}_{i'}(\underline{P}-\underline{q}-\underline{q}',\underline{q}') + U^{\lambda'}_{i}(\underline{P}-\underline{q},\underline{q}')\big]\Big\} \\
&\qquad = M^2 G^{\lambda}_{ijs}(\underline{q}) \quad (13)
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{m_i^2+(\vec{q}_{1\perp}+\vec{q}_{2\perp})^2}{1-y_1-y_2} + \frac{\mu_j^2+q_{1\perp}^2}{y_1} + \frac{\mu_k^2+q_{2\perp}^2}{y_2}\right] \\
&\times D^{\lambda_1\lambda_2}_{ijks}(\underline{q_1},\underline{q_2}) + \sum_{i'} (-1)^{i'} \frac{\sqrt{1+\delta_{jk}}}{2} P^+ \\
&\times \Big\{ G^{\lambda_1}_{i'j,-s}(\underline{q_1})\big[V^{\lambda_2 *}_{-2s}(\underline{P}-\underline{q_1}-\underline{q_2},\underline{q_2}) \\
&+ V^{\lambda_2}_{2s}(\underline{P}-\underline{q_1},\underline{q_2})\big] + G^{\lambda_1}_{i'js}(\underline{q_1})\big[U^{\lambda_2}_{i}(\underline{P}-\underline{q_1}-\underline{q_2},\underline{q_2}) \\
&+ U^{\lambda_2}_{i'}(\underline{P}-\underline{q_1},\underline{q_2})\big] + G^{\lambda_2}_{i'k,-s}(\underline{q_2})\big[V^{\lambda_1 *}_{-2s}(\underline{P}-\underline{q_1}-\underline{q_2},\underline{q_1}) \\
&+ V^{\lambda_1}_{2s}(\underline{P}-\underline{q_2},\underline{q_1})\big] + G^{\lambda_2}_{i'ks}(\underline{q_2})\big[U^{\lambda_1}_{i}(\underline{P}-\underline{q_1}-\underline{q_2},\underline{q_1}) \\
&+ U^{\lambda_1}_{i'}(\underline{P}-\underline{q_2},\underline{q_1})\big]\Big\} + \ldots \\
&\qquad = M^2 D^{\lambda_1\lambda_2}_{ijks}(\underline{q_1},\underline{q_2}) \quad (14)
\end{aligned}$$

And a diagrammatic representation of this equations are:

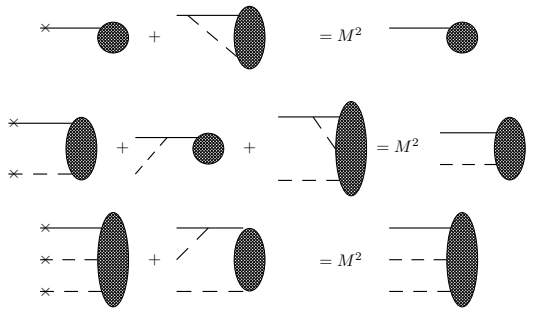

Figure 1. Diagrammatic representation of the first three coupled equations for the wave functions of the dressed electron state. The large blobs represent wave functions and the crosses represent light-cone energies. The solid line indicates the electron constituent, and the dashed lines correspond to photons.

These coupled equations can be reduced to 32 equations for the two-particle amplitudes. They have the form:

$$\left[M^2-\frac{m_i^2+q_\perp^2}{1-y}-\frac{\mu_j^2+q_\perp^2}{y}\right]G^\lambda_{ijs}(y,q_\perp)$$
$$=\frac{e^2}{8\pi^3}\sum_l I_{ijl}(y,q_\perp)\,G^\lambda_{ljs}(y,q_\perp)$$
$$+\frac{e^2}{4\pi^2}\sum_{n,k,s',\lambda'}\int_0^1 dy'\int_0^{+\infty} q'_\perp dq'_\perp$$
$$\times J^{(0)\lambda\lambda'}_{ijs,nks'}(y,q_\perp;y',q'_\perp)\,G^{\lambda'}_{nks'}(y',q'_\perp)$$
$$+\frac{e^2}{8\pi^3}\sum_{n,k,s',\lambda'}\int_0^{1-y} dy'\int_0^{+\infty} q'_\perp dq'_\perp$$
$$\times J^{(2)\lambda\lambda'}_{ijs,nks'}(y,q_\perp;y',q'_\perp)\,G^{\lambda'}_{nks'}(y',q'_\perp) \quad (15)$$

Here I_{ija} is a computable self-energy and $J^{(n)}_{ijs,abs'}$ is the kernel due to n-boson intermediate states.

One of the 2-boson kernels, for example, looks like:

$$J^{(2)[-](-)}_{ij+\frac{1}{2},nk-\frac{1}{2}}(y,q_\perp;y',q'_\perp)=\sum_l\frac{(-1)^{l+n+k}}{\sqrt{yy'}}$$
$$\times\int_0^{2\pi}d\varphi'\frac{-1}{D_{ljk}+Fcos(\varphi-\varphi')}\frac{1}{(1-y')(1-y-y')^2}$$
$$\times\left\{e^{-i(\varphi+\varphi')}\Big[\big(q_\perp e^{i\varphi}+q'_\perp e^{i\varphi'}\big)\big(q_\perp m_n e^{i\varphi'}\right.$$
$$\left.+q'_\perp e^{i\varphi}(m_n-m_l)\big)\Big]+\frac{\Big[m_l(1-y)-m_i(1-y-y')\Big]}{1-y}\right.$$
$$\left.\times\Big[q_\perp q'_\perp+e^{i(\varphi+\varphi')}(m_l m_n+{q'_\perp}^2)\Big]\right\}, \quad (16)$$

where
$D_{ljk}=$
$$\frac{m_l^2+q_\perp^2+{q'_\perp}^2}{1-y-y'}+\frac{\mu_j^2+q_\perp^2}{y}+\frac{\mu_k^2+{q'_\perp}^2}{y'}-M^2,$$
$$F=\frac{2q_\perp q'_\perp}{1-y-y'}.$$

We solve this eigenvalue problem numerically by converting the integral equations into a discrete matrix equation using Gauss-Legendre quadrature. Then we will apply the Lanczos diagonalization scheme developed in [3].

To succeed in this we first have to learn how to do the integrals with singularities. They arise in the terms which contain only physical fields. The denominator looks like $-M^2y(1-y)+m^2y+\mu^2(1-y)+q^2$, where M is a physical mass, m is the bare mass of electron and μ is a photon mass. When the bare mass is less than the physical mass, which is the case in QED, the denominator could be zero. In previous work [8] we used principal-value prescription for this singularity but we did not need to perform such a huge numerical calculation there. Now we employ some subtle procedure to do integrals with singularities. We divided the interval of integration over transverse momentum into two parts: first, from zero to twice the location of the pole point, and second, from this point to infinity. When the number of points of the quadrature is taken to be even, the divergence from the pole will be cancelled out. This ensures us that applying the discretization to the integral equations will give the eigenstates with the desired accuracy.

3. ELECTRON-POSITRON LOOP

The next step that we wish to take is to include another portion of representation space — two electron and one positron states. Then we will have an electron-positron loop and this will require further extension of the method.

As was discussed in the work [8], there are a few problems in our approach that must be solved in order to produce successful calculations. One of them is the problem of uncanceled divergences, which always arise when we truncate Fock space. We will not go into particulars here, the detailed description could be found in [8]. The solution of this problem is to keep the Pauli-Villars masses finite. More precisely, to keep the Pauli-Villars photon mass finite while taking the Pauli-Villars electron mass to infinity. This is the best choice, because with the presence of negative metric electrons, we have flavor changing currents which violate gauge invariance. Taking the Pauli-Villars electron mass to infinity will restore the gauge symmetry. But only the Pauli-Villars electrons regularize the electron-positron loop and we must keep its mass finite. When we do so, we break the

gauge invariance and have to add counter terms in Lagrangian to correct this.

4. CONCLUSION

The generalized Pauli-Villars method produces a satisfactory answer for the electron's anamolous magnetic moment when the Fock space is truncated to one electron and one electron - one photon states. We are extending these calculations and include one electron - two photons states and two electrons - one positron states. This extended computation will further test the accuracy that method could achieve and will further check our proposed prescription for the singularities that occur in nonperturbative calculations but not in perturbation theory.

The coupled integral equations have been derived and the method for properly treating the singularities numerically has been found. Currently the programming of the code to make actual numerical calculations is being performed. Later we shall also add states with two electrons - one positron. This extension will require adding counter terms to Lagrangian.

ACKNOWLEDGMENTS

This work is carried out in collaboration with G. McCartor and J.R. Hiller.

REFERENCES

1. S.J. Brodsky, J.R. Hiller, and G. McCartor, Phys. Rev. D **58** (1998), 025005.
2. S.J. Brodsky, J.R. Hiller, and G. McCartor, Phys. Rev. D **60** (1999), 054506.
3. S.J. Brodsky, J.R. Hiller, and G. McCartor, Phys. Rev. D **64** (2001), 114023.
4. S.J. Brodsky, J.R. Hiller, and G. McCartor, Ann. Phys. **296** (2002), 406.
5. S.J. Brodsky, J.R. Hiller, and G. McCartor, Ann. Phys. **305** (2003), 266.
6. S.A. Paston and V.A. Franke, Theor. Math. Phys. **112** (1997), 1117 [Teor. Mat. Fiz. **112** (1997), 399] [arXiv:hep-th/9901110].
7. S.A. Paston, V.A. Franke, and E.V. Prokhvatilov, Theor. Math. Phys. **120** (1999), 1164 [Teor. Mat. Fiz. **120** (1999), 417] [arXiv:hep-th/0002062].
8. S.J. Brodsky, V.A. Franke, J.R. Hiller, G. McCartor, S.A. Paston, and E.V. Prokhvatilov, Nucl. Phys. B **703** (2004), 333 [arXiv:hep-ph/0406325].
9. S.J. Brodsky, S.D. Drell, Phys. Rev. D **22** (1980), 2236.