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Regularization, Renormalization, and Dimensional Analysis:
Dimensional Regularization meets Freshman E&M[0]

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We illustrate the dimensional regularization technique using a simple problem from elementary electrostatics. We contrast this approach with the cutoff regularization approach, and demonstrate dimensional regularization preserves the translational symmetry. We then introduce a Minimal Subtraction (MS) and a Modified Minimal Subtraction ($\overline{MS}$) scheme to renormalize the result. Finally, we consider dimensional transmutation as encountered in the case of compact extra-dimensions.

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I. DIMENSIONAL REGULARIZATION

A. Introduction and Motivation

In 1999, Gerardus 't Hooft and Martinus J.G. Veltman received the Nobel Prize in Physics[1] "for elucidating the quantum structure of electroweak interactions in physics." In particular, they demonstrated that the non-abelian electroweak theory could be consistently renormalized to yield unique and precise predictions.

An key ingredient for their demonstration was the development of the dimensional regularization technique. That is, instead of working in precisely D=4 space-time dimensions, they generalized the dimension to be a continuous variable so they could compute the theory in D=4.01 or D=3.99 dimensions.

An important property of the dimensional regularization is that it respects gauge and Lorentz symmetries;[2] this is in contrast to the older regularization schemes (e.g., cutoff schemes, etc.) which violates these symmetries. The symmetries of the electroweak theory play an critical role in determining the dynamics of the particles and their interactions. Because it respects these symmetries, dimensional regularization has become a essential tool for the calculation of field theories.

While dimensional regularization is an powerful and elegant technique, most examples and applications of di-

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[0] This work is based on lectures presented a the CTEQ Summer Schools on QCD Analysis and Phenomenology. http://www.cteq.org
[1] See Ref. [5] and also the webpage: http://nobelprize.org/nobel_prizes/physics/laureates/1999/
[2] Note, for chiral symmetries there are some subtle difficulties that must be handled carefully. In particular, the properties of the parity operator are dependent on the dimensionality of space-time.

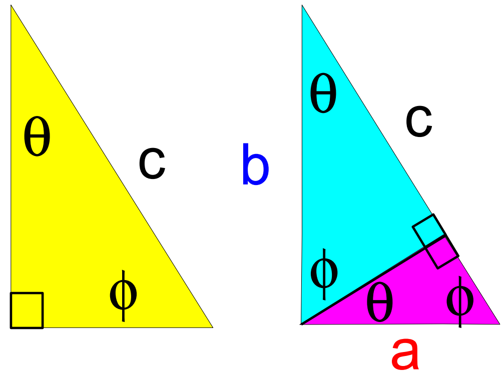

Figure 1: a) A right triangle specified by angles $\{\theta, \phi\}$ and hypotenuse c. b) The same triangular area can be described by two similar triangles of hypotenuse a and b.

mensional regularization use complex higher-order Quantum Field Theory (QFT) calculations involving gauge and Lorentz symmetries.

In the present paper, we will apply the dimensional regularization method to a problem from an elementary undergraduate physics course, namely the electric potential of an infinite line of charge.[2–4] The example is simple enough for the undergraduate to understand, yet contains many of concepts we encounter in a true QFT calculation. We will contrast the symmetry-preserving dimensional regularization approach with a symmetry-violating cutoff approach.

Imagining a variable number of dimensions can be a productive exercise. To explain the weak nature of the gravitational force physicists have recently posited the existence of "Extra Dimensions." After we consider space-time dimensions in the neighborhood of $D = 4$, we briefly consider wider excursions of $D = 4, 5, 6, \ldots$ and the implications for electric potential and field.

II. DIMENSION ANALYSIS: THE PYTHAGOREAN THEOREM

To illustrate utility of dimensional regularization and dimensional analysis, we warm-up with a pre-example. Our goal will be to demonstrate the Pythagorean Theorem, and our method will be dimensional analysis.

We consider the right triangle displayed in Fig. 1-a). From the Angle-Side-Angle (ASA) theorem, this can be uniquely specified using the two angles $\{\theta, \phi\}$ and the hypotenuse c. We now construct a formula for the area of the triangle, A_c, using only these variables: $\{c, \theta, \phi\}$. Note that c has dimensions of length, and $\{\theta, \phi\}$ are dimensionless. From dimensional analysis, the area of the triangle must have dimensions of length squared. As c is the only dimensional quantity, the formula for A_c must be of the form:

$$A_c = c^2 f(\theta, \phi) \tag{1}$$

where $f(\theta, \phi)$ is an unknown *dimensionless* function. Note that $f(\theta, \phi)$ cannot depend on the length c as this would spoil the dimensionless nature of $f(\theta, \phi)$.

We now observe that we can divide the original triangle of Fig. 1-a) into two similar triangles of hypotenuse a and b as displayed in Fig. 1-b). Again, using the ASA theorem, we can represent the area of these triangles, A_a and A_b, in terms of the variables $\{a, \theta, \phi\}$ and $\{b, \theta, \phi\}$, respectively. Again from dimensional considerations, these areas must be proportional to a^2 and b^2; thus, we obtain:

$$A_a + A_b = a^2 f(\theta, \phi) + b^2 f(\theta, \phi) \tag{2}$$

Because all three triangles are similar, their areas are described by the same $f(\theta, \phi)$. It is important to note that the function $f(\theta, \phi)$ is *universal*, dimensionless, and scale-invariant.

Finally, we use "conservation of area" to obtain our result. Specifically, since the area of the original triangle A_c is equal to the sum of the combined A_a and A_b,

$$A_a + A_b \;=\; A_c \tag{3}$$

We can substitute Eqs. 1 and 2 to obtain our desired result:

$$\begin{aligned} a^2 f(\theta, \phi) + b^2 f(\theta, \phi) \;&=\; c^2 f(\theta, \phi) \\ a^2 + b^2 \;&=\; c^2 \end{aligned} \tag{4}$$

The last equation is, of course, the Pythagorean Theorem. Clearly, there are much simpler methods to prove this theorem; however, this method does illustrate the power of the dimensional analysis approach. Additionally, we gain a new perspective on the Pythagorean Theorem in this proof as it is linked to conservation of area. There are instances, such as renormalizable field theory, where use of dimensional analysis tools are essential to making certain calculations tractable. The following example will illustrate some of these features.

III. AN INFINITE LINE OF CHARGE

A. Statement of the problem

For our next example we consider the calculation of the electric potential V for the case of an infinite line of charge with linear charge density $\lambda = Q/L$. The contribution to the electric potential from an infinitesimal charge dQ is given by:[3]

[3] We will use MKS units here so that our results reduce to the usual undergraduate textbook expressions.

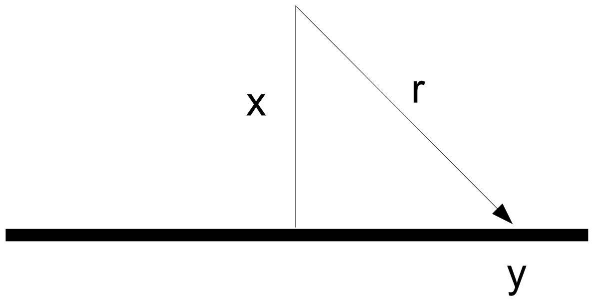

Figure 2: Coordinate system for an infinite line of charge running in the y-direction. We compute the potential $V(x)$ at a fixed perpendicular distance x from the line of charge. The distance to the element of charge dQ is $r = \sqrt{x^2+y^2}$.

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r}$$

We choose our coordinate system (cf., Fig. 2) such that x specifies the perpendicular distance from the wire, y is the coordinate along the wire, and $r = \sqrt{x^2+y^2}$. Given $\lambda = Q/y$ we have $dQ = \lambda dy$ and can integrate along the length of the wire to obtain:

$$V(x) \;=\; \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} dy\;\frac{1}{\sqrt{x^2+y^2}} = \infty \qquad (5)$$

Unfortunately, this integral is logarithmically divergent and a we obtain an infinite result.

B. Scale invariance:

If we take a closer look at this integral, we notice that it is scale invariant; that is, if we rescale the argument x by a constant factor k, $(x \to k\,x)$, the result is invariant.

$$\begin{aligned} V(k\,x) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} dy\;\frac{1}{\sqrt{(k\,x)^2+y^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} d(y/k)\;\frac{1}{\sqrt{x^2+(y/k)^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty} dz\;\frac{1}{\sqrt{x^2+z^2}} \qquad (6) \\ &= V(x) \qquad (7) \end{aligned}$$

In the above we have implemented the rescaling $z = y/k$; since both y and z are dummy variables and the integration limits are infinite, the integral is unchanged. This equation implies the notable result that $V(x)$ is scale invariant:

$$V(x_1) \;=\; V(x_2) \qquad (8)$$

At first glance, this result appears to be a disaster since the usual purpose of the electric potential is to compute the work W via the formula

$$W/Q = \Delta V = V(x_2) - V(x_1)$$

or to compute the electric field via

$$\vec{E} = -\vec{\nabla} V$$

As Eq. 8 suggests $V(x_2) - V(x_1) = 0$, this implies that our attempts to compute the work W or the electric field $\vec{E}$ will be meaningless.

We now understand why it is fortunate that $V(x)$ is infinite as infinite numbers have some unusual properties. For example, for a finite constant c we can write (schematically) $\infty + c = \infty$ which implies $\infty - \infty = c$. We now understand that even though we have $V(x_1) = V(x_2)$, because these quantities are infinite we can still find that the difference is non-zero: $V(x_2) - V(x_1) \neq 0$. The challenge is that the difference of two infinite quantities is ambiguous; that is, how can tell if $\infty - \infty = c_1$ or $\infty - \infty = c_2$ is the correct physical result?

The solution is that we must regularize the infinite quantities so that we can uniquely extract the difference.

IV. CUTOFF REGULARIZATION:

A. Cutoff Regularization Computation

We will first regularize the integral using a simple cutoff method. That is, instead of considering an infinite wire, we will compute the potential for a finite wire of length $2L$. In this instance, the potential becomes:[4]

$$\begin{aligned} V(x) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{+L} dy \frac{1}{\sqrt{x^2+y^2}} \\ &= \frac{\lambda}{4\pi\epsilon_0}\mathrm{Log}\left[\frac{+L+\sqrt{L^2+x^2}}{-L+\sqrt{L^2+x^2}}\right] \qquad (9) \end{aligned}$$

We make the following observations.

- The result is finite.
- In addition to the physical length scale x, $V(x)$ depends on an artificial regulator L.
- We cannot remove the regulator L without $V(x)$ becoming singular.
- The result for $V(x)$ violates a symmetry of the original problem—translation invariance.

[4] For simplicity, we will calculate the potential at the mid-point of the wire; the general case is more complicated algebraically, but yields the same result in the $L \to \infty$ limit.

B. Computation of E and δV

Even though $V(x)$ depends on the artificial regulator L, we observe that all physical quantities are independent of this regulator in the limit $L \to \infty$. Specifically, for the electric field we have:

$$E(x) = \frac{-\partial V(x)}{\partial x} = \frac{\lambda}{2\pi\epsilon_0 x}\frac{L}{\sqrt{L^2+x^2}} \underset{L\to\infty}{\longrightarrow} \frac{\lambda}{2\pi\epsilon_0 x}$$

and for the potential difference (proportional to the electric work W) we have:

$$\delta V = V(x_1) - V(x_2) \underset{L\to\infty}{\longrightarrow} \frac{\lambda}{4\pi\epsilon_0}\mathrm{Log}\left[\frac{x_2^2}{x_1^2}\right] \quad (10)$$

C. Broken translational symmetry:

Notice that the presence of the cutoff L breaks the translation symmetry of the original problem. That is, for a truly infinite wire, our position in the y-direction is inconsequential; however, for a finite wire this is no longer the case. Specifically, if we shift our y-position by a constant c to $y \to y' = y + c$, our result becomes:

$$\begin{aligned} V(x) &= \frac{\lambda}{4\pi\epsilon_0}\int_{-L+c}^{+L+c} dy \frac{1}{\sqrt{x^2+y^2}} \quad (11)\\ &= \frac{\lambda}{4\pi\epsilon_0}\mathrm{Log}\left[\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right] \end{aligned}$$

Clearly we have lost the translation invariance $y \to y' = y + c$.

While preserving symmetries is not of paramount importance in this simple example, it is essential for certain field theory calculations. We now repeat the this calculation, but instead using dimensional regularization which will preserve the translational symmetry.

D. Recap

In summary, we find that our problem is solved at the expense of 1) an extra scale L which serves to both regulates the infinities and provide an auxiliary length scale, and 2) a broken symmetry—translational invariance.

V. DIMENSIONAL REGULARIZATION

A. Generalization to arbitrary dimension

The central idea of dimensional regularization is to compute $V(x)$ in n-dimensions where n is not necessarily an integer. We can generalize the integration of Eq. 5 by replacing the one-dimensional integration $dy = d^1y$ by the general n-dimension result:

n	$\Omega(n)$	$\Gamma(n/2)$
1	2	$\sqrt{\pi}$
2	2π	1
3	4π	$\frac{\sqrt{\pi}}{2}$
4	$2\pi^2$	1

Table I: Angular integration measure as a function of dimension n. We recognize $\Omega(1)$ as the 1-dimensional integration measure of $\int_{-1}^{+1} dr$, $\Omega(2)$ as the circumference of the unit circle, $\Omega(3)$ as the surface area of the unit sphere, and $\Omega(4)$ as the 3-volume of the 4-dimensional unit hypercube.

$$dy \longrightarrow d^n y = \frac{d\Omega_n}{2}\ y^{n-1}\ dy \quad (12)$$

where the angular integration measure is given by

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \quad (13)$$

It is instructive to verify that Ω_n yields the expected result for integer dimensions as tabulated in Table I.

B. Computation V in arbitrary dimensions

The generalized formula for $V(x)$ now reads:

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\int_0^{+\infty} d\Omega_n\ \frac{y^{n-1}}{\mu^{n-1}}\ \frac{dy}{\sqrt{x^2+y^2}} \quad (14)$$

Note that we have introduced an auxiliary scale factor of μ^{n-1}, where μ has units of length, to ensure $V(x)$ has the correct mass dimension.[5] Replacing $n = 1 - 2\epsilon$ to facilitate expanding about $n = 1$ we obtain

$$\begin{aligned} V(x) &= \frac{\lambda}{4\pi\epsilon_0}\frac{\Gamma\left[\frac{1-n}{2}\right]}{\left(\frac{x}{\mu}\sqrt{\pi}\right)^{1-n}}\\ &= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\ \frac{\Gamma[\epsilon]}{\pi^{\epsilon}}\right) \quad (15) \end{aligned}$$

We make the following observations about the dimensionally regularized result.

[5] Since the factor $\lambda/(4\pi\epsilon_0)$ has units of potential, the integral must be dimensionless. Also note we have changed the integration limits from $[-\infty, +\infty]$ to $[0, +\infty]$, and the compensating factor of 2 cancels the factor of 2 in $d\Omega_n/2$.

- $V(x)$ depends on an artificial regulator ϵ which is dimensionless.
- $V(x)$ depends on an auxiliary scale μ which has dimensions of length.
- If we remove either the regulator ϵ or the auxiliary scale μ then $V(x)$ will become ill-defined.
- The dimensional regularization preserves the translation invariance of the original problem.

It is interesting to contrast this result with the cutoff regularization method where L serves as both the regulator and the auxiliary scale.

C. Computation of E and δV

As before, we observe that all physical quantities are independent of both the regulator ϵ and the auxiliary scale μ. For the potential difference we find

$$\delta V \;=\; V(x_1) - V(x_2) \underset{\epsilon \to 0}{\longrightarrow} \frac{\lambda}{4\pi\epsilon_0} \mathrm{Log}\left[\frac{x_2^2}{x_1^2}\right] \qquad (16)$$

and for the electric field we obtain:

$$\begin{aligned} E \;&=\; \frac{-\partial V(x)}{\partial x} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\mu^{2\epsilon}\Gamma[\epsilon]}{\pi^{\epsilon}x^{1+2\epsilon}}\right] \\ &\underset{\epsilon\to 0}{\longrightarrow}\; \frac{\lambda}{2\pi\epsilon_0}\,\frac{1}{x} \end{aligned} \qquad (17)$$

D. The Renormalization Group Equation

The fact that the physical observables are independent of the un-physical auxiliary scale μ is simply a consequence of the renormalization group equation:

$$\mu \frac{d\sigma}{d\mu} \;=\; 0 \qquad (18)$$

where σ represents any physical observable. Thus, the renormalization group equation implies that the electric field $\vec{E} = \vec{\nabla}V$ and the work $W = \delta V$ are also independent of the μ scale:

$$\mu \frac{dE}{d\mu} = 0 \qquad\qquad \mu \frac{dW}{d\mu} = 0$$

E. Recap

In conclusion we find that the problem for $V(x)$ is solved at the expense of an artificial regulator ϵ and an auxiliary scale μ. Also note the regulator ϵ and auxiliary scale μ are separate entities in contrast to the cutoff regularization method where the length L plays both roles. Additionally, translational invariance symmetry is preserved; the fact that dimensional regularization respects symmetry makes this technique essential for field theory calculations involving gauge symmetries and Lorentz symmetries.

VI. RENORMALIZATION

Having demonstrated two separate methods to regularize the infinities that enter the calculation of $V(x)$, we now turn to renormalization.

While physical quantities such as the work $W \sim \delta V$ and the electric field $\vec{E} \sim -\vec{\nabla}V$ are derived from $V(x)$, the potential itself is not a physical quantity. In particular, we can shift the potential by a constant c, $V \to V+c$, and the physical quantities will be unchanged.

To illustrate this point, let's expand $V(x)$ of Eq. 15 in powers of ϵ:

$$V(x) \;=\; \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right] + \mathcal{O}(\epsilon)\right]$$

Let us now invent a Minimal Subtraction (MS) prescription. I have the freedom to shift $V(x)$ by a constant, and I design this to eliminate the $1/\epsilon$ term:

$$V_{MS}(x) \;=\; \frac{\lambda}{4\pi\epsilon_0}\left[\qquad \ln\left[\frac{e^{-\gamma_E}}{\pi}\right] + \ln\left[\frac{\mu^2}{x^2}\right] + \mathcal{O}(\epsilon)\right]$$

I can go even further and invent a Modified Minimal Subtraction ($\overline{MS}$) prescription to eliminate the $\ln[e^{-\gamma_E}/\pi]$ term as well:

$$V_{\overline{MS}}(x) \;=\; \frac{\lambda}{4\pi\epsilon_0}\left[\qquad\qquad\qquad \ln\left[\frac{\mu^2}{x^2}\right] + \mathcal{O}(\epsilon)\right]$$

After renormalization we can remove the regulator ($\epsilon \to 0$), but not the auxiliary scale μ; recall that without an auxiliary scale to generate a dimensionless ratio μ/x we could not have any substantive x-dependence.

In addition to the μ-dependence we will also have renormalization scheme dependence in $V(x)$. However, physical observables *must* be independent of the auxiliary scale μ and the particular renormalization scheme. For example, the computed potential differences yield identical results when calculated consistently in a single renormalization scheme.

$$V_{MS}(x_1) - V_{MS}(x_2) \;=\; \delta V = V_{\overline{MS}}(x_1) - V_{\overline{MS}}(x_2)$$

Here, the results of the Minimal Subtraction (MS) and the Modified Minimal Subtraction ($\overline{MS}$) are identical for *physical* quantities.

D_{eff}	$E(r)$	$V(r)$	Example
3	$\frac{1}{r^2}$	$\frac{1}{r}$	Point charge
2	$\frac{1}{r^1}$	$\ln r$	Line charge
1	$\frac{1}{r^0}$	r	Sheet charge

Table II: Example charge configurations that illustrate $D_{eff} = \{3, 2, 1\}$ effective dimensions.

However, if you mix renormalization schemes inconsistently you will obtain non-sensible results that *are* dependent on the choice of scheme:[6]

$$V_{\overline{MS}}(x_1) - V_{MS}(x_2) \;\neq\; \delta V \neq V_{MS}(x_1) - V_{\overline{MS}}(x_2)$$

A. Connection to QFT

This elementary problem of the infinite line charge contains all the key concepts of the dimensional regularization and renormalization that we encounter in the full QFT radiative calculations. For example, in the radiative Quantum Chromodynamics (QCD) calculation of the Drell-Yan process ($q\bar{q} \to \gamma^* \to \mu^+\mu^-$) we encounter the following infinite expression:[7]

$$\begin{aligned}\frac{D(\epsilon)}{\epsilon} &= \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\sim \frac{1}{\epsilon} - \ln\left(\frac{e^{+\gamma_E}}{4\pi}\right) + \ln\left(\frac{\mu^2}{Q^2}\right)\end{aligned}$$

In this equation, Q represents the characteristic energy scale; this is the independent variable that is analogous to x in our example. While this is for a 4-dimensional QCD calculation, the structure of the divergent term is remarkably similar to our simple one-dimensional example above. For the QCD calculation, the Minimal Subtraction (MS) prescription for this Drell-Yan calculation eliminates the $1/\epsilon$ term, and the Modified Minimal Subtraction ($\overline{MS}$) prescription for this Drell-Yan calculation eliminates the $1/\epsilon - \ln[e^{+\gamma_e}/(4\pi)]$ so that only the $\ln[\mu^2/Q^2]$ remains.

[6] The reader is invited to verify that the computation of the electric field $\vec{E}(x)$ in a consistent renormalization scheme yields the previous results of Eq. 17, and a inconsistent application of the schemes does not.

[7] Cf., Ref. [6], Eq. (46) and Eq. (47).

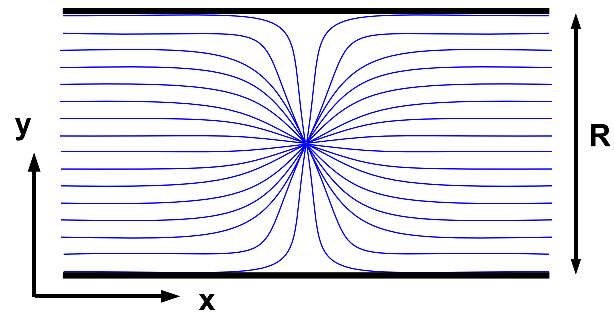

Figure 3: Electric field for a point charge confined in one infinite dimension (x) and one finite dimension (y) of scale R.

VII. EXTRA DIMENSIONS

A. E and V in arbitrary dimensions

In the above example, we used the mathematical trick of generalizing the number of integration dimensions from an integer to a continuous parameter. While we only let the dimension stray by 2ϵ, it is useful to consider more drastic shifts as in the case of "Extra-Dimensions" which have recently been hypothesized.[1, 7] In this section, we provide an example of a dimensional transmutation; that is where the effective dimension D_{eff} changes from one integer to another as we probe the system at different scales.

For example, we can generalize the r-dependence of the potential and electric field in for the case of D-dimensions as:[8]

$$V(r) \sim \frac{1}{r^{D-2}} \qquad\qquad E(r) \sim \frac{1}{r^{D-1}}$$

A quick check will verify that this reproduces the usual expressions in ordinary $D = 3$ spacial dimensions. Additionally, in 3-dimensions we can create charge distributions that mimic lower order spatial dimensions; this is illustrated in Table II. For a (zero-dimensional) point-charge in 3-dimensions, according to Gauss's law the electric field lines spread out on a surface of $D-1 = 2$ dimensions, and we observe $E(r) \sim 1/r^2$. Similarly, for a (one-dimensional) line-charge, our space is now effectively $D = 2$ dimensional; hence the electric field lines spread out on a surface of $D - 1 = 1$ dimension, and we observe $E(r) \sim 1/r$. Finally, for a (two-dimensional) sheet-charge, our space is now effectively $D = 1$ dimension; hence the electric field lines spread out on in $D - 1 = 0$ dimensions, and we observe $E(r) \sim 1/r^0 = constant$.

[8] Note, for the special case D=2 the potential $V(r)$ has a logarithmic form; see Table II for details.

B. Relation to compactified dimensions

Figure 3 displays the electric field lines for a point charge confined to one infinite dimension (x) and one finite (or compact) dimension (y) of scale R. We observe that if we examine the electric field at scales small compared to the compact dimension R ($r \ll R$), we find the the electric field lines spread out in 2 dimensions and we obtain the usual 2-dimensional result $\vec{E}(r) \sim 1/r$; conversely, if we examine the electric field at distance scales large compared to the compact dimension R ($r \gg R$), we find the 1-dimensional result $\vec{E}(r) \sim \textit{constant}$. In this example, the effective dimension of our space changes as we move from small ($D = 2$) to large length scales ($D = 1$).

VIII. CONCLUSIONS

In this paper we have computed the potential of an infinite line of charge using dimensional regularization. By contrasting this calculation with the conventional cutoff approach, we demonstrated that dimensional regularization respects the symmetries of the problem—namely, translational invariance. The dimensional regularization requires that we introduce a regulator ϵ and an auxiliary length scale μ. We then renormalized the potential to eliminate the $1/\epsilon$ singularities; this potential was finite and independent of the regulator ϵ, but it depended on the particular renormalization scheme. However, we demonstrated that all physical observables ($E, \delta V$) were scheme and scale invariant.

As this example exhibits many of the key features of dimensional regularization as applied to QFT, it provides an excellent opportunity to understand the features of this regularization method without the complications of gauge symmetries. As such, this example serves as an ideal pedagogical study.

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