Southern Methodist University

Remote Video Seminar

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The Simplicity of Scattering Amplitudes

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New Physics At Hadron Colliders

What new signals do we seek in current high-energy experiments?

- Higgs boson (electroweak symmetry breaking)
- Supersymmetry (and its breaking)
- Dark matter candidate
- Extra spatial dimensions
- $\cdots$

The relevant experiments are at the Tevatron (Fermilab) and at the Large Hadron Collider (CERN).

Tevatron at Fermilab collides protons with antiprotons at 2 TeV. Discovered the top quark in 1995. Collecting more data in “Run II” until 2009. Instantaneous luminosity $10^{32}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$.

Large Hadron Collider (LHC) at CERN expected to start test run at 450 GeV in May 2008; full 14 TeV to start mid-2008. Instantaneous luminosity 10-100 times larger than Tevatron.

LHC should see new signals:

Higgs boson must be found – or face big theoretical puzzle.

Supersymmetry arises most naturally at this scale.

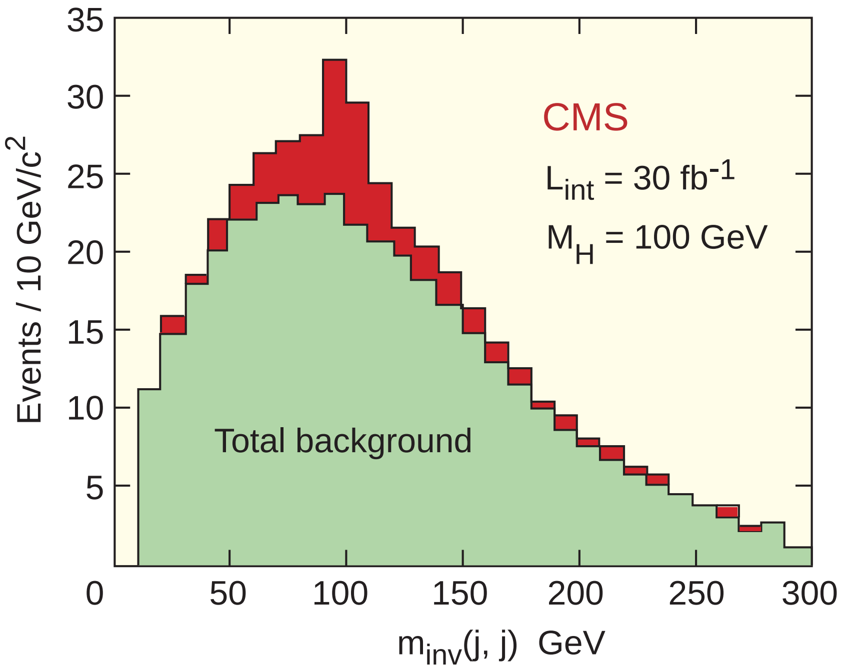
Hadron colliders come with a very large QCD background to signals of new physics.

Higgs Signal With Background

$H^0 \to b\bar{b}$ in $t\bar{t}H$

$t\bar{t}H \to \ell\ \nu\, b\, j\bar{j}\, b\, b\bar{b}$

E_t (jet) > 15 GeV



in: 90 GeV < m_{inv}(j,j) < 130 GeV

$\to$ S / B = 0.3

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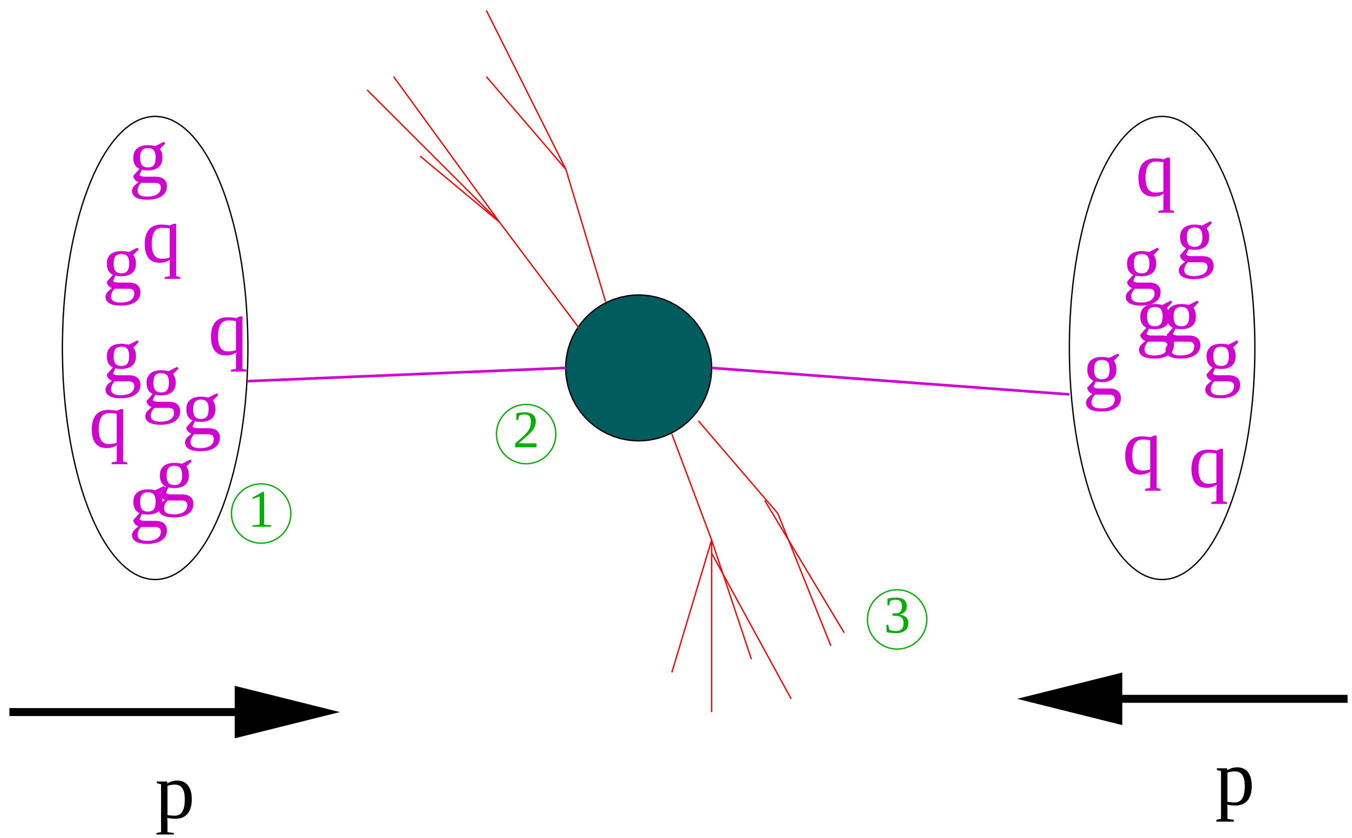
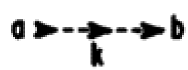
A Proton-Proton Scattering Event



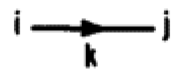
1. Parton distribution function (pdf) of quarks and gluons
2. Hard scattering from collision of 2 partons
3. Showering and hadronization

Feynman rules in QCD: Propagators and Vertices

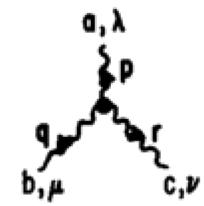
$$-i\delta^{ab}\left[\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)/k^2 + \alpha k_\mu k_\nu/k^4\right]$$



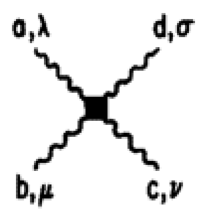
$$-i\delta^{ab}/k^2$$



$$i\delta^{ij}\not{k}/k^2$$



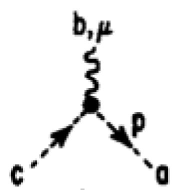
$$-gf^{abc}\left[(p-q)_\nu g_{\lambda\mu} + (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}\right]$$



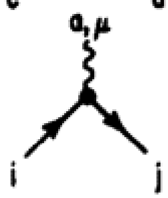
$$-ig^2 f^{abe} f^{cde} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu})$$

$$-ig^2 f^{ace} f^{bde} (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\sigma} g_{\mu\nu})$$

$$-ig^2 f^{ade} f^{cbe} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\mu} g_{\sigma\nu})$$



$$gf^{abc}p^\mu$$



$$-ig\gamma^\mu T^a_{ij}$$

(graphic: Wolfram 1978)

Too many Feynman diagrams

Already at tree level in pure gauge theory: (Weinzierl)

#gluons	Diagrams
4	4
5	25
6	220
7	2,485
8	34,300
9	559,405
10	10,525,900

These numbers are derived from intricate diagram combinatorics.

Factorial growth.

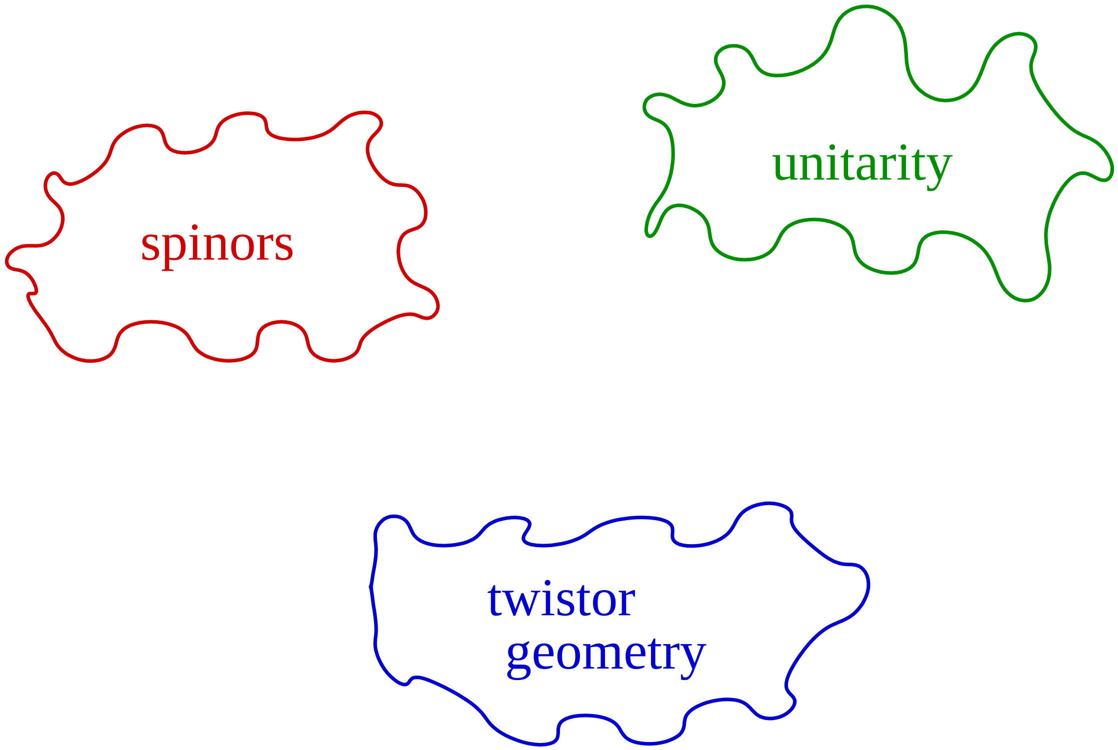
Each diagram is tedious...

Traditional approach: Feynman diagrams

In the end, one finds many cancellations ending with a gauge-symmetric result.

Symmetry offers clues for improvement. Amplitudes have remarkable structure!

Ingredients



We are discovering new perspectives.

Complex-valued momentum: find poles and branch cuts.

Construct amplitude from global properties.

Spinor-Helicity Formalism (1980's)

Quantum scattering amplitudes have elegant expressions in terms of spinors.

Unitarity (1990's)

Unitarity of the scattering matrix gives targeted information about loop amplitudes.

Twistor Space Geometry (2000's)

Amplitudes can be built from lines and planes in twistor space.

Twistor string theory and its duality to gauge theory suggested this geometry.

Helicity Amplitudes Look Nice

Gluon amplitudes at tree level:

$$A(p_1^+, p_2^+, p_3^+, p_4^+, \dots, p_n^+) = 0 \tag{1}$$

$$A(p_1^-, p_2^+, p_3^+, p_4^+, \dots, p_n^+) = 0 \tag{2}$$

Maximally Helicity Violating (MHV) Amplitudes:

$$A(p_1^-, p_2^+, p_3^-, p_4^+, \dots, p_n^+) = \frac{\langle 1\, 3\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \dots \langle n-1\; n\rangle\langle n\, 1\rangle} \tag{3}$$

(Parke, Taylor 1986; Berends, Giele 1989)

Momentum and spinors

$$(p_0, p_1, p_2, p_3) = (\frac{1}{c}E, p_x, p_y, p_z)$$

Change to spinor indices with Pauli matrices:

$$p_{a\dot{a}} = \sigma^{\mu}_{a\dot{a}}\, p_{\mu} \qquad a, \dot{a} = 1, 2$$

For a null vector (massless particle):

$$0 = p^{\mu} p_{\mu} = \det(p_{a\dot{a}}).$$

$$p_{a\dot{a}} = \begin{pmatrix} \lambda_1 \tilde{\lambda}_{\dot{1}} & \lambda_1 \tilde{\lambda}_{\dot{2}} \\ \lambda_2 \tilde{\lambda}_{\dot{1}} & \lambda_2 \tilde{\lambda}_{\dot{2}} \end{pmatrix}.$$

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}.$$

Spinors:

$$\lambda_a = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \qquad \tilde{\lambda}_{\dot{a}} = \begin{pmatrix} \tilde{\lambda}_{\dot{1}} \\ \tilde{\lambda}_{\dot{2}} \end{pmatrix}$$

Lorentz-invariant spinor products:

$$\begin{aligned} \langle \lambda\, \lambda' \rangle &\equiv \epsilon_{ab}\, \lambda^a\, \lambda'^b \\ [\tilde{\lambda}\, \tilde{\lambda}'] &\equiv \epsilon_{\dot{a}\dot{b}}\, \tilde{\lambda}^{\dot{a}}\, \tilde{\lambda}'^{\dot{b}} \end{aligned}$$

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Spinor formulas are elegant

Gluons:

$$A(p_1^-, p_2^+, p_3^-, p_4^+, \ldots, p_n^+) = \frac{\langle 1\, 3\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n-1\, n\rangle\langle n\, 1\rangle}$$

(Parke, Taylor 1986; Berends, Giele 1989)

So, there is also a purely theoretical motivation to study gauge theory amplitudes.

$A(1^-, 2^-, 3^-, 4^+, \ldots, n^+) =$

(Kosower 1990)

$$
\begin{aligned}
&\frac{1}{[n\,1][1\,2][2\,3][3\,4]\prod_{k=3}^{n}\langle k\;k+1\rangle} \\
&\times\Bigg(\frac{(1\,2)\langle 3\,4\rangle[1\,n]\langle 1|P_{1,3}|4]^2}{P_{5,1}^2}+\frac{(3\,2)\langle 1\,n\rangle[3\,4]\langle 3|P_{1,3}|n]^2}{P_{3,n-1}^2}+\frac{\langle 2\,1\rangle\langle 3\,1\rangle\langle 3\,n\rangle[3\,2][3\,4][1\,n]\langle 3|P_{1,3}|n]}{P_{3,n-1}^2} \\
&+\frac{\langle 2\,3\rangle\langle 1\,3\rangle\langle 1\,4\rangle[1\,2][3\,4][1\,n]\langle 1|P_{1,3}|4]}{P_{5,1}^2}-\langle 3\,1\rangle^2P_{4,n}^2[3\,4][1\,n]-\frac{(1\,2)(3\,2)\langle 3\,1\rangle\langle 3|P_{1,3}|n][3\,4]}{P_{3,n-1}^2} \\
&-[2\,3][2\,1][3\,4][1\,n]\times\sum_{i=5}^{n-1}\Bigg[\frac{\langle 2\,1\rangle^2\langle 1\,3\rangle\langle 3|P_{3,i-1}k_i|3\rangle}{P_{3,i-1}^2P_{3,i}^2}+\frac{\langle 2\,3\rangle^2\langle 3\,1\rangle\langle 1|P_{i+1,1}k_i|1\rangle}{P_{i+1,1}^2P_{i,1}^2} \\
&-\frac{\langle 2\,3\rangle\langle 2\,1\rangle\langle 1\,3\rangle\langle 1|P_{i+1,2}k_i|3\rangle}{P_{3,i}^2P_{i,1}^2}-\frac{\langle 2\,3\rangle\langle 2\,1\rangle^2\langle 1|P_{i+1,2}k_i|3\rangle\langle 3|P_{i,2}|2]}{P_{3,i-1}^2P_{3,i}^2P_{i,1}^2} \\
&-\frac{\langle 2\,3\rangle^2\langle 2\,1\rangle\langle 1|P_{i+1,1}k_i|3\rangle\langle 1|P_{i+1,2}|2]}{P_{3,i}^2P_{i+1,1}^2P_{i,1}^2}\Bigg]\Bigg)
\end{aligned}
$$

Twistor Space

(Penrose 1967)

$$p_\mu = (p_1, p_2, p_3, p_4) \quad \longrightarrow \quad (\lambda^1, \lambda^2, \tilde{\lambda}^{\dot{1}}, \tilde{\lambda}^{\dot{2}}) \tag{4}$$

$$\longrightarrow \quad (\lambda^1, \lambda^2, \mu^{\dot{1}}, \mu^{\dot{2}}), \tag{5}$$

with

$$\mu^{\dot{a}} \equiv -i \frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}}. \tag{6}$$

Breaks symmetry between λ and $\tilde{\lambda}$.

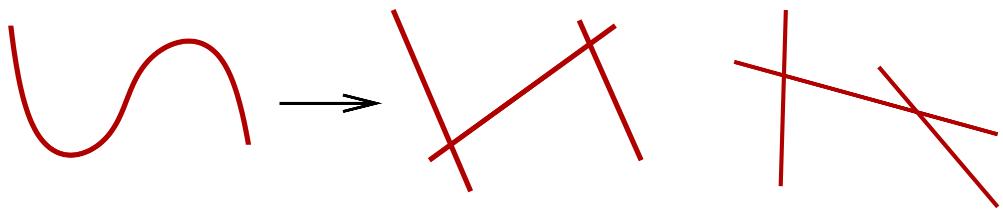
Nonlocal transformation.

Simplicity In Twistor Space

Amplitudes are localized on algebraic curves in twistor space (Witten 2003)...

$$\#\text{negative helicities} - 1 + \#\text{loops} = \text{degree of curve}$$

...and further, on intersections of lines and planes (Cachazo, Svrcek, Witten 2004).



Twistor String Theory

Witten (2003) proposed a duality between a topological string theory on twistor space, and supersymmetric Yang-Mills theory. (see also Nair, 1988)

Consequence: gauge theory amplitudes are computable in string theory.

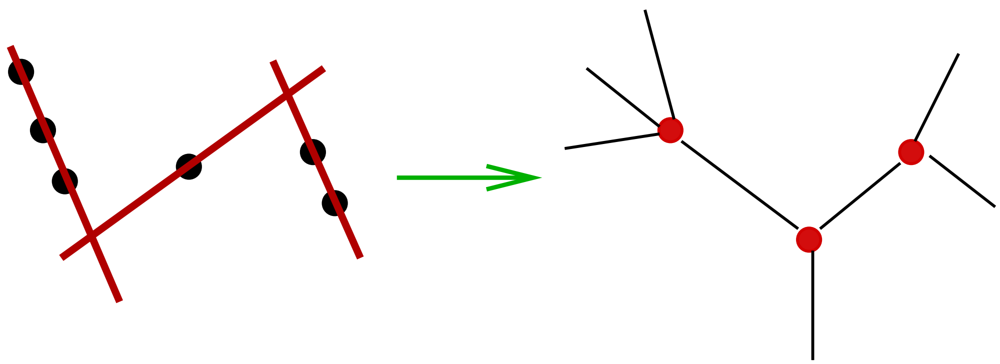
Amplitude computation was confirmed, but not improved.

However: it suggested the localization to lines and planes.

Twistor Geometry

Points in momentum space $\longleftrightarrow$ lines in twistor space.

So, the lines in twistor space correspond to vertices (points) of a Feynman-like diagram. Motivates the degeneration of the curve describing the amplitude.



MHV Diagrams are a field theory prescription with an interpretation in twistor-space geometry. MHV **amplitudes** become **vertices** of these diagrams. (Cachazo, Svrček, Witten)

Twistor-geometric “MHV Diagrams” extend to quarks, electroweak vector boson currents, QED, Higgs,...and even loop amplitudes.

(Georgiou, Glover, Khoze; Dixon, Glover, Khoze; Badger, Glover, Khoze; Schwinn, Weinzierl; Bern, Forde, Kosower, Mastrolia; Bedford, Brandhuber, Spence, Travaglini; Quigley, Rozali; Ozeren, Stirling;...)

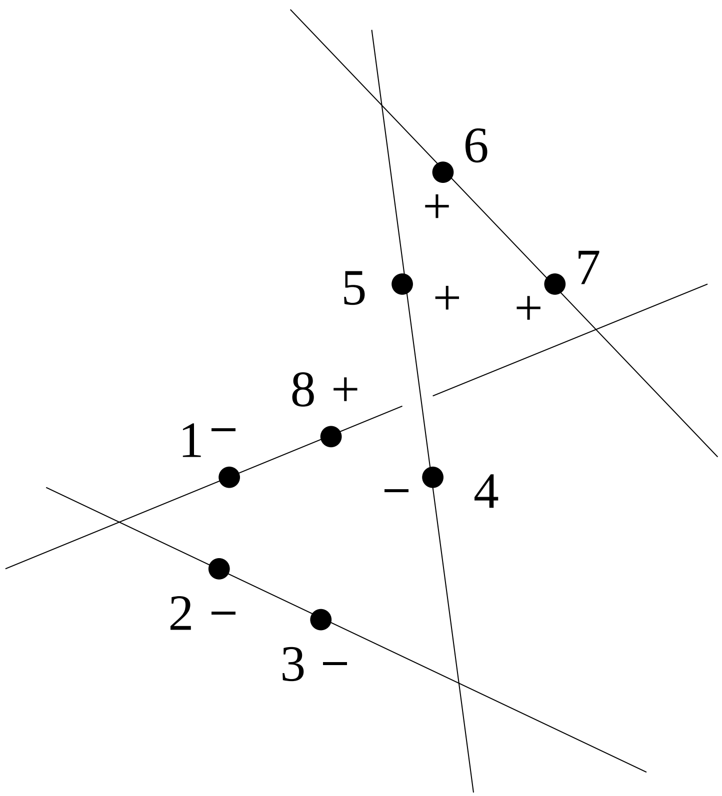
By now there is even a Lagrangian formulation and related twistor-space actions.

(Mansfield; Ettle, Morris; Boels, Mason, Skinner)

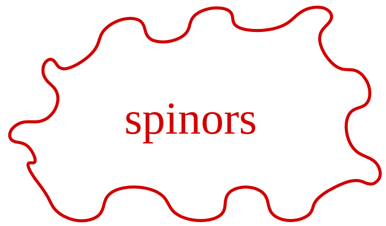
One loop in twistor space

Elements of loop amplitudes are localized as well. (Brandhuber, Spence, Travaglini; Cachazo, Svrcek, Witten)

This offered clues for new constructions, in particular for all next-to-MHV amplitudes. (Bena, Bern, Kosower, Roiban; Cachazo; RB, Cachazo, Feng)

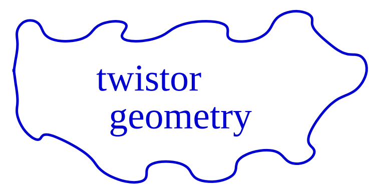


Helicity amplitudes look nice in terms of spinors.



Bring spinors into complexified momentum space.

λ and $\tilde{\lambda}$ are independent complex spinors.



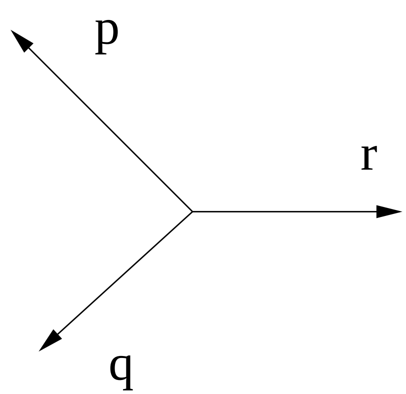
This means that we treat the amplitude as a complex function of the spinor products $\langle \lambda\ \lambda' \rangle$ and $[\widetilde{\lambda}\ \widetilde{\lambda}']$. We can examine its analytic structure.

We did this and derived new recursion relations for amplitudes.

One new feature: three-point amplitudes do not vanish.

Complexified Momenta and Three-Particle Amplitudes

(Witten)



Momentum conservation: $r^2 = 0 \;\Rightarrow\; p \cdot q = 0.$

$$2p \cdot q = \langle p\, q\rangle [p\, q] \;\Rightarrow\; \langle p\, q\rangle = 0 \quad \text{or} \quad [p\, q] = 0.$$

$$A(p^-, q^-, r^+) = \frac{\langle p\, q\rangle^3}{\langle q\, r\rangle \langle r\, p\rangle}$$

Construction of tree-level amplitudes.

(RB, Cachazo, Feng, Witten)

- Define the following function of a complex variable z:

$$A(z) = A(p_1, \ldots, p_{k-1}, p_k(z), p_{k+1}, \ldots, p_{n-1}, p_n(z)),$$

where

$$p_k(z) = p_k - z\lambda_k\tilde{\lambda}_n, \qquad p_n(z) = p_n + z\lambda_k\tilde{\lambda}_n.$$

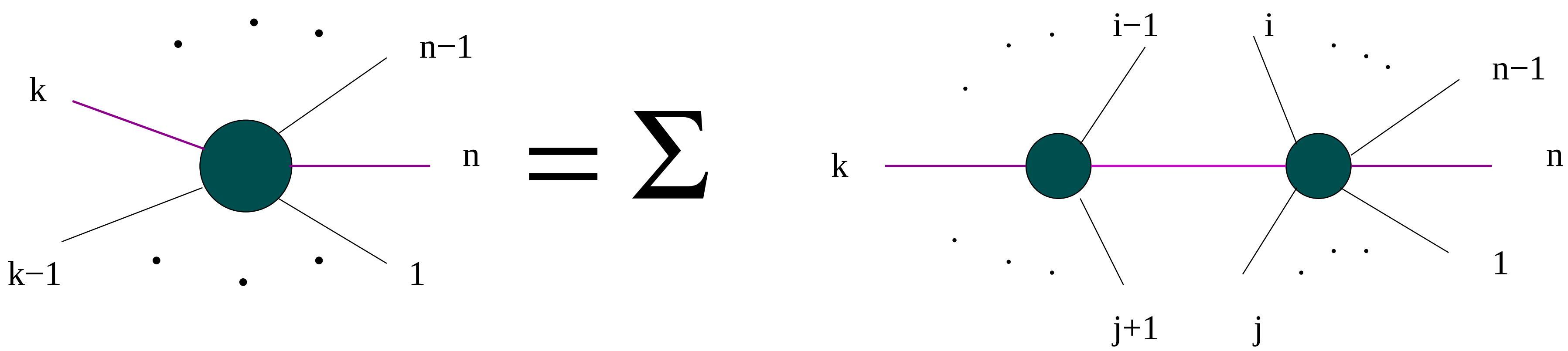
$$p_k = \lambda_k\tilde{\lambda}_k, \quad p_n = \lambda_n\tilde{\lambda}_n.$$

The original amplitude is $A(0)$.

- Tree amplitudes have only simple poles! These come from propagators.

- When we apply the residue theorem, we get an expression for this amplitude in terms of lower-point amplitudes.

Calculation of Residues



$$A(0) = \sum_{i,j} A_L(z_{ij}) \frac{1}{P^2} A_R(z_{ij}),$$

where z_{ij} is the solution to $P_{i,j}(z_{ij})^2 = 0$.

This gives a recursion relation for amplitudes.

(RB, Cachazo, Feng)

$$A_n(1,\ldots,(n-1)^-,n^+) =$$

$$\sum_{i=1}^{n-3}\sum_{h=+,-} A_{i+2}(\hat{n},1,\ldots,i,-\hat{P}^h_{n,i})\frac{1}{P^2_{n,i}}A_{n-i}(+\hat{P}^{-h}_{n,i},i+1,\ldots,n-2,\widehat{n-1})$$

- The hatted momenta are shifted in such a way that all of these momenta are on mass shell *and* momentum is conserved at each node.
- This is possible only with complexified momenta.
- Build arbitrary amplitudes out of 3-point amplitudes.
- Formulas that come from the recursion relation are the most compact ones available.

Example: $A(1^-, 2^-, 3^-, 4^+, \ldots, n^+)$

Spinor-helicity:

(Kosower 1990)

$$
\begin{aligned}
&\frac{1}{[n\,1][1\,2][2\,3][3\,4]\prod_{k=3}^{n}\langle k\; k+1\rangle} \\
&\times\Bigg(\frac{(1\,2)\langle 3\,4\rangle[1\,n]\langle 1|P_{1,3}|4]^2}{P_{5,1}^2}+\frac{(3\,2)\langle 1\,n\rangle[3\,4]\langle 3|P_{1,3}|n]^2}{P_{3,n-1}^2}+\frac{\langle 2\,1\rangle\langle 3\,1\rangle\langle 3\,n\rangle[3\,2][3\,4][1\,n]\langle 3|P_{1,3}|n]}{P_{3,n-1}^2} \\
&+\frac{\langle 2\,3\rangle\langle 1\,3\rangle\langle 1\,4\rangle[1\,2][3\,4][1\,n]\langle 1|P_{1,3}|4]}{P_{5,1}^2}-\langle 3\,1\rangle^2 P_{4,n}^2[3\,4][1\,n]-\frac{(1\,2)(3\,2)\langle 3\,1\rangle\langle 3|P_{1,3}|n][3\,4]}{P_{3,n-1}^2} \\
&-[2\,3][2\,1][3\,4][1\,n]\times\sum_{i=5}^{n-1}\Bigg[\frac{\langle 2\,1\rangle^2\langle 1\,3\rangle\langle 3|P_{3,i-1}k_i|3\rangle}{P_{3,i-1}^2P_{3,i}^2}+\frac{\langle 2\,3\rangle^2\langle 3\,1\rangle\langle 1|P_{i+1,1}k_i|1\rangle}{P_{i+1,1}^2P_{i,1}^2} \\
&-\frac{\langle 2\,3\rangle\langle 2\,1\rangle\langle 1\,3\rangle\langle 1|P_{i+1,2}k_i|3\rangle}{P_{3,i}^2P_{i,1}^2}-\frac{\langle 2\,3\rangle\langle 2\,1\rangle^2\langle 1|P_{i+1,2}k_i|3\rangle\langle 3|P_{i,2}|2]}{P_{3,i-1}^2P_{3,i}^2P_{i,1}^2} \\
&-\frac{\langle 2\,3\rangle^2\langle 2\,1\rangle\langle 1|P_{i+1,1}k_i|3\rangle\langle 1|P_{i+1,2}|2]}{P_{3,i}^2P_{i+1,1}^2P_{i,1}^2}\Bigg]\Bigg)
\end{aligned}
$$

Twistor geometry:

(Cachazo, Svrček, Witten 2004)

$$\frac{1}{\prod_{k=3}^{n} \langle k\, k+1\rangle}$$
$$\times \left[\sum_{i=4}^{n-1} \frac{\langle i\, i+1\rangle}{\langle i|P_{2,i}|2]\langle i+1|P_{i+1,2}|2]\langle 2|P_{2,i}|2]} \left(\frac{\langle 3\, 2\rangle^3 \langle 1|P_{2,i}|2]^3}{P_{2,i}^2} + \frac{\langle 1\, 2\rangle^3 \langle 3|P_{i+1,2}|2]^3}{P_{i+1,2}^2} \right) \right.$$
$$\left. -\langle 1\, 3\rangle^2 \left(\frac{(1\, 3) + 2(1\, 2) + 2(2\, 3)}{[3\, 2][1\, 2]} + \frac{\langle 1\, 2\rangle\langle n\, 3\rangle}{[1\, 2]\langle n\, 1\rangle} + \frac{\langle 3\, 2\rangle\langle 1\, 4\rangle}{[3\, 2]\langle 3\, 4\rangle} \right) \right]$$

Construction from residues:

(RB, Feng, Roiban, Spradlin, Volovich 2005)

$$\frac{1}{\prod_{k=3}^{n}\langle k\ k+1\rangle}\sum_{i=4}^{n-1}\frac{\langle 1|P_{2,i}\,P_{i+1,2}|3\rangle^{3}}{P_{2,i}^{2}\,P_{i+1,2}^{2}}\,\frac{\langle i{+}1\ i\rangle}{[2|P_{2,i}|i{+}1\rangle\langle i|P_{i+1,2}|2]}$$

Another Example: Split-Helicity Amplitudes

(RB, Feng, Roiban, Spradlin, Volovich 2005)

$$A(1^-,\ldots,q^-,(q+1)^+,\ldots,n^+) = \sum_k \sum_{A_k,B_{k+1}} \frac{N_1 N_2 N_3}{D_1 D_2 D_3} \,.$$

$$\begin{aligned}
N_1 &= \langle 1| P_{2,b_1} P_{b_1+1,a_1} P_{a_1+1,b_2} \cdots P_{b_{k+1}+1,q-1} |q\rangle^3 \,, \\
N_2 &= \langle b_1{+}1\, b_1\rangle \langle b_2{+}1\, b_2\rangle \cdots \langle b_{k+1}{+}1\, b_{k+1}\rangle \,, \\
N_3 &= [a_1\, a_1{+}1] \cdots [a_k\, a_k{+}1] \,, \\
D_1 &= P^2_{2,b_1} P^2_{b_1+1,a_1} P^2_{a_1+1,b_2} \cdots P^2_{b_{k+1}+1,q-1} \,, \\
D_2 &= \langle q\; q+1\rangle \cdots \langle n\; 1\rangle [2\,3][3\,4] \cdots [q-2\; q-1] \,, \\
D_3 &= [2| P_{2,b_1} |b_1{+}1\rangle \langle b_1| P_{b_1+1,a_1} |a_1] \cdots \langle b_{k+1}| P_{b_{k+1}+1,q-1} |q{-}1]
\end{aligned}$$

$$\begin{aligned}
A_k &= \{a_1, \ldots, a_k\} \subset \{2, \ldots, q-2\} \\
B_{k+1} &= \{b_{k+1}, \ldots, b_1\} \subset \{q+1, \ldots, n-1\}
\end{aligned}$$

The recursion gives new and compact expressions for tree-level amplitudes in pure gauge theory. (Luo, Wen; RB, Feng, Roiban, Spradlin, Volovich)

The same construction has been applied to loop amplitudes, gravitons, and massive particles including Higgs, gauge bosons, and fermions.

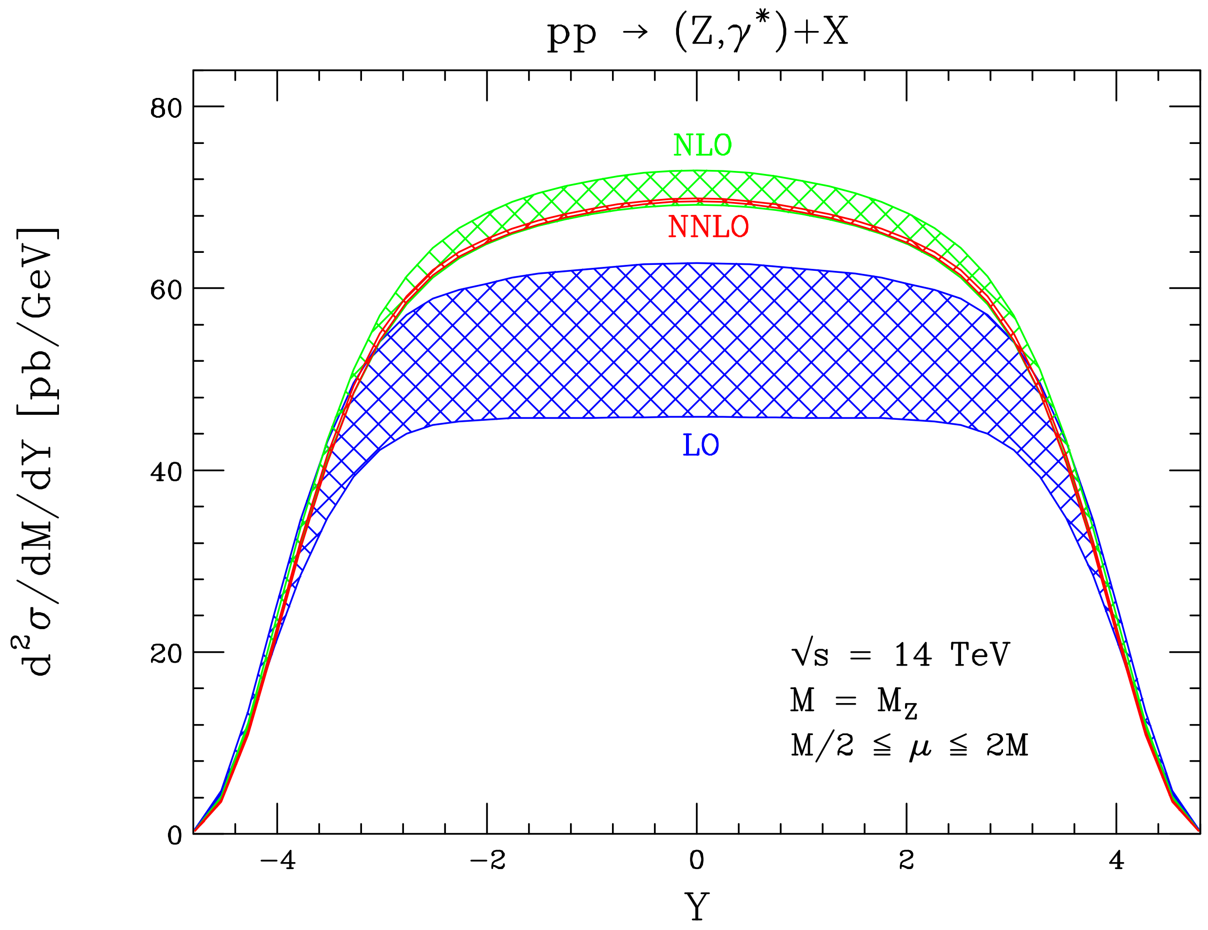
(Badger, Glover, Khoze, Svrcek; Forde, Kosower; Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Bern, Dixon, Kosower; Bern, Bjerrum-Bohr, Dunbar, Ita; Ferrario, Rodrigo, Talavera; Quigley, Rozali; Ozeren, Stirling; Berger, Bern, Dixon, Forde, Kosower; Benincasa, Boucher-Veronneau, Cachazo; Schwinn, Weinzierl;...)

We have had to expand our physical intuition to complexified momentum space.

Construction from the residue theorem is very elegant, but this is not how the recursion relations were first understood. In fact, they arose mysteriously in the context of loop amplitudes!

Loop amplitudes have been the principal target of activity for collider physics: QCD at NLO.

Next-to-Leading-Order Effects Are Large



Rapidity distribution of a Z boson at LHC. $\alpha_s = 0.121$ at M_Z.

(Anastasiou, Dixon, Melnikov, Petriello)

Status at Next-to-Leading-Order

Complexity of a calculation increases with the number of kinematic variables.

2 $\longrightarrow$ 3 processes are state of the art in QCD at next-to-leading order.

$$\begin{aligned} p\,p \;\; \longrightarrow \qquad & 3\,\mathrm{jets},\; Z+2\,\mathrm{jets},\; W+2\,\mathrm{jets}, \\ & Z\,b\,\bar{b},\; H+2\,\mathrm{jets},\; \gamma\,\gamma\,\mathrm{jet}, \\ & t\,\bar{t}\,H,\; b\,\bar{b}\,H,\; t\,\bar{t}\,h^0,\; t\,\bar{b}\,H^-,\; t\,\bar{t}\,\mathrm{jet} \end{aligned}$$

(Bern, Dixon, Kosower (x2); Kunszt, Signer, Trocsanyi; Kilgore, Giele; Nagy; Campbell, Ellis, Rainwater; Campbell, Ellis, Maltoni, Willenbrock; Ellis, Veseli; Campbell, Ellis; Campbell; Han, Valencia, Willenbrock; Figy, Oleari, Zeppenfeld; Berger, Campbell; Figy, Zeppenfeld; Ellis, Giele, Zanderighi; Del Duca, Maltoni, Nagy, Trocsanyi; Binoth, Guillet, Mahmoudi; Dittmaier, Krämer, Spira; Dawson, Jackson, Reina, Wackeroth (x3); Beenakker, Dittmaier, Krämer, Pluemper, Spira, Zerwas (x2); Dawson, Orr, Reina, Wackeroth; Reina, Dawson, Wackeroth; Wu, Ma, Hou, Zhang, Han, Jiang; Wu, Ma, Zhang, Jiang, Han, Guo; Dittmaier, Uwer, Weinzierl)

Getting the cross section

- Generate diagrams.
- Real radiation corrections: tree amplitudes and pole subtraction.
- Virtual corrections: one-loop amplitudes.
- Integration over phase space.

One-loop amplitudes are the bottleneck.

By improving methods for computing amplitudes, we will pin down signals and backgrounds of new physics.

Status at Next-to-Leading-Order

2006: complexity of $2 \longrightarrow 4$ in QCD was achieved in the complete 6-gluon amplitude:

- Semi-numerical method

  Ellis, Giele, Zanderighi

- Analytic results completed

  Bern, Dixon, Dunbar, Kosower (x2); Bidder, Bjerrum-Bohr, Dixon, Dunbar; RB, Buchbinder, Cachazo, Feng; Bidder, Bjerrum-Bohr, Dunbar, Perkins (x2); Bedford, Brandhuber, Spence, Travaglini; Bern, Bjerrum-Bohr, Dunbar, Ita; Bern, Dixon, Kosower; RB, Feng, Mastrolia; Berger, Bern, Dixon, Forde, Kosower (x2); Xiao, Yang, Zhu (x2)

  The analytic techniques apply to any number of particles!

Several $2 \longrightarrow 4$ processes will be needed for LHC physics, especially as backgrounds to Higgs production.

Experimenter's Wish List at Next-to-Leading Order

(Knuteson, Campbell 2001)

single boson	diboson	triboson	heavy flavor
$W + \leq 5j$	$WW + \leq 5j$	$WWW + \leq 3j$	$t\bar{t} + \leq 3j$
$Wb\bar{b} + \leq 3j$	$WWb\bar{b} + \leq 3j$	$WWWb\bar{b} + \leq 3j$	$t\bar{t}\gamma + \leq 2j$
$Wc\bar{c} + \leq 3j$	$WWc\bar{c} + \leq 3j$	$WWW\gamma\gamma + \leq 3j$	$t\bar{t}W + \leq 2j$
$Z + \leq 5j$	$ZZ + \leq 5j$	$Z\gamma\gamma + \leq 3j$	$t\bar{t}Z + \leq 2j$
$Zb\bar{b} + \leq 3j$	$ZZb\bar{b} + \leq 3j$	$WZZ + \leq 3j$	$t\bar{t}H + \leq 2j$
$Zc\bar{c} + \leq 3j$	$ZZc\bar{c} + \leq 3j$	$ZZZ + \leq 3j$	$t\bar{b} + \leq 2j$
$\gamma + \leq 5j$	$\gamma\gamma + \leq 5j$		$b\bar{b} + \leq 3j$
$\gamma b\bar{b} + \leq 3j$	$\gamma\gamma b\bar{b} + \leq 3j$		$b\bar{b}t\bar{t}$
$\gamma c\bar{c} + \leq 3j$	$\gamma\gamma c\bar{c} + \leq 3j$		
	$WZ + \leq 5j$		
	$WZb\bar{b} + \leq 3j$		
	$WZc\bar{c} + \leq 3j$		
	$W\gamma + \leq 3j$		
	$Z\gamma + \leq 3j$		

LHC “priority” wishlist at Next-to-Leading Order

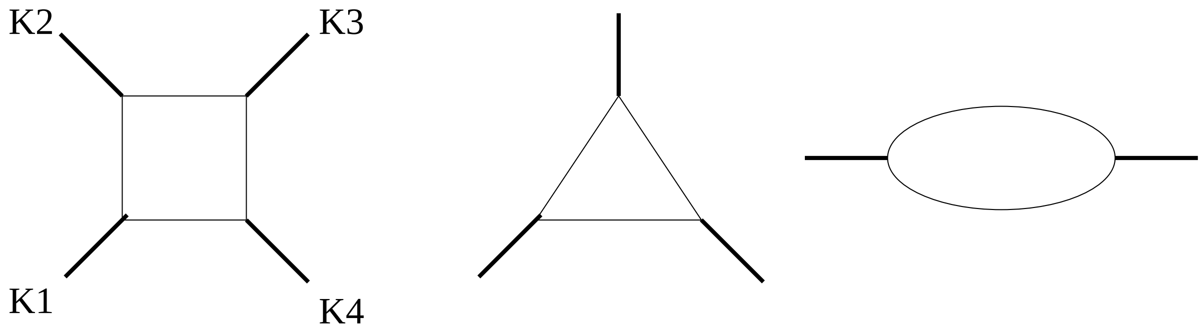
(Les Houches physics at TeV colliders 2005, standard model and Higgs working group: Summary report)

$$p\,p \;\longrightarrow\; V\,V + \text{ jet}$$
$$p\,p \;\longrightarrow\; t\,\bar{t}\,b\,\bar{b}$$
$$p\,p \;\longrightarrow\; t\,\bar{t} + 2\,\text{jets}$$
$$p\,p \;\longrightarrow\; V\,V\,b\,\bar{b}$$
$$p\,p \;\longrightarrow\; V\,V + 2\,\text{jets}$$
$$p\,p \;\longrightarrow\; V + 3\,\text{jets}$$
$$p\,p \;\longrightarrow\; V\,V\,V$$

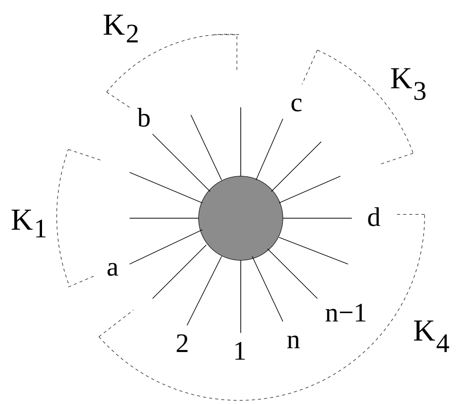
$$V \in \{Z, W, \gamma\}$$

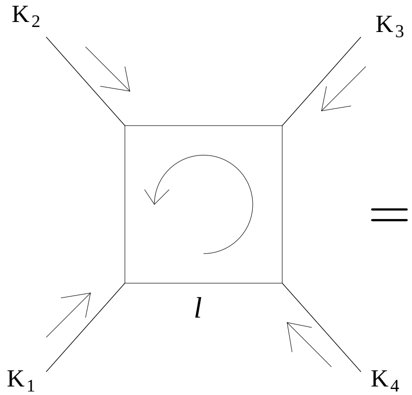
Reduction of Loop Integrals

Early shortcuts to computing one-loop amplitudes reduced the tensor structure of the momentum integral, so there were fewer types of integrals to carry out. (Brown, Feynman; Passarino, Veltman; 't Hooft, Veltman; Stuart; Van Neerven, Vermaseren; Melrose; Van Oldenborgh, Vermaseren)

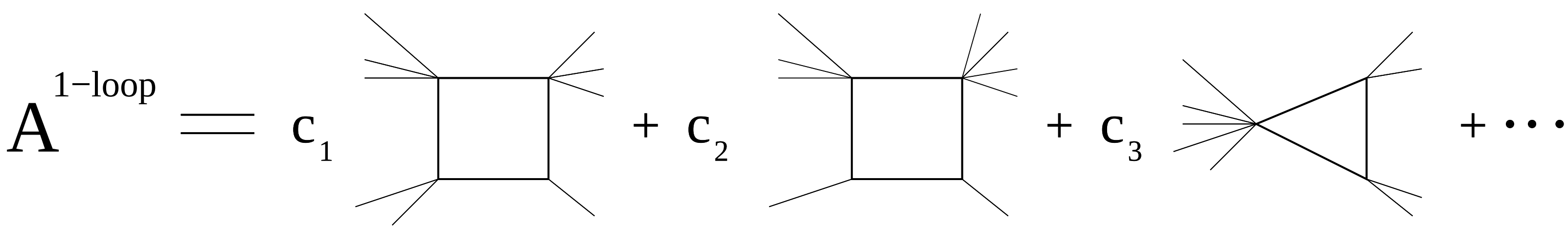


box triangle bubble



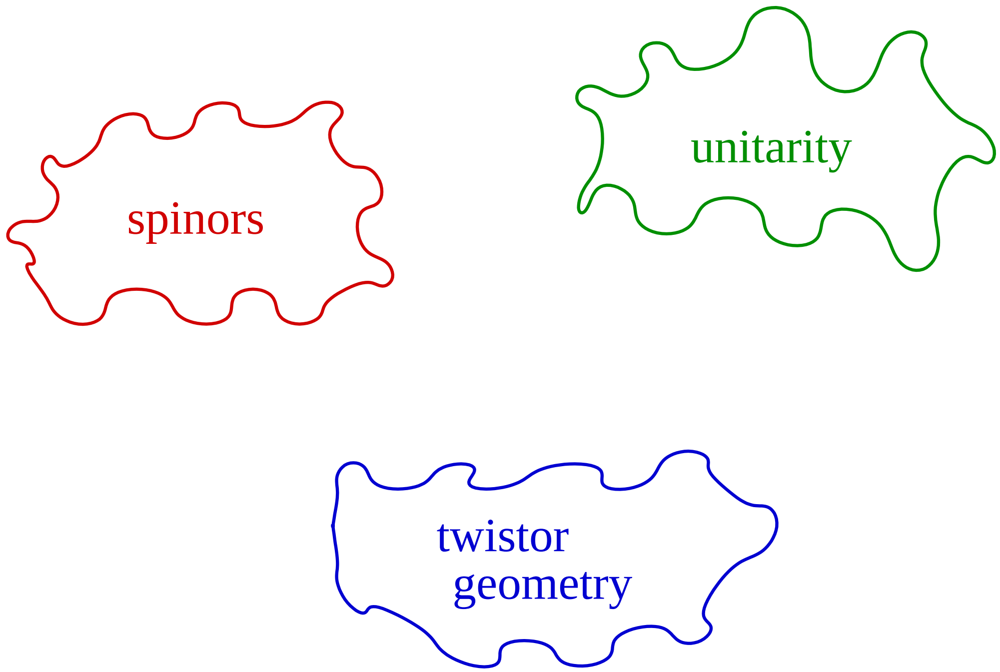


$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2(\ell - K_1)^2(\ell - K_1 - K_2)^2(\ell + K_4)^2} \quad (7)$$



$$A^{1-\mathrm{loop}} = \sum c_i I_i \quad (8)$$

For amplitudes involving many particles, this is not yet enough simplification, although improved reductions exist.



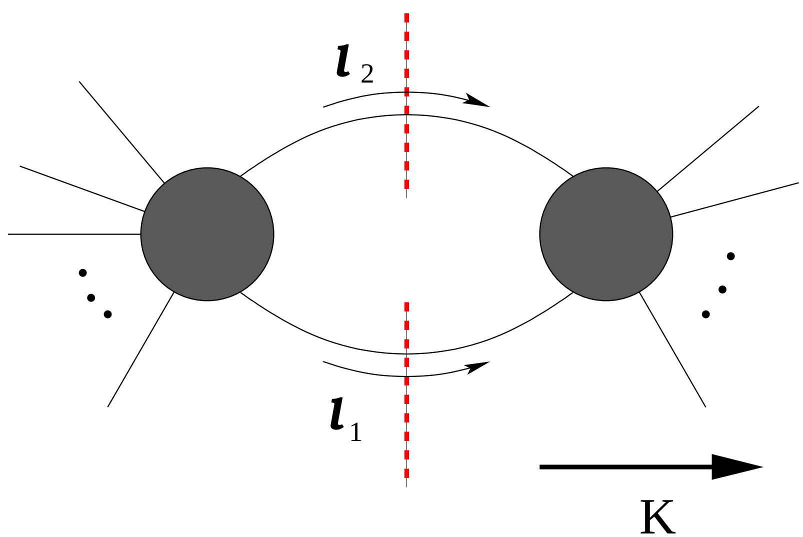
All the ingredients come together for general, complete amplitudes at one loop!

Unitarity Cuts: Loops from Trees

$$\Delta A^{1-\text{loop}} = \int d\mu \;\; A^{\text{tree}}_{\text{Left}} \times A^{\text{tree}}_{\text{Right}} \tag{9}$$

where

$$d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - K)\; \delta(\ell_1^2)\; \delta(\ell_2^2) \tag{10}$$



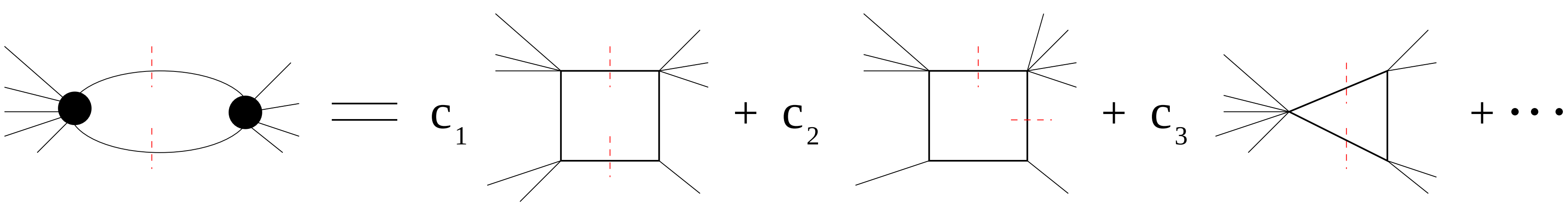
By unitarity, this is the discontinuity of the amplitude across a branch cut, in a kinematic region selecting the cut momentum K. (Cutkosky 1960)

Amplitudes from unitarity cuts

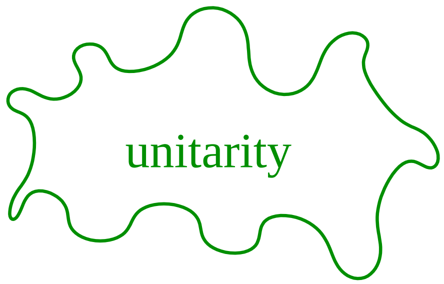
4-dimensional cuts suffice to determine certain one-loop amplitudes! (Bern, Dixon, Dunbar, Kosower 1994)

Match cuts of amplitudes with cuts of master integrals from Passarino-Veltman reduction: essence of loop momentum integral is done once and for all.

$$\Delta A^{1-\mathrm{loop}} = \sum c_i \Delta I_i \tag{11}$$



Earlier versions of the “unitarity method” still involved integral reduction, and an intelligent ansatz for a given coefficient based on singularities. (Bern, Dixon, Dunbar, Kosower; Bern, Del Duca, Dixon, Kosower)

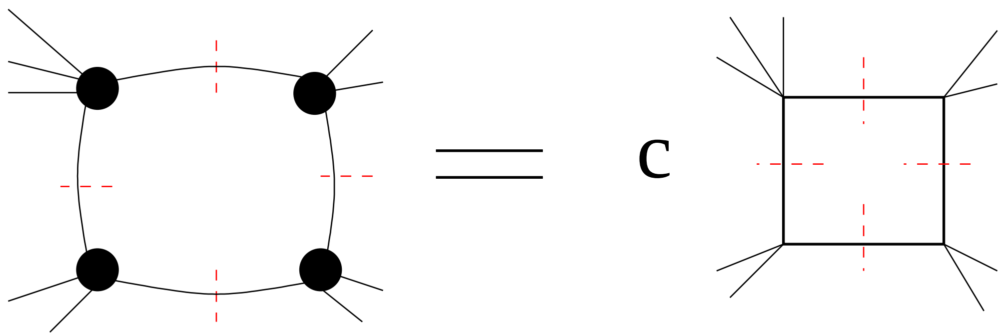


Apply unitarity method to tree amplitudes that have been continued to complex momentum space.

Generalized unitarity gives immediate, algebraic results for “box” parts.

Box Coefficients from Quadruple Cuts

(RB, Cachazo, Feng)



Generalized Unitarity: Try replacing all four propagators by delta functions.

This operation isolates any given box.

In four dimensions, these four delta functions localize the integral completely. This computation is very easy!

The solutions of loop momenta

The box coefficients computed from quadruple cuts are given by

$$\frac{1}{2}\sum_{\mathcal{S}} A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree} \tag{12}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

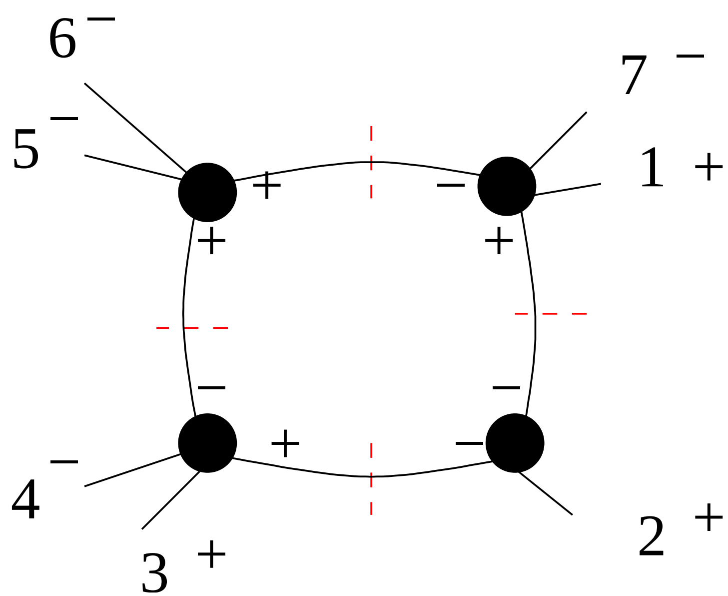
$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\} \tag{13}$$

Can these equations always be solved?

In complexified momentum space, there are exactly 2 solutions.

(Again: nonvanishing 3-point amplitudes.)

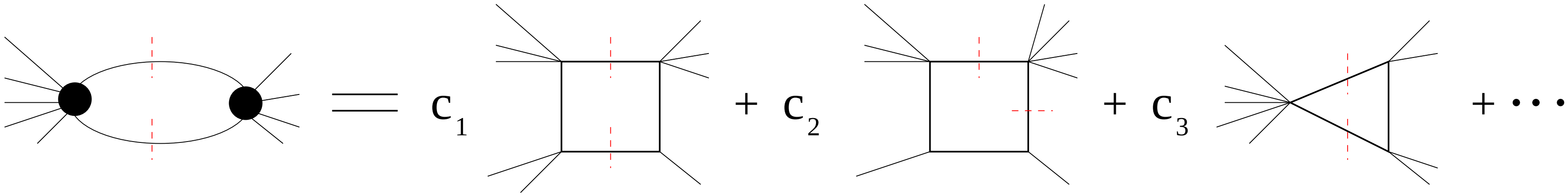
Box Coefficient from Quadruple Cut



$$\text{coeff} = \frac{1}{2}\frac{[\ell_1\,\ell_4]^3}{[\ell_1\,2][2\,\ell_4]}\frac{[4\,\ell_2]^3}{[\ell_2\,\ell_1][\ell_1\,3][3\,4]}\frac{[5\,6]^3}{[6\,\ell_3][\ell_3\,\ell_2][\ell_2\,5]}\frac{[\ell_3\,7]^3}{[7\,1][1\,\ell_4][\ell_4\,\ell_3]} \quad (14)$$

$$= -\frac{\langle 1\,2\rangle^3\langle 2\,3\rangle^3[5\,6]^3}{\langle 7\,1\rangle\langle 3\,4\rangle\langle 2|P_{3,4}|5]\langle 2|P_{7,1}|6]\langle 2|P_{3,4}P_{5,6}|7\rangle\langle 2|P_{7,1}P_{5,6}|4\rangle} \quad (15)$$

Constructing amplitudes from unitarity cuts



We have a systematic way to evaluate these cuts using spinors. Then apply the residue theorem to avoid integration. (RB, Buchbinder, Cachazo, Feng; RB, Feng, Mastrolia)

Integrating with spinors instead of momentum 4-vectors makes use of the simplicity and structure we saw with tree amplitudes.

This is how we finished computing the cut-constructible part of the 6-gluon amplitude!

Spinor integration

(RB, Buchbinder, Cachazo, Feng; RB, Feng, Mastrolia; RB, Feng; Anastasiou, RB, Feng, Kunszt, Mastrolia)

- Change loop momentum to spinor variables in unitarity cut integral.

$$\ell \longrightarrow \lambda, \tilde{\lambda} \tag{16}$$

- Each term of integrand takes the form:

$$\frac{(K^2)^{n+1}}{\langle\lambda|K|\tilde{\lambda}]^{n+2}} \frac{\prod_{i=1}^{n+k} \langle\lambda|R_i|\tilde{\lambda}]}{\prod_{j=1}^{k} \langle\lambda|Q_j|\tilde{\lambda}]} \tag{17}$$

- Evaluate with residue theorem.
- Identify expressions with cuts of basis integrals and read off coefficients.
- We have given formulas for the resulting coefficients.

Triangle coefficients

$$C[Q_s, K] = \frac{(K^2)^{1+n}}{2} \frac{1}{(\sqrt{\Delta_s})^{n+1}} \frac{1}{(n+1)!\langle P_{s,1}\, P_{s,2}\rangle^{n+1}}$$

$$\times \frac{d^{n+1}}{d\tau^{n+1}} \left(\frac{\prod_{j=1}^{k+n} \langle P_{s,1} - \tau P_{s,2} | R_j Q_s | P_{s,1} - \tau P_{s,2}\rangle}{\prod_{t=1, t\neq s}^{k} \langle P_{s,1} - \tau P_{s,2} | Q_t Q_s | P_{s,1} - \tau P_{s,2}\rangle} + \{P_{s,1} \leftrightarrow P_{s,2}\} \right)\Bigg|_{\tau=0}$$

$$P_{s,1} = Q_s + \left(\frac{-2Q_s \cdot K + \sqrt{\Delta_s}}{2K^2} \right) K$$

$$P_{s,2} = Q_s + \left(\frac{-2Q_s \cdot K - \sqrt{\Delta_s}}{2K^2} \right) K$$

$$\Delta_s = (2Q_s \cdot K)^2 - 4Q_s^2 K^2$$

Bubble coefficients

$$C[K] = (K^2)^{1+n} \sum_{q=0}^{n} \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left(\mathcal{B}^{(0)}_{n,n-q}(s) + \sum_{r=1}^{k} \sum_{a=q}^{n} \left(\mathcal{B}^{(r;a-q;1)}_{n,n-a}(s) - \mathcal{B}^{(r;a-q;2)}_{n,n-a}(s) \right) \right) \Bigg|_{s\to 0}$$

$$\mathcal{B}^{(0)}_{n,t}(s) \equiv \frac{d^n}{d\tau^n} \left(\frac{1}{n![\eta|\eta' K|\eta]^n} \frac{(2\eta \cdot K)^{t+1}}{(t+1)(K^2)^{t+1}} \frac{\prod_{j=1}^{n+k} \langle \ell | R_j (K+s\eta) | \ell \rangle}{\langle \ell\, \eta \rangle^{n+1} \prod_{p=1}^{k} \langle \ell | Q_p (K+s\eta) | \ell \rangle} \right) \Bigg|_{\substack{|\ell\rangle \to |K-\tau\eta'|\eta] \\ \tau\to 0}}$$

$$\mathcal{B}^{(r;b;1)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2}\rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\langle P_{r,1} - \tau P_{r,2} |\eta| P_{r,1}]^{t+1}}{\langle P_{r,1} - \tau P_{r,2} |K| P_{r,1}]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,1} - \tau P_{r,2} | Q_r \eta | P_{r,1} - \tau P_{r,2}\rangle^b \prod_{j=1}^{n+k} \langle P_{r,1} - \tau P_{r,2} | R_j (K+s\eta) | P_{r,1} - \tau P_{r,2} \rangle}{\langle P_{r,1} - \tau P_{r,2} | \eta K | P_{r,1} - \tau P_{r,2}\rangle^{n+1} \prod_{p=1,p\neq r}^{k} \langle P_{r,1} - \tau P_{r,2} | Q_p (K+s\eta) | P_{r,1} - \tau P_{r,2}\rangle} \right) \Bigg|_{\tau\to 0}$$

$$\mathcal{B}^{(r;b;2)}_{n,t}(s) \equiv \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{\,b+1} \langle P_{r,1}\, P_{r,2}\rangle^b} \frac{d^b}{d\tau^b} \left(\frac{1}{(t+1)} \frac{\langle P_{r,2} - \tau P_{r,1} |\eta| P_{r,2}]^{t+1}}{\langle P_{r,2} - \tau P_{r,1} |K| P_{r,2}]^{t+1}} \right.$$

$$\left. \times \frac{\langle P_{r,2} - \tau P_{r,1} | Q_r \eta | P_{r,2} - \tau P_{r,1}\rangle^b \prod_{j=1}^{n+k} \langle P_{r,2} - \tau P_{r,1} | R_j (K+s\eta) | P_{r,2} - \tau P_{r,1} \rangle}{\langle P_{r,2} - \tau P_{r,1} | \eta K | P_{r,2} - \tau P_{r,1}\rangle^{n+1} \prod_{p=1,p\neq r}^{k} \langle P_{r,2} - \tau P_{r,1} | Q_p (K+s\eta) | P_{r,2} - \tau P_{r,1}\rangle} \right) \Bigg|_{\tau\to 0}$$

What next?

- Wider applications to massive particles (tree level input!)
- Better programmability
- Increase efficiency
- LHC cross sections

Summary

- Observing new physics at hadron colliders like LHC requires better hard-scattering (amplitude) computation techniques.
- Insights from complex analysis and twistor space, spinor-helicity formalism, unitarity.
- Combine these tools in new and practical ways.

  Simple formulas for new amplitudes, better expressions for old ones.
- Complex-analytic constructions give recursion relations for amplitudes, and relate loop amplitudes to tree amplitudes.
- Spinor integration is an efficient, systematic alternative to Feynman diagrams.