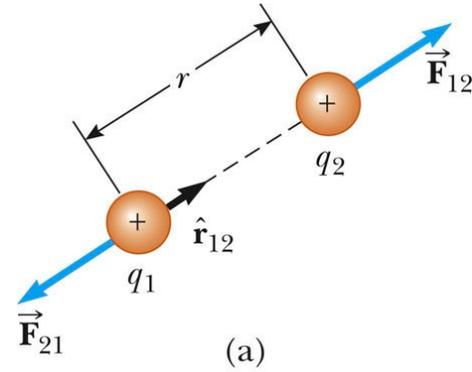


Gauss Law

1. Review on
 - 1) Coulomb's Law (charge and force)
 - 2) Electric Field (field and force)
2. Gauss's Law: connects charge and field
3. Applications of Gauss's Law

Coulomb's Law and Electric Field

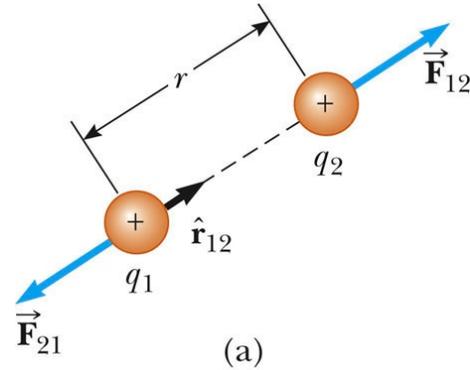
- Coulomb's Law: the force between two point charges



Coulomb's Law and Electric Field

- Coulomb's Law: the force between two point charges

$$\vec{F}_e = K_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

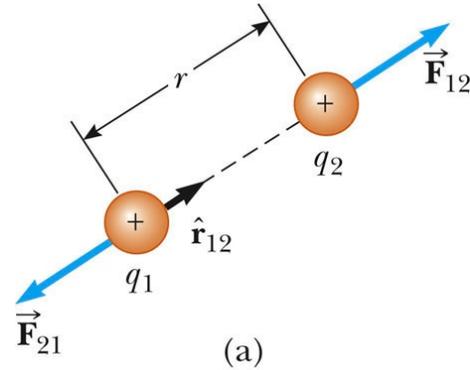


Coulomb's Law and Electric Field

- Coulomb's Law: the force between two point charges

$$\vec{F}_e = K_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

- The electric field is defined as



Coulomb's Law and Electric Field

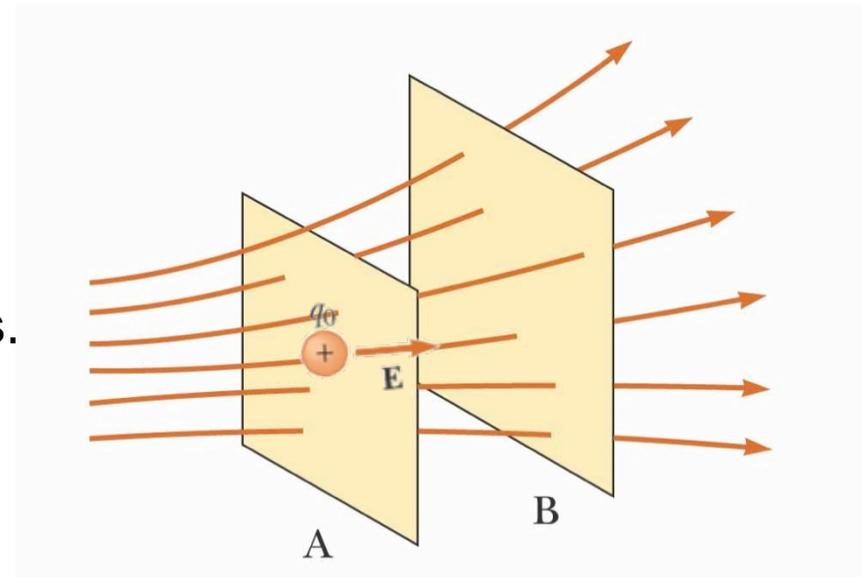
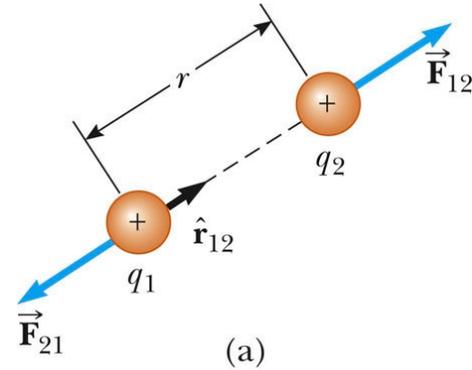
- Coulomb's Law: the force between two point charges

$$\vec{F}_e = K_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

- The electric field is defined as

$$\vec{E} \equiv \frac{\vec{F}}{q_0}$$

and is represented through field lines.



Coulomb's Law and Electric Field

- Coulomb's Law: the force between two point charges

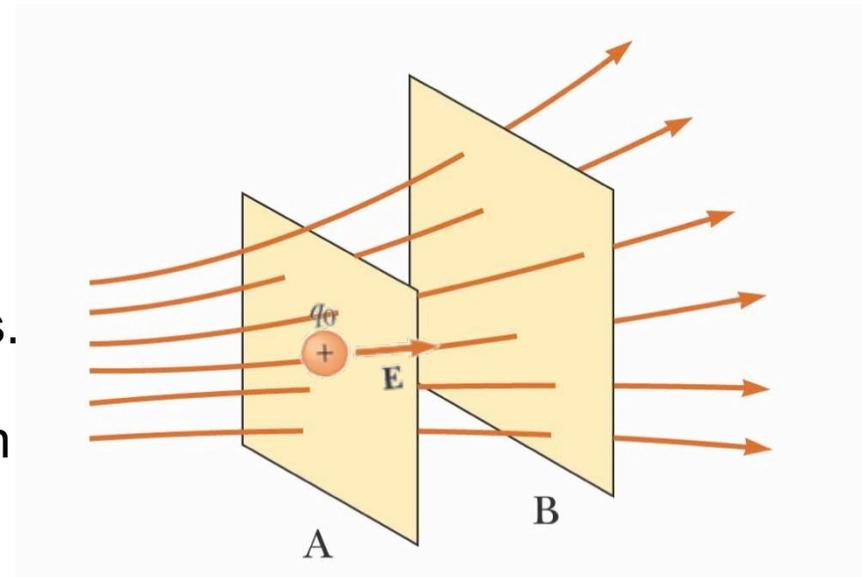
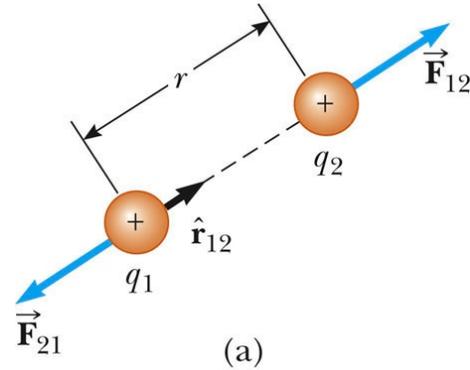
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- The force a charge experiences in an electric field is



Coulomb's Law and Electric Field

- Coulomb's Law: the force between two point charges

$$\vec{F}_e = K_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

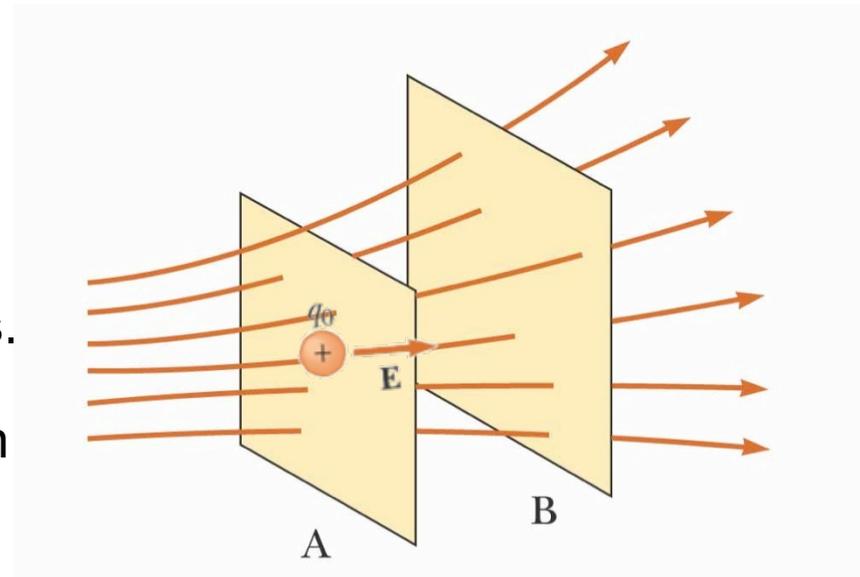
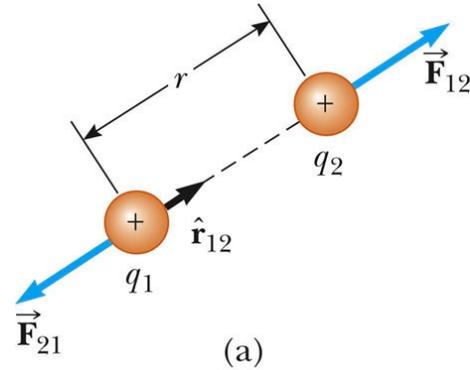
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$$\vec{F} = q_0 \vec{E}$$



Coulomb's Law and Electric Field

- Coulomb's Law: the force between two point charges

$$\vec{F}_e = K_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

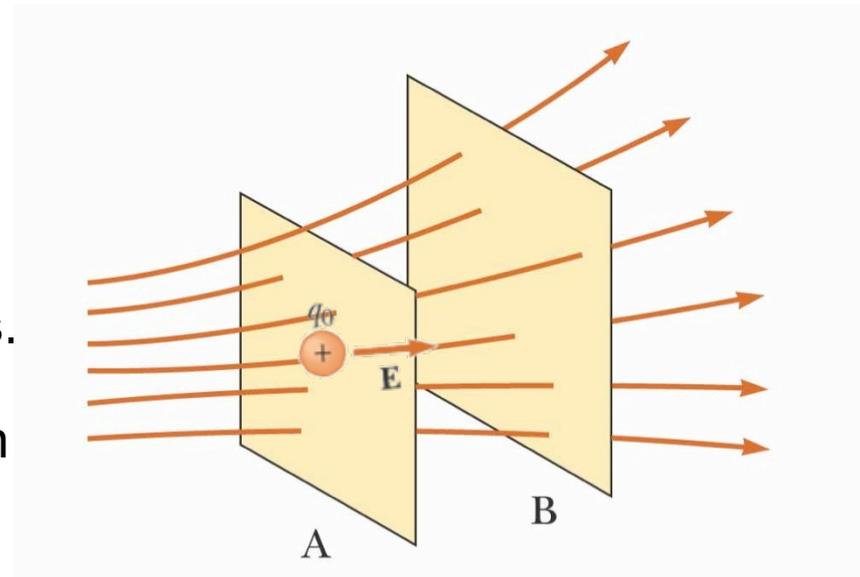
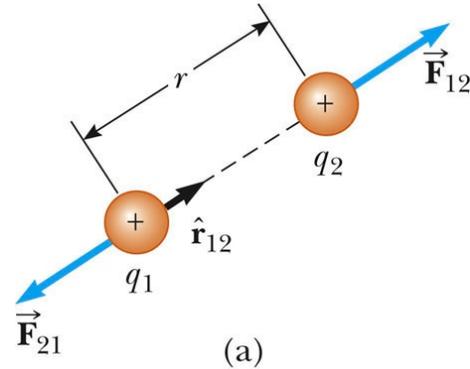
- The electric field is defined as

$$\vec{E} \equiv \frac{\vec{F}}{q_0}$$

and is represented through field lines.

- The force a charge experiences in an electric field is

$$\vec{F} = q_0 \vec{E}$$



A one page summary of two chapters

From Coulomb's Law to Gauss's law

- Try to calculate the electric field generated by
 - a point charge ← easy
 - an infinitely long straight wire with evenly distributed charge ← hard
 - a wire loop ← only at special locations
 - a round disk ← only at special locations
 - an infinitely large plane ← what???
 - a solid sphere with evenly distributed charge
- Are there other ways to calculate electric field generated from a charge distribution?
- Electric field is generated by source charges, are there ways to connect electric field directly with these source charges?



The answer is YES!

Some preparation: Electric Flux, a special case

- When the electric field \vec{E} is perpendicular to the chosen surface A , the **electric flux** is the product of the magnitude of \vec{E} and the surface area A :

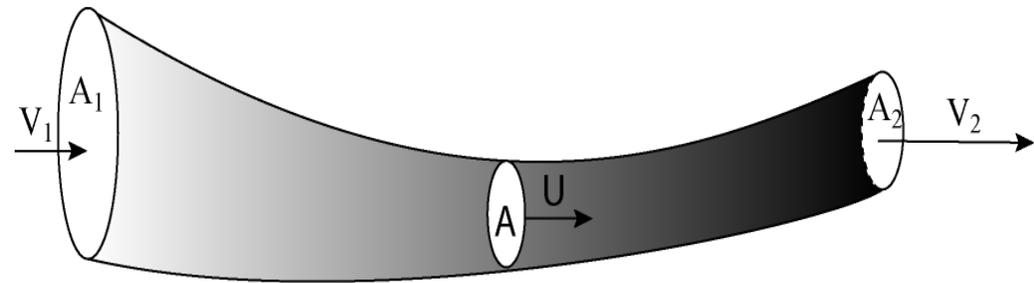
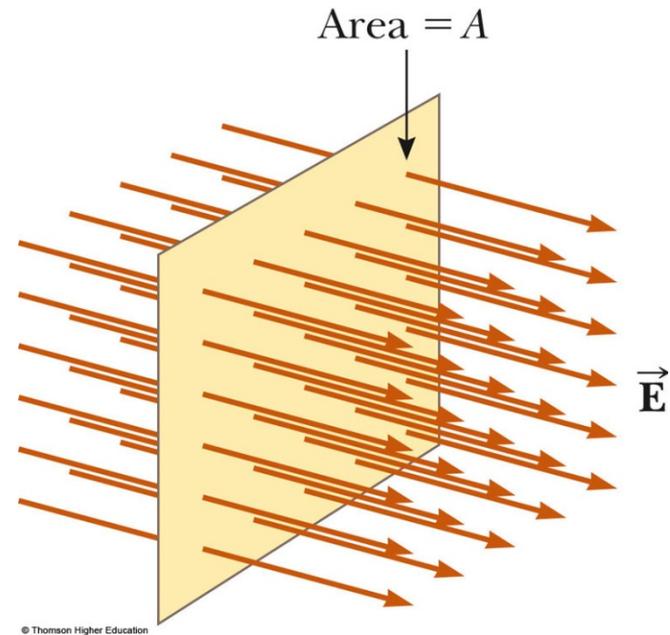
$$\Phi_E = E \cdot A$$

The unit of Φ_E is $\text{N}\cdot\text{m}^2/\text{C}$

- Compare to a water flux in a tube:

$$\Phi_W = -V_1 \cdot A_1 = V_2 \cdot A_2$$

→ This sign means water flows into the tube, by convention.



Electric Flux when E and A is at an angle θ

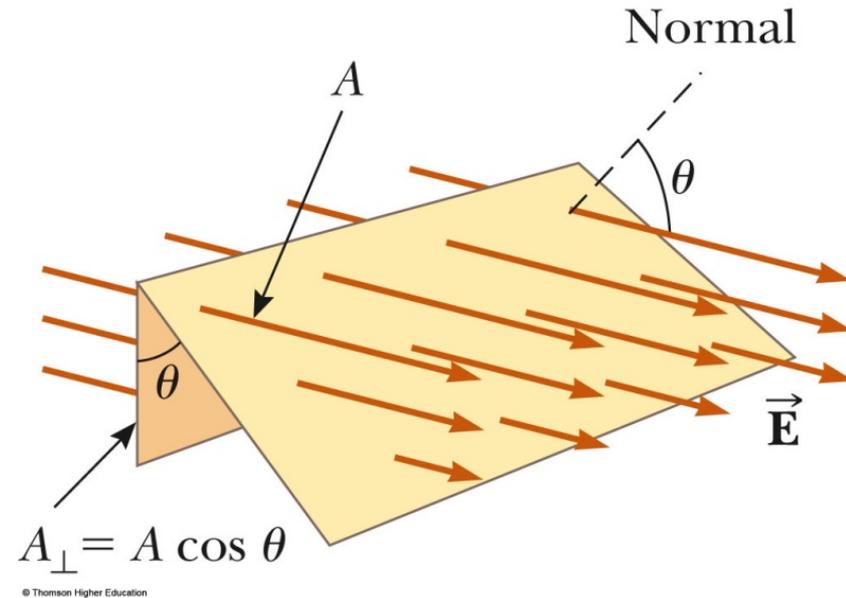
- When the electric field lines make an angle θ with the direction (i.e., the normal) of the surface, the flux is calculated as:

$$\Phi_E = EA_{\perp} = EA \cos \theta = \vec{E} \cdot \vec{A}$$

And the electric field \vec{E} has to be a constant all over the \vec{A} area .

Question: when this (electric field is constant) is not the case, what do you do to get the flux?

$$\Phi_E = \int d\Phi_E = \int \vec{E} \cdot d\vec{A}$$



Review on math:

1. direction of a surface is defined as the (outwards) normal to that surface.
2. Dot product of two vectors.

Electric Flux, the general case

- In a more general case, look at a small area element

$$\Delta\Phi_E = \vec{E}_i \cdot \Delta\vec{A} = E_i \Delta A_i \cos\theta_i$$

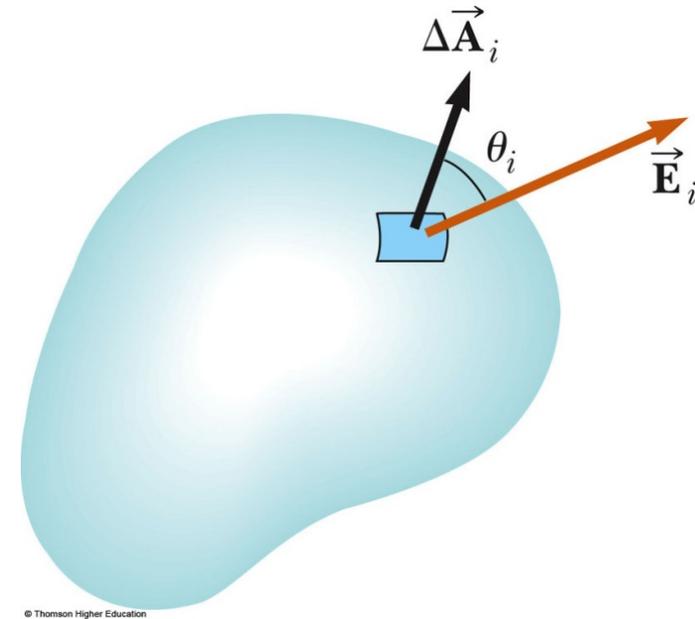
- this becomes

$$\Delta\Phi_E = \lim_{\Delta A_i \rightarrow 0} \vec{E}_i \cdot \Delta\vec{A}$$

$$\Phi_E = \int_{\text{surface}} \vec{E} \cdot d\vec{A} \quad \Rightarrow$$

Review on math:
Integral over a surface.

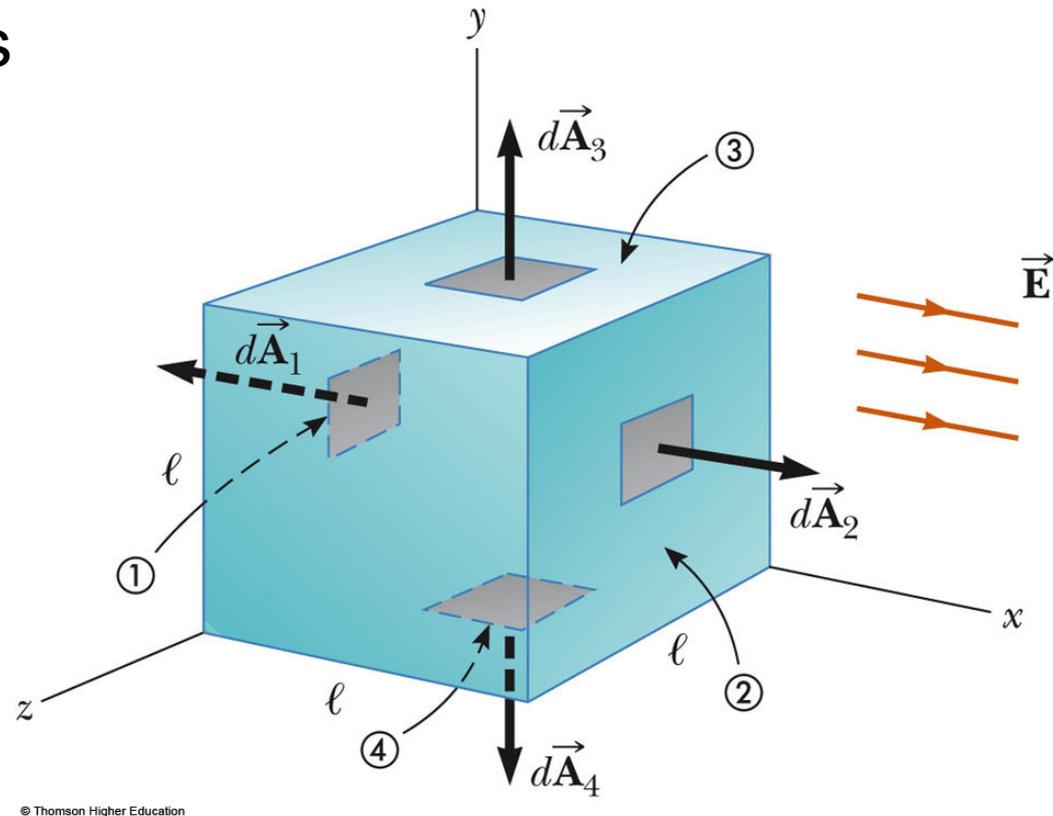
- The surface integral means the integral must be evaluated over the surface in question
- In general, the value of the flux will depend both on the field pattern and on the shape of the surface
- When the surface is closed, the direction of the surface (i.e. the normal of it) points outwards.



flux through a cube in a uniform electric field

- The field lines that pass through surfaces 1 and 2 perpendicularly and are parallel to the other four surfaces
- For side 1, $\Delta\Phi_E = -El^2$
- For side 2, $\Delta\Phi_E = El^2$
- For the other sides, $\Delta\Phi_E = 0$. Why?
- Therefore:

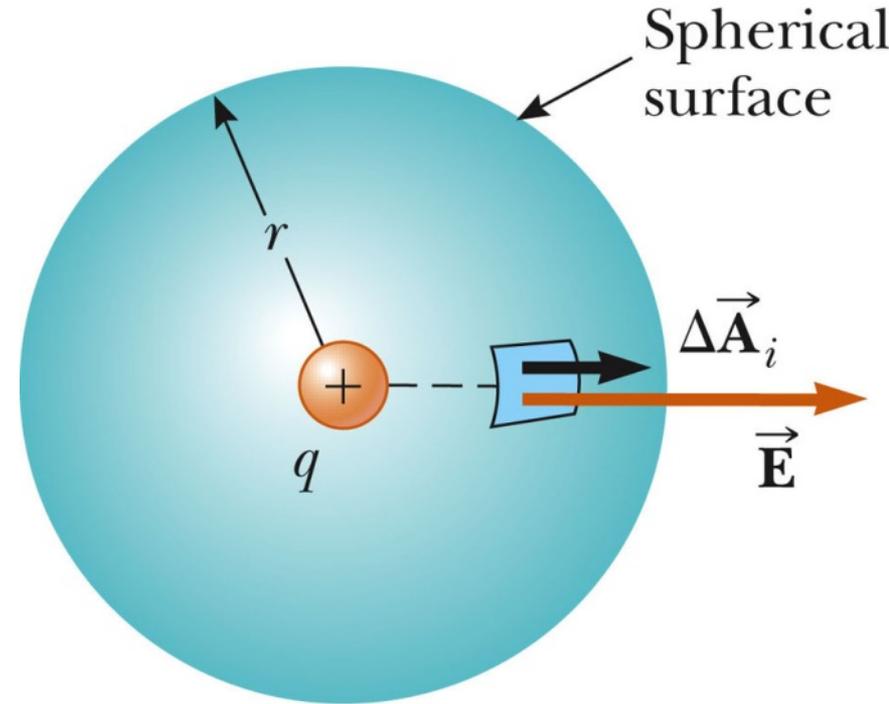
$$\Phi_E = 0$$



flux through a sphere with a charge at its center

- A positive point charge, q , is located at the center of a sphere of radius r
- According to Coulomb's Law, the magnitude of the electric field everywhere on the surface of the sphere is

$$E = k_e \frac{q}{r^2}$$



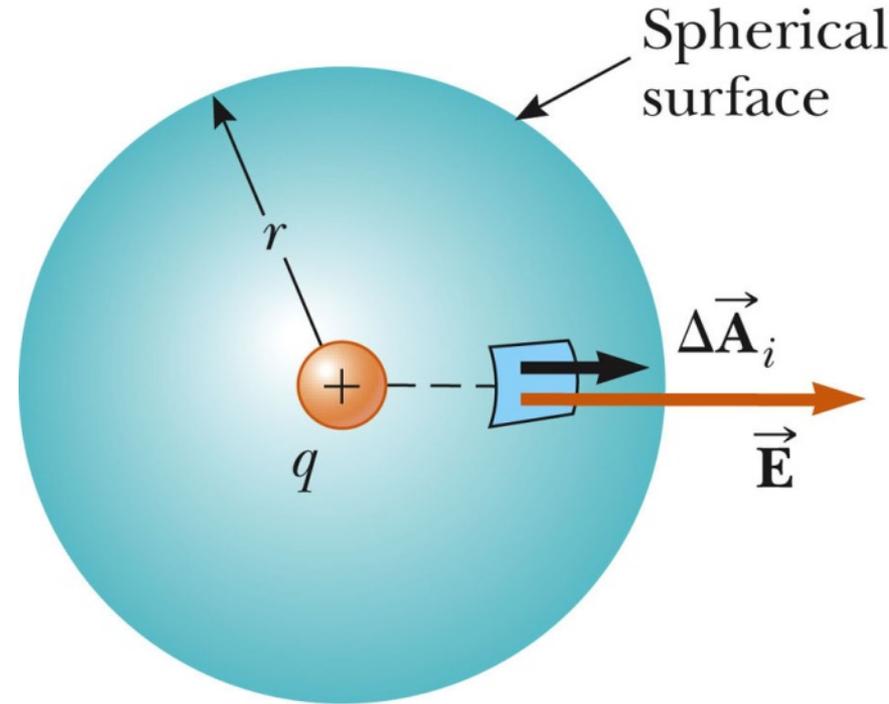
flux through a sphere with a charge at its center

- The field lines are directed radially outwards and are perpendicular to the surface at every point, so

$$\begin{aligned}\Phi_E &\equiv \oint \vec{E} \cdot d\vec{A} = \oint E_n dA \\ &= \oint E dA = E \oint dA = E \cdot 4\pi r^2\end{aligned}$$

- Combine these two equations, we have

$$\Phi_E = E \cdot 4\pi r^2 = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

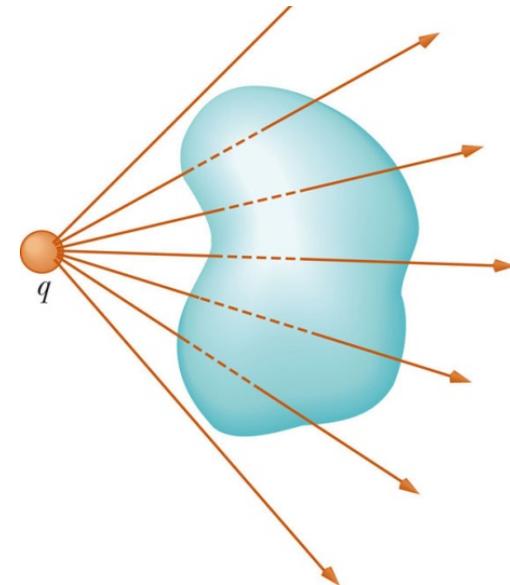
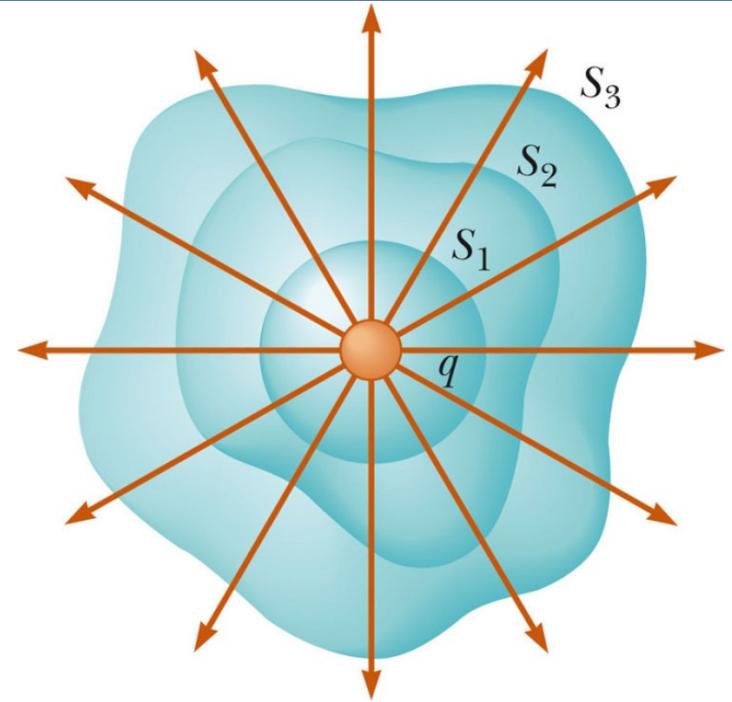


The Gaussian Surface and Gauss's Law

- You choose a **closed surface** and call it a Gaussian Surface.
- This Gaussian Surface can be any shape. It may or may not enclose charges.
- Gauss's Law (Karl Friedrich Gauss, 1777 – 1855) states:

The net flux through any closed surface surrounding a charge q is given by q/ϵ_0 and is independent of the shape of that surface.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$



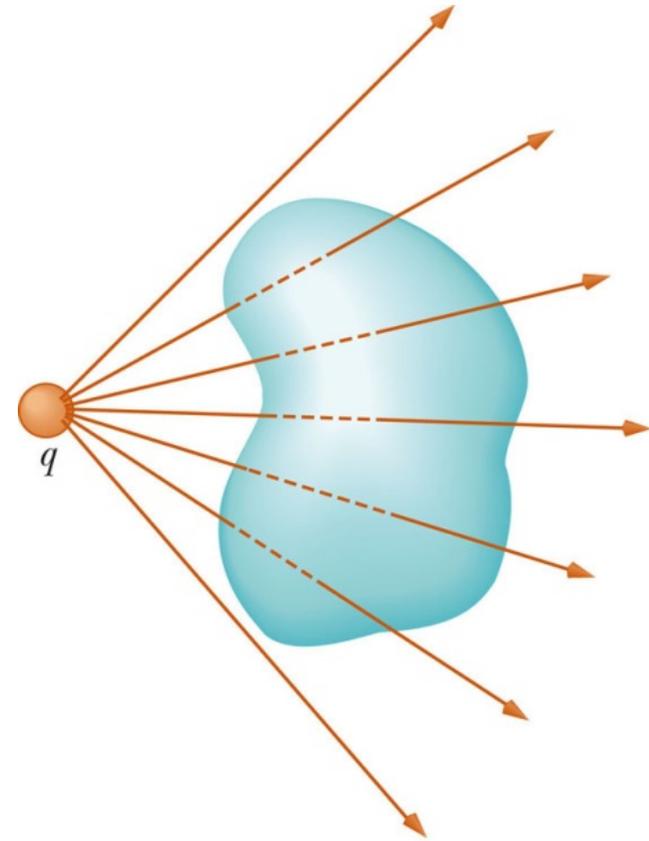
The Gaussian Surface and Gauss's Law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad \Rightarrow \quad \text{The net}$$

electric flux through a closed surface that surrounds no charge is zero.

The electric field due to many charges is the vector sum of the electric fields produced by the individual charges \Rightarrow the flux through any closed surface can be expressed as

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint \left(\sum_i \vec{E}_i \right) \cdot d\vec{A} = \frac{\sum_i q_i}{\epsilon_0}$$



Gauss's Law connects electric field with its source charge

Gauss's Law – Summary

- Gauss's law states $\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$

q_{in} is the net charge **inside** the Gaussian surface

- \vec{E} represents the electric field at any point on the surface

\vec{E} is the *total electric field* at a point in space and may have contributions from charges both inside and outside of the surface

- Although Gauss's law can, in theory, be solved to find \vec{E} for any charge configuration, in practice it is limited to **a few symmetric situations**

Recap: Electric Flux and Gauss Law

- Electric flux, definition

$$\Phi_E = EA_{\perp} = EA \cos \theta = \vec{E} \cdot \vec{A}$$

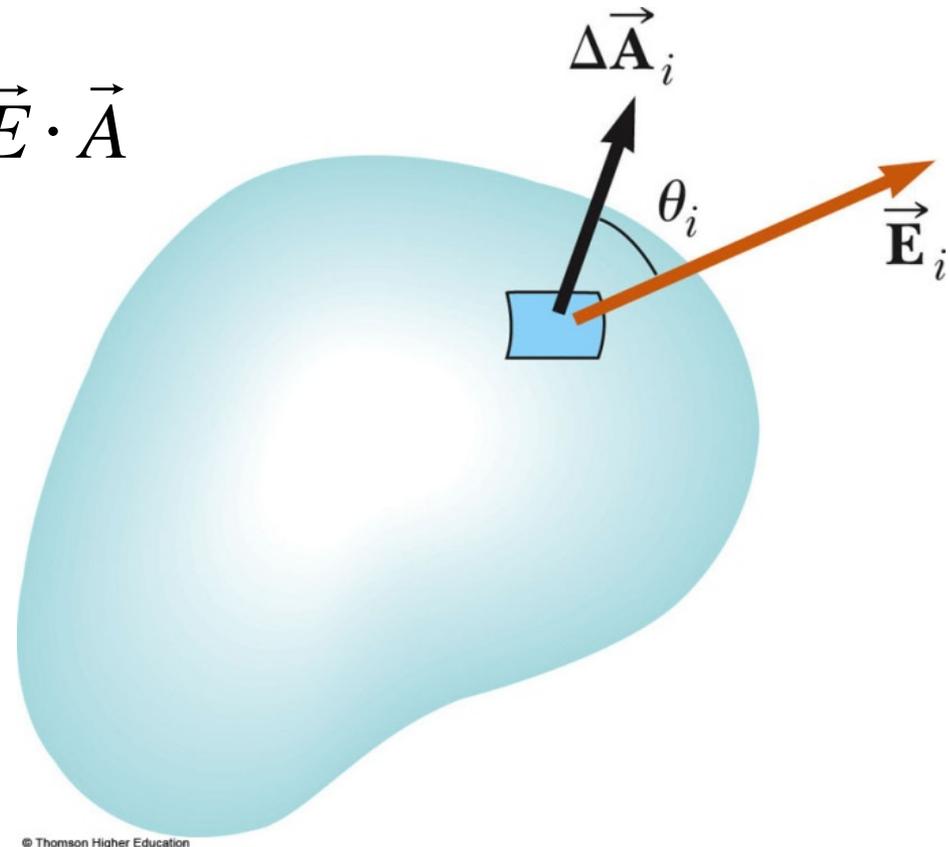
$$\Phi_E = \lim_{\Delta A_i \rightarrow 0} \sum_i E_i \cdot \Delta A_i$$

$$\Phi_E = \int_{\text{surface}} \vec{E} \cdot d\vec{A}$$

- Gauss law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$

q_{in} is the net charge **inside** the surface



Applying Gauss Law

- To use Gauss law, you need to choose a Gaussian surface over which the surface integral can be simplified and the electric field determined
- Take advantage of symmetry
- Remember, the Gaussian surface is a surface you choose, it does not have to coincide with a real surface

Conditions for a Gaussian Surface

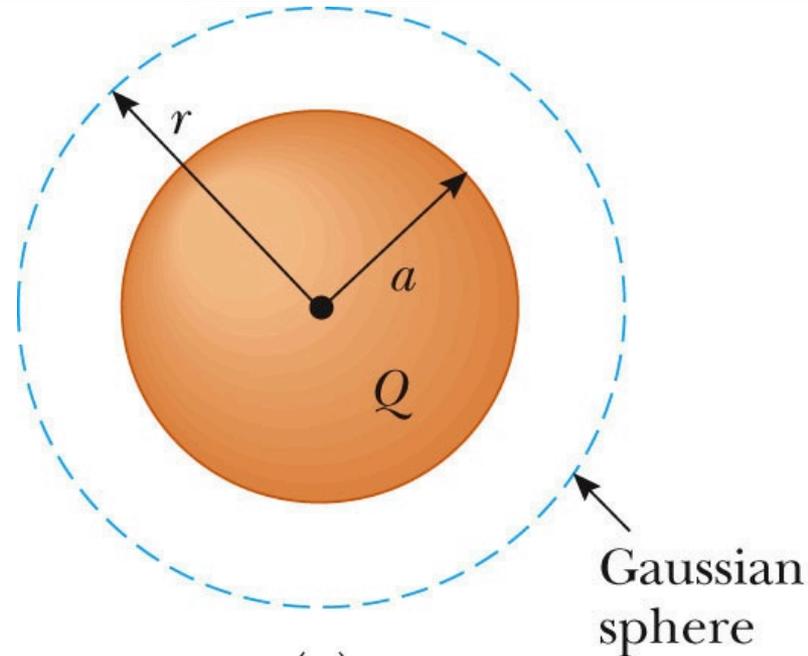
- Try to choose a surface that satisfies one or more of these conditions:
 - The value of the electric field can be argued from symmetry to be constant over the surface
 - The dot product of $\vec{E} \cdot d\vec{A}$ can be expressed as a simple algebraic product $E dA$ when \vec{E} and $d\vec{A}$ are parallel .
 - The dot product is 0 when \vec{E} and $d\vec{A}$ are perpendicular

Still no clue about how to use Gauss Law?
There are only three types of problems.
See examples in the following pages.

Problem type I:

Field Due to a Spherically Symmetric Even Charge Distribution, including a point charge.

- The field must be different inside ($r < a$) and outside ($r > a$) of the sphere.
- For $r > a$, select a sphere as the Gaussian surface, with radius r and concentric to the original sphere. Because of this symmetry, the electric field direction must be radially along r , and at a given r , the field's magnitude is a constant.

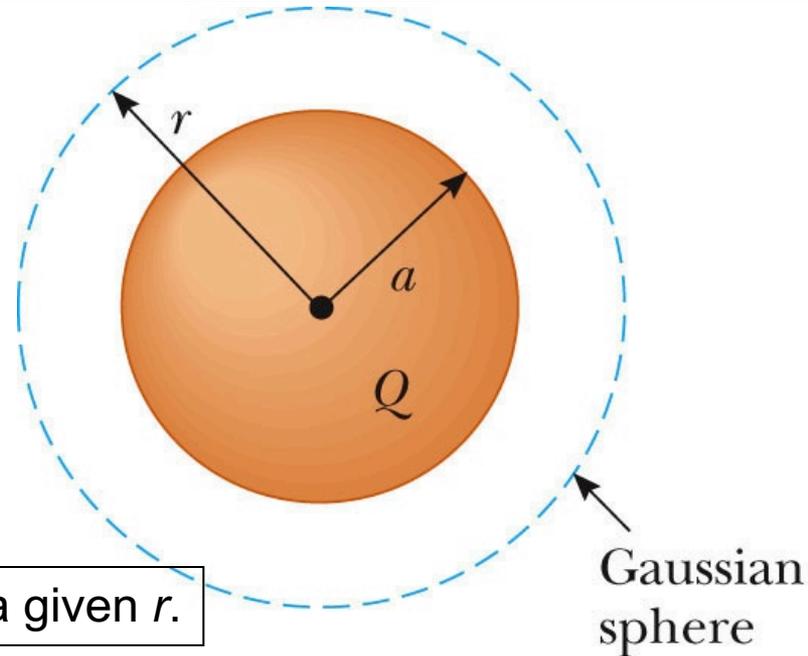


Can you write down the mathematical expression based on the above reasoning?

Problem type I:

Field Due to a Spherically Symmetric Even Charge Distribution, including a point charge.

- The field must be different inside ($r < a$) and outside ($r > a$) of the sphere.
- For $r > a$, select a sphere as the Gaussian surface, with radius r and concentric to the original sphere. Because of this symmetry, the electric field direction must be radially along r , and at a given r , the field's magnitude is a constant.



E is constant at a given r .

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \cdot dA = E \cdot 4\pi r^2 = \frac{q_{in}}{\epsilon_0}$$

Gauss Law

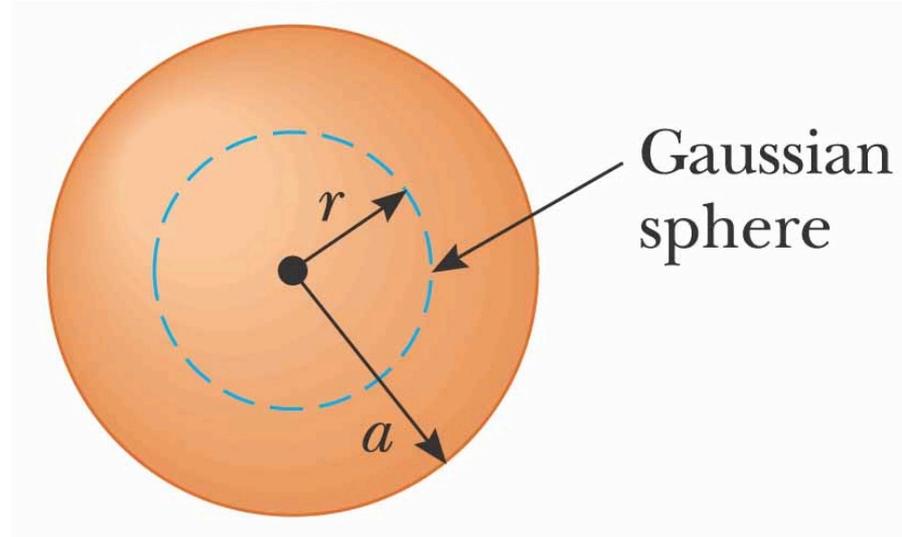
$$E = \frac{1}{4\pi\epsilon_0} \frac{q_{in}}{r^2} = k_e \frac{Q}{r^2}$$

$$\vec{E} = k_e \frac{Q}{r^2} \hat{r}$$

As if the charge is a point charge Q

Field inside the sphere

- For $r < a$, select a sphere as the Gaussian surface.
- All the arguments are the same as for $r > a$. The only difference is here $q_{in} < Q$
- Find out that $q_{in} = Q(r/a)^3$ (How?)



$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E dA = E \cdot 4\pi r^2 = \frac{q_{in}}{\epsilon_0}$$

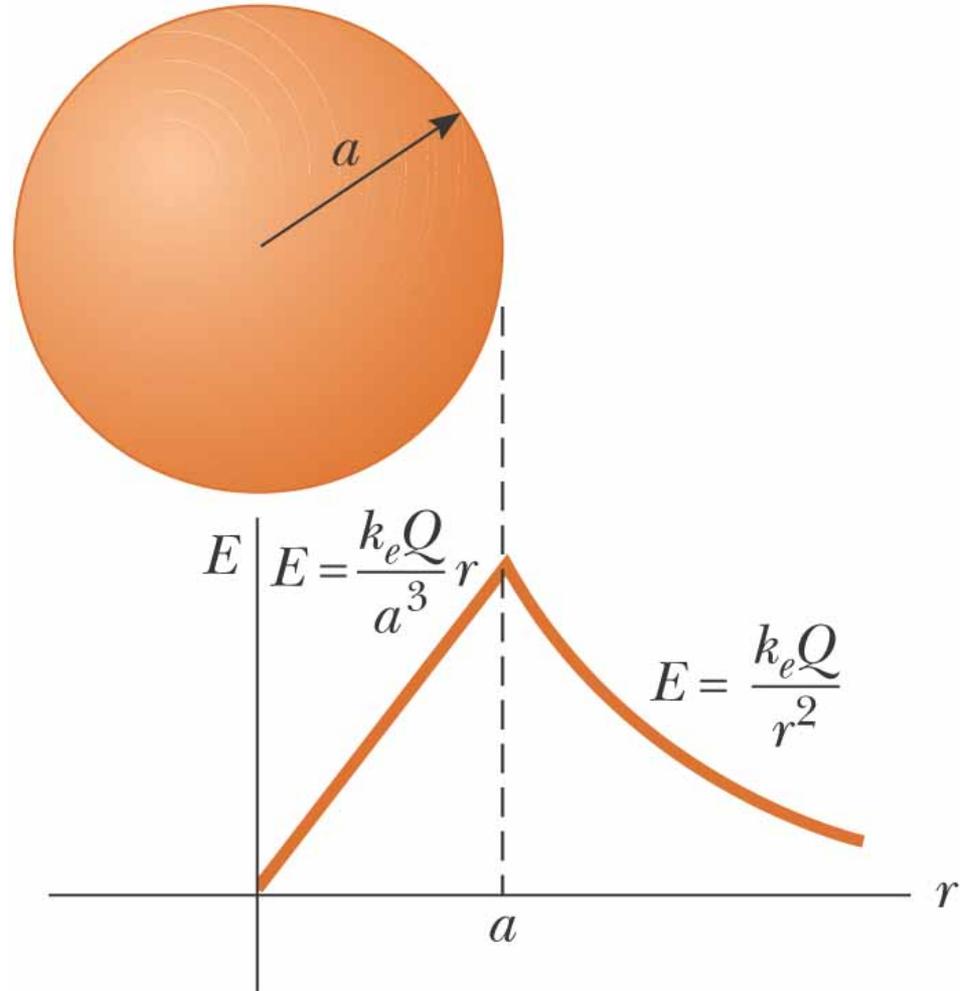
$$E = \frac{1}{4\pi\epsilon_0} \frac{q_{in}}{r^2} = k_e \frac{q_{in}}{r^2} = k_e \frac{1}{r^2} \frac{r^3}{a^3} Q = k_e \frac{Q}{a^3} r$$

$$\vec{E} = k_e \frac{Q}{a^3} \vec{r}$$

← Increase linearly with r , not with $1/r^2$

Plot the results (assume positive Q)

- Inside the sphere, E varies linearly with r
 $E \rightarrow 0$ as $r \rightarrow 0$
- The field outside the sphere is equivalent to that of a point charge located at the center of the sphere

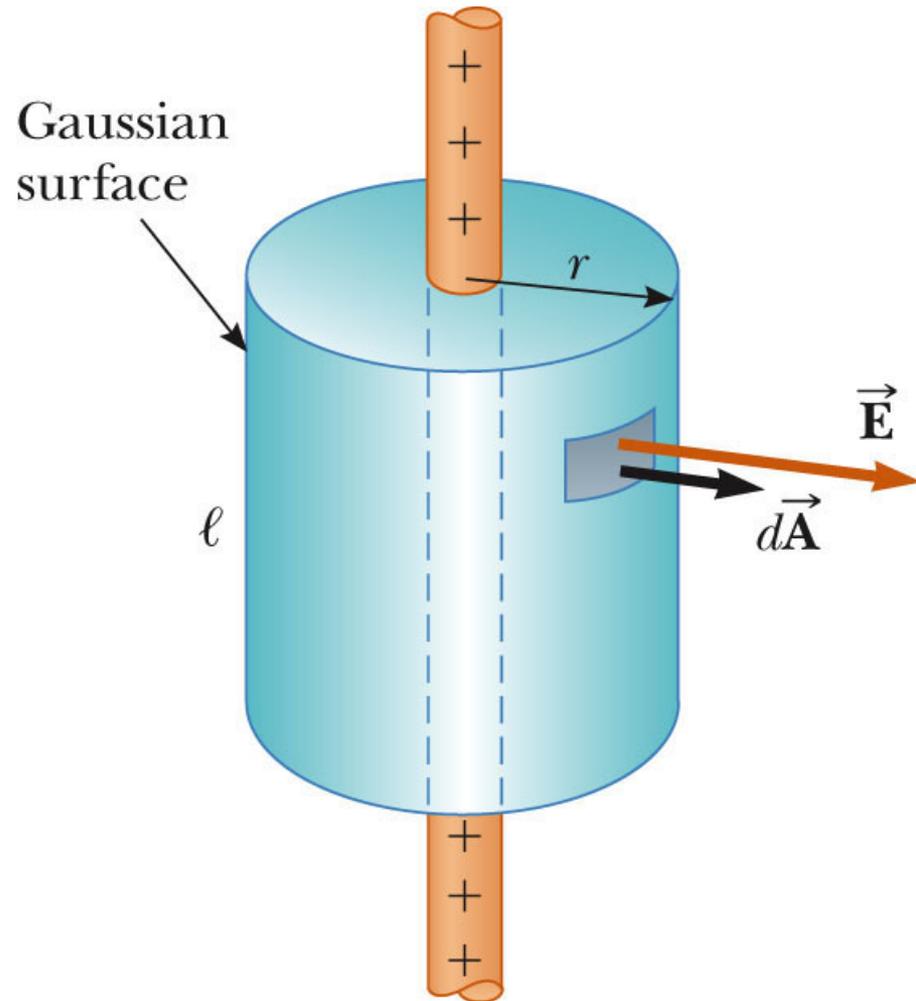


This concludes Gauss Law type I problems.

Problem type II:

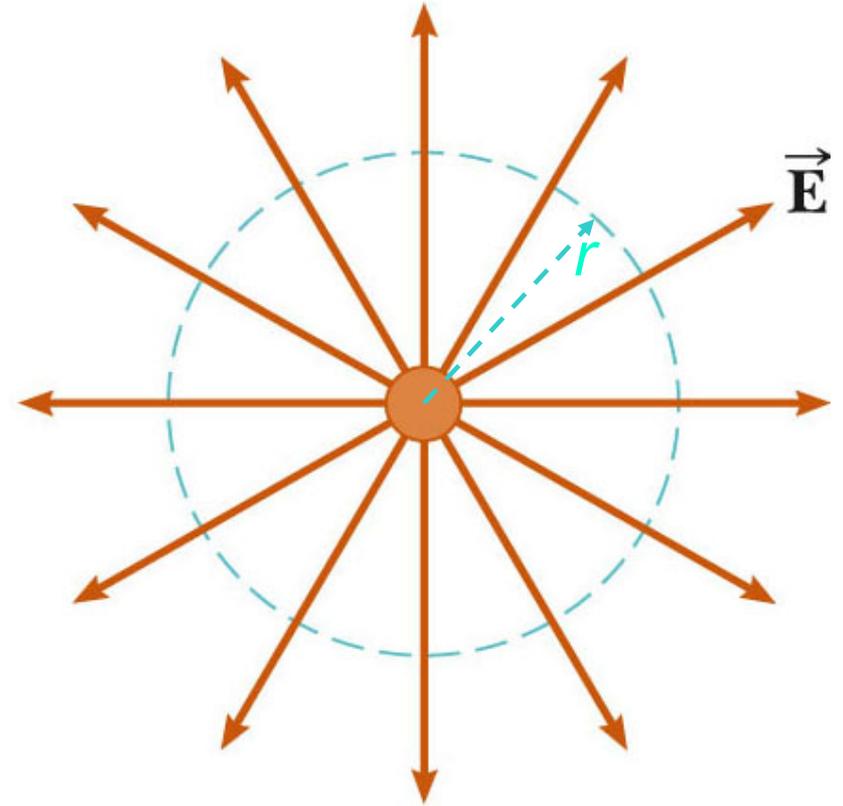
Field at a Distance from a Straight Line of Charge

- Select a cylinder as the Gaussian surface. The cylinder has a radius of r and a length of ℓ
- \vec{E} is constant in magnitude and parallel to the surface (the direction of a surface is its normal!) at every point on the curved part of the surface (the body of the cylinder).



Calculate the flux

- Because of this line symmetry, the end view illustrates more clearly that the field is parallel to the curved surface, and constant at a given r , so the flux is $\Phi_E = E \cdot 2\pi r \ell$
- The flux through the ends of the cylinder is 0 since the field is perpendicular to these surfaces.



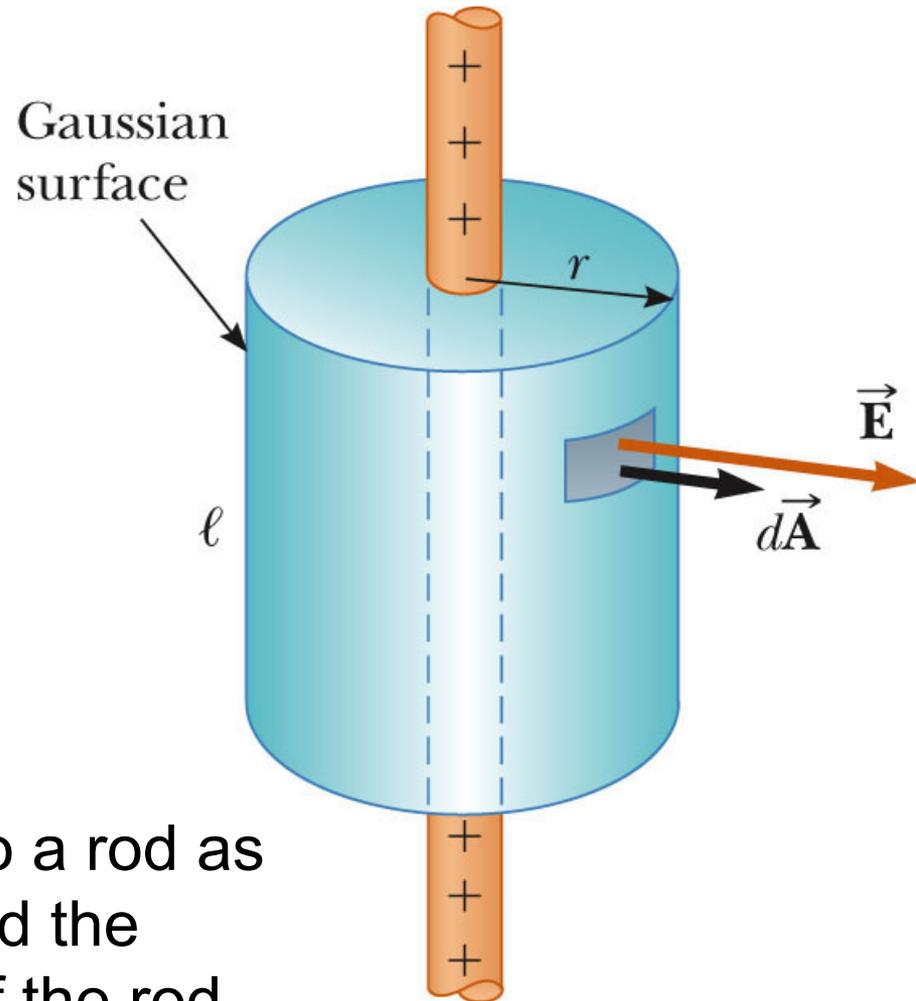
Electric field from Gauss Law

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E dA = \frac{q_0}{\epsilon_0}$$

$$E \cdot 2\pi r \cdot l = \frac{\lambda l}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k_e \frac{\lambda}{r}$$

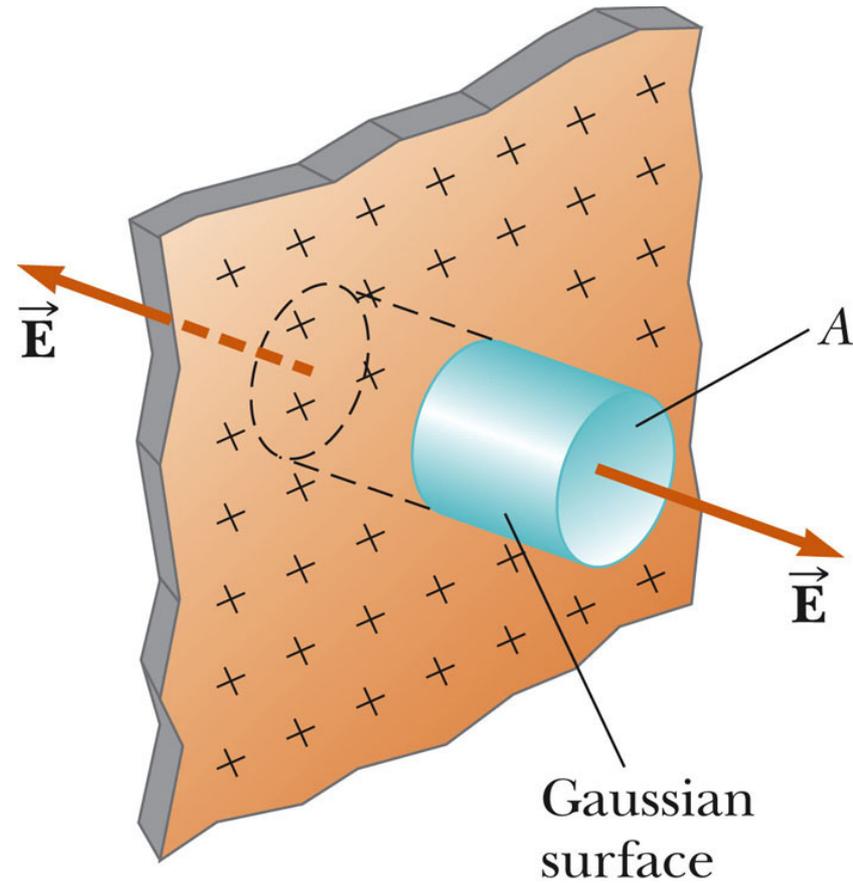
One can change the thin wire into a rod as we did in the sphere case and find the electric field inside and outside of the rod.



This concludes Gauss Law type II problems.

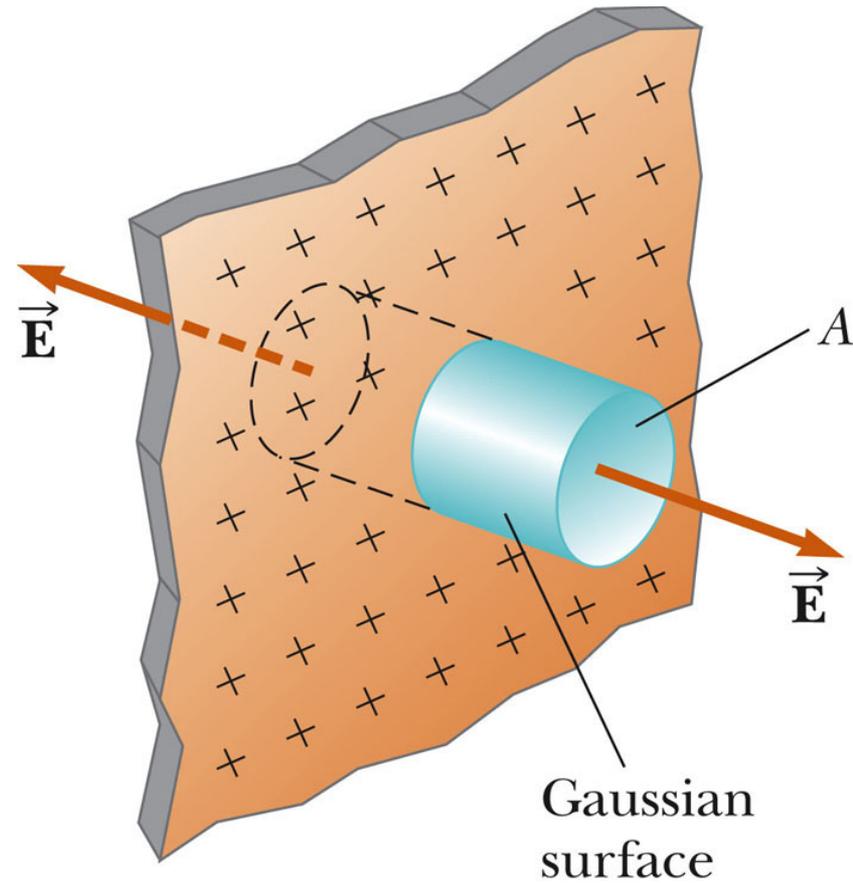
Problem type III: Field Due to an Infinitely Large Plane of Charge

- Argument about the electric field: Because the plane is infinitely large, any point can be treated as the center point of the plane, so at that point \vec{E} must be parallel to the plane direction (again this is its normal) and must have the same magnitude at all points equidistant from the plane.
- Choose the Gaussian surface to be a small cylinder whose axis is parallel to the plane's direction (third time, this is the normal of the plane).



Find out the flux

- \vec{E} is perpendicular to the curved surface direction, so the flux through this surface is 0, because $\cos(90^\circ) = 0$.
- \vec{E} is parallel to the ends, so the flux through each end of the cylinder is EA and the total flux is $2EA$



Electric field from Gauss Law

- The total charge in the surface is $\sigma \cdot A$
- Applying Gauss's law

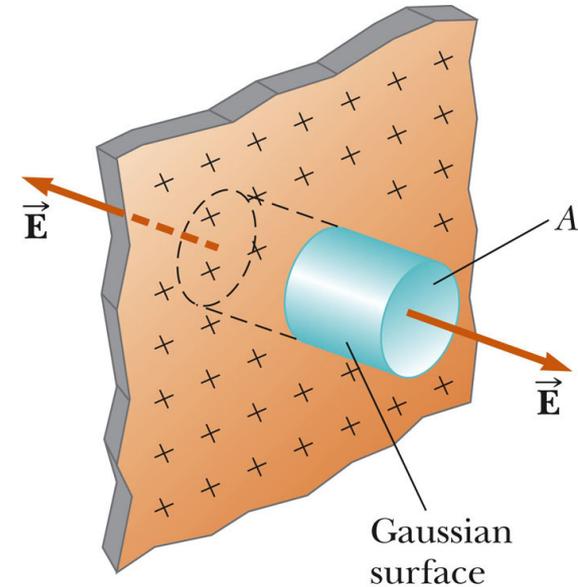
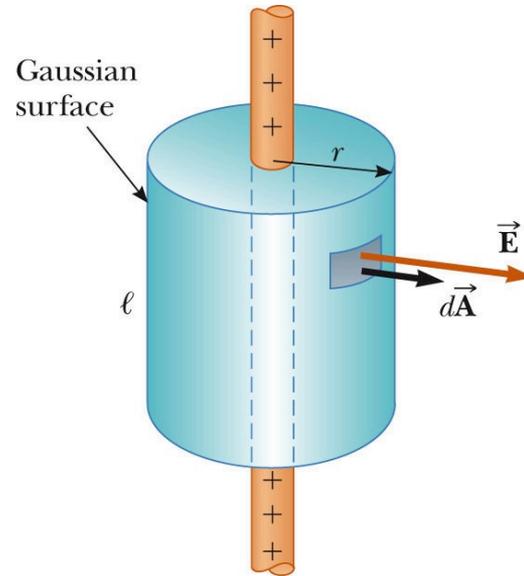
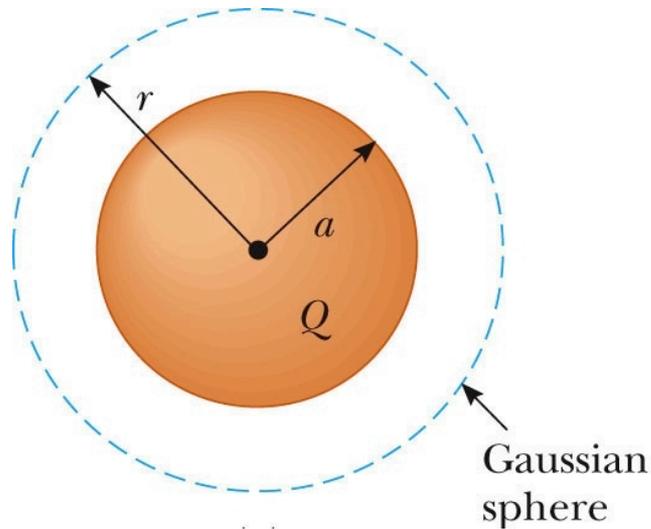
$$\Phi_E = 2EA = \frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{2\epsilon_0}$$

- Note, this does not depend on r , the distance from the point of interest to the charged plane. **Why?**
- Therefore, the field is uniform everywhere

One can also change the plane (without thickness) into a plate with thickness d , and follow the discussion in example I.

This concludes Gauss Law type III problems.

To summarize: the 3 types of Gauss Law problems



$$\vec{E} = \left(k_e \frac{Q}{a^3} r \right) \hat{r}, \quad \text{when } r < a$$

$$\vec{E} = \left(k_e \frac{Q}{r^2} \right) \hat{r}, \quad \text{when } r \geq a$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = 2k_e \frac{\lambda}{r}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

Other applications for Gauss Law: Electrostatic Equilibrium

- **Definition:**

When there is no net motion of charge within a conductor, the conductor is said to be in **electrostatic equilibrium**

- **When in electrostatic equilibrium, the properties:**

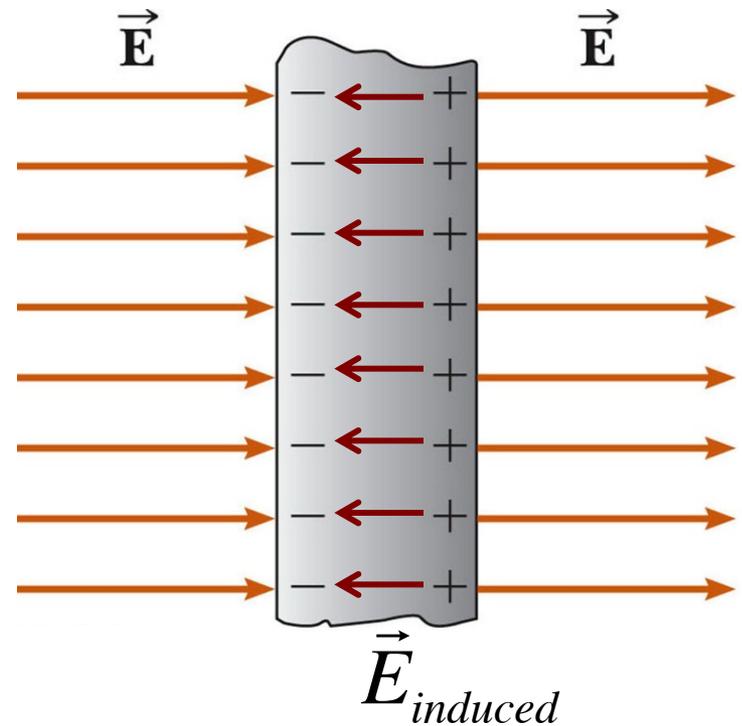
- The electric field is zero everywhere inside the conductor, whether the conductor is solid or hollow ← **Faraday Cage**
- If an isolated conductor carries a charge, the charge resides on its surface
- The electric field just outside a charged conductor is perpendicular to the surface and has a magnitude of σ/ϵ_0 , σ is the surface charge density at that point
- On an irregularly shaped conductor, the surface charge density is inversely proportional to the radius at that local surface, so σ is greatest at locations where the radius of curvature is the smallest ← **Lightning Rod**



More discussions about electrostatic equilibrium properties.

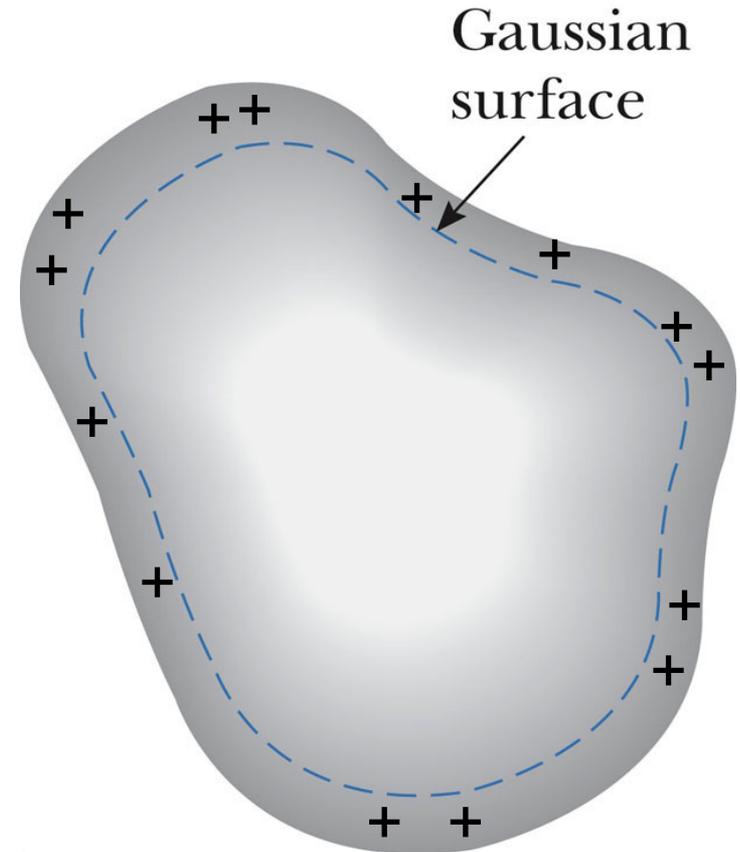
Property 1: for a conductor, $\text{Field}_{\text{inside}} = 0$

- Consider a neutral conducting slab, when there is no external field, charges are distributed throughout the conductor, experience no force and are in electrostatic equilibrium.
- When there is an external field \vec{E} This external field will exert a force on the charges inside the conductor and redistribute them in such a way that the internal electric field generated by these redistributed charges cancel the external field so that **the net field inside the conductor is zero** to prevent further motion of charges.
- Hence the conductor reaches again electrostatic equilibrium
- This redistribution takes about 10^{-16} seconds and can be considered instantaneous



Property 2: For a charged conductor, charge resides only on the surface, and the field inside the conductor is still zero.

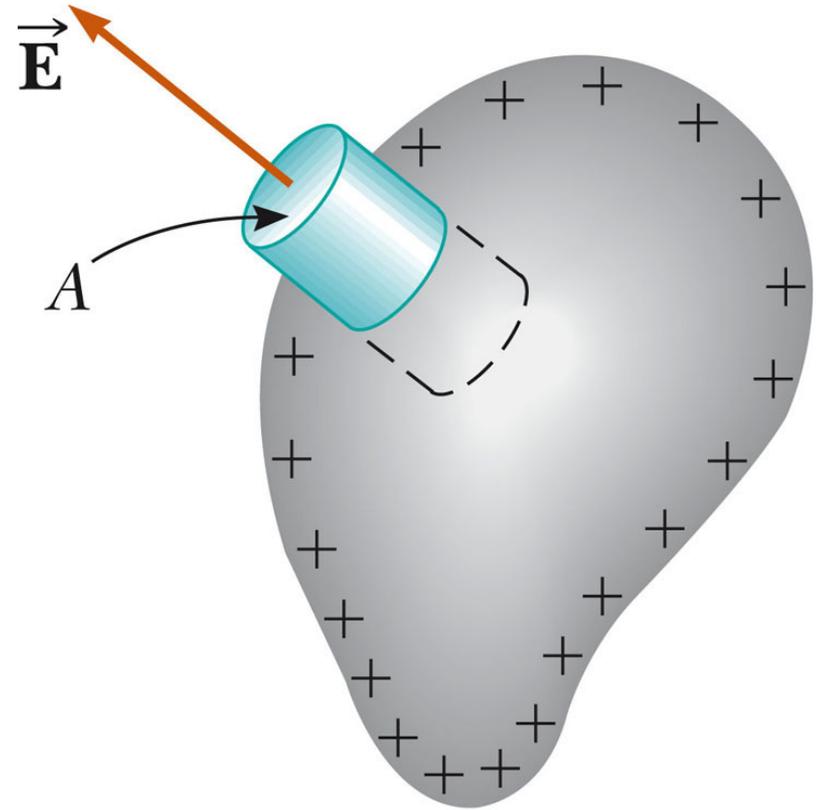
- Charges (have to be the same sign, why?) repel and move away from each other until they reach the surface and can no longer move out: charge resides only on the surface because of Coulomb's Law.
- Choose a Gaussian surface inside but close to the actual surface
- Since there is no net charge inside this Gaussian surface, there is no net flux through it.
- Because the Gaussian surface can be any where inside the volume and as close to the actual surface as desired, the electric field inside this volume is zero anywhere.



Property 3: Field's Magnitude and Direction on the surface

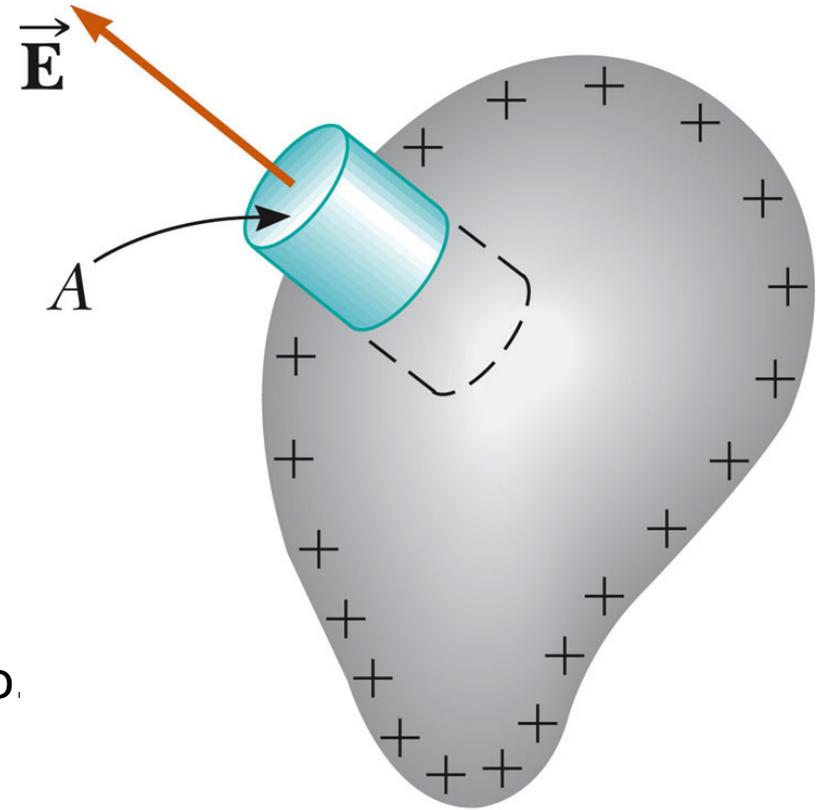
Direction:

- Choose a cylinder as the gaussian surface
- The field must be parallel to the surface (again this is its normal)
 - If there were an angle ($\theta \neq 0$), then there were a component E_{\perp} from \vec{E} and tangent to the surface that would move charges along the surface. Then the conductor would not be in equilibrium (no charge motions)



Property 3: Field's Magnitude and Direction on the surface

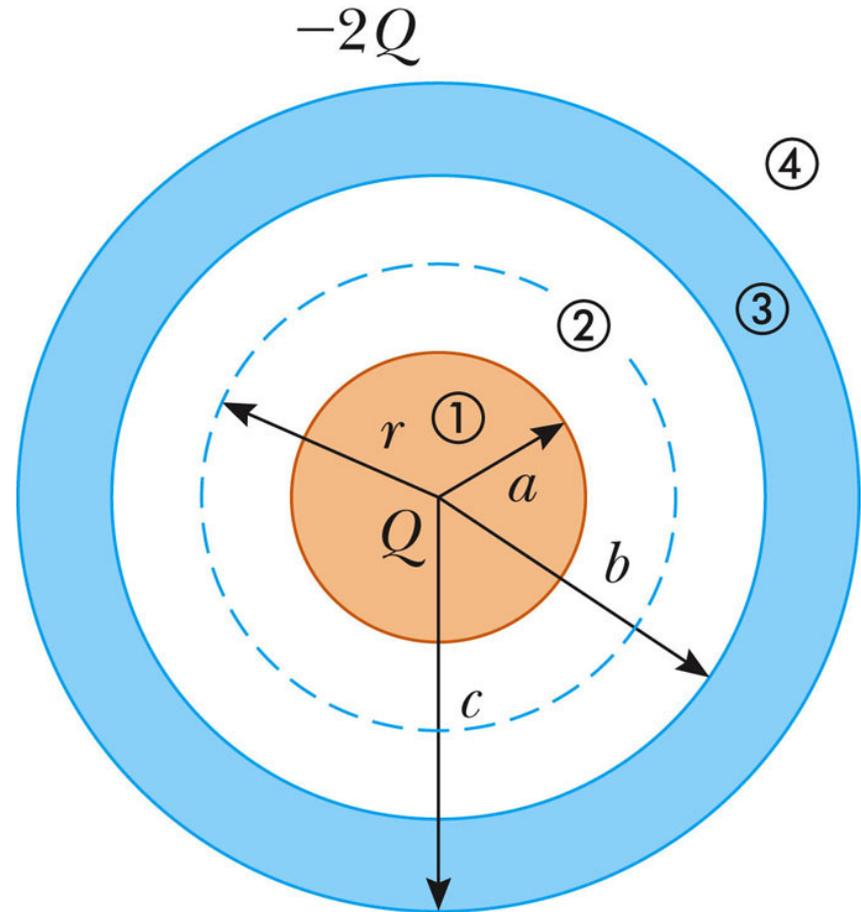
- **Magnitude:**
- Choose a Gaussian surface as an infinitesimal cylinder with its axis parallel to the conductor surface, as shown in the figure. The net flux through the Gaussian surface is that only through the flat face outside the conductor
 - The field here is parallel to the surface
 - The field on all other surfaces of the Gaussian cylinder is either perpendicular to that surface, or zero.
- Applying Gauss's law, we have



$$\Phi_E = EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{\epsilon_0}$$

Another example: Electric field generated by a conducting sphere and a conducting shell

- *Charge and dimensions as marked*
- *Analyze:*
 - System has spherical symmetry, Gauss Law problem type I.
 - Electric field inside conductors is zero
 - There are two other ranges, $a < r < b$ and $b < r$ that need to be considered
- *Arguments for electric field*
 - Similar to the sphere example, because the spherical symmetry, the electrical field in these two ranges $a < r < b$ and $b < r$ is only a function of r , and goes along the radius.



Construct Gaussian surface and calculate the flux, and use Gauss Law to get the electric field

- $E = 0$ when $r < a$, and $b < r < c$
- Construct a Gaussian sphere with its center coincides with the center of the inner sphere

- When $a < r < b$:

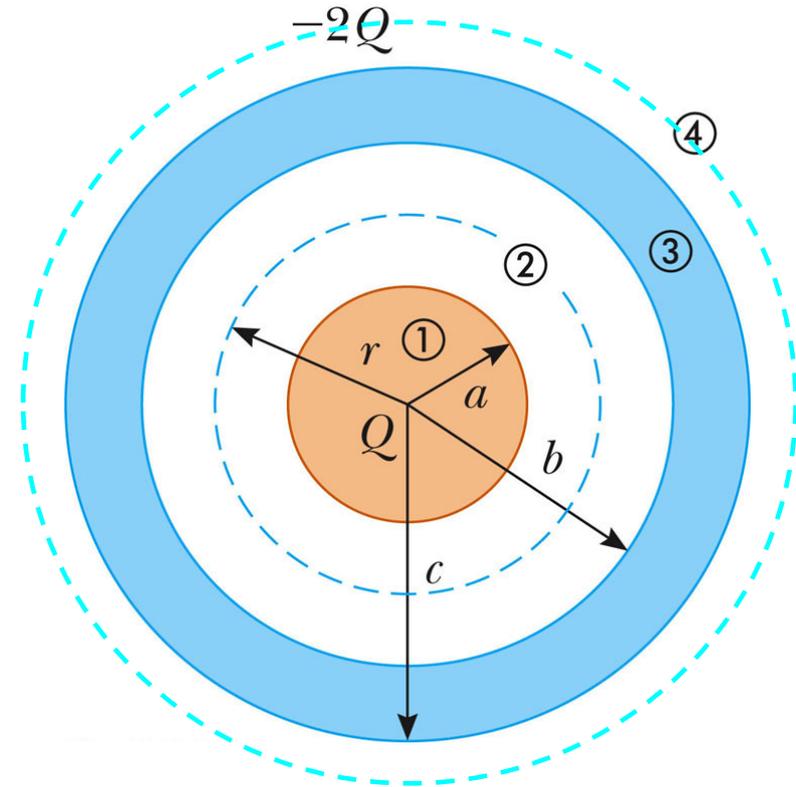
- The flux $\Phi_E = E \cdot 4\pi r^2$
- Apply Gauss Law $\Phi_E = Q/\epsilon_0$

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} = k_e \frac{Q}{r^2} \quad \text{or} \quad \vec{E} = k_e \frac{Q}{r^2} \hat{r}$$

- When $b < r$

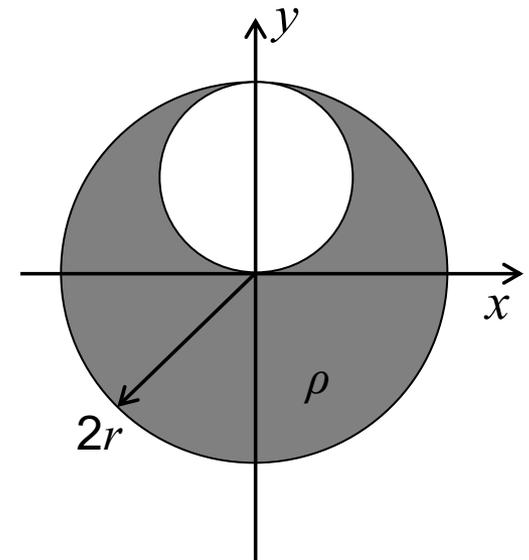
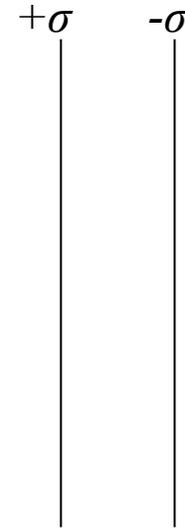
- The flux $\Phi_E = E \cdot 4\pi r^2$
- Apply Gauss Law $\Phi_E = (-2Q + Q)/\epsilon_0$

$$E = \frac{1}{4\pi\epsilon_0} \frac{-2Q + Q}{r^2} = k_e \frac{-Q}{r^2} \quad \text{or} \quad \vec{E} = k_e \frac{-Q}{r^2} \hat{r}$$



Two more example problems

- Example 1: find the electric field between and outside of two parallel plates that are infinitely large and oppositely charged with equal surface density of σ
- Example 2: a solid sphere of radius $2r$ with a charge density ρ has a cavity of radius r . Find the electrical field along the y -axis from 0 to $2r$.



Reading material and Homework assignment

Please watch this video (about 50 minutes):

http://videolectures.net/mit802s02_lewin_lec03/

Please check wileyplus webpage for homework assignment.