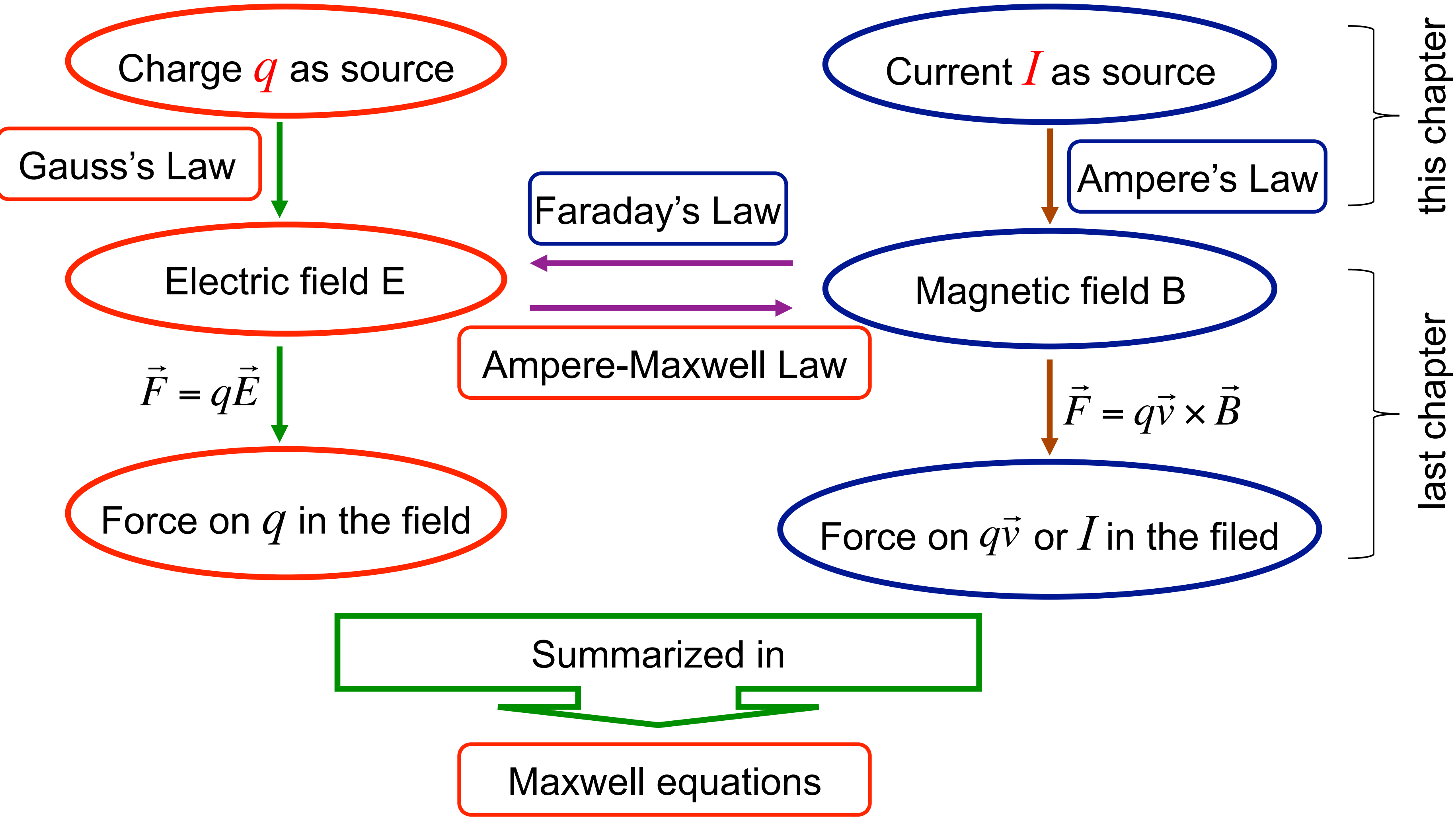
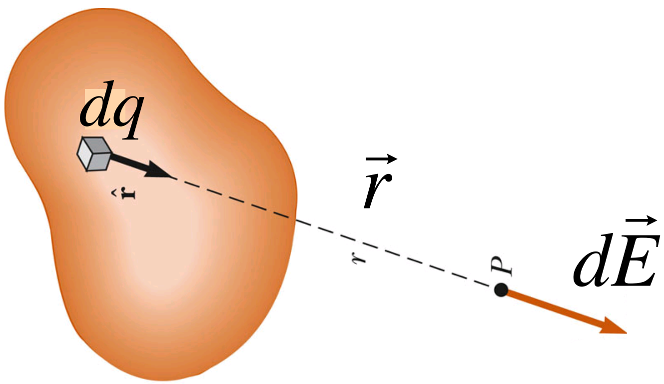
Sources of Magnetic Fields



Sources of electric field and magnetic field

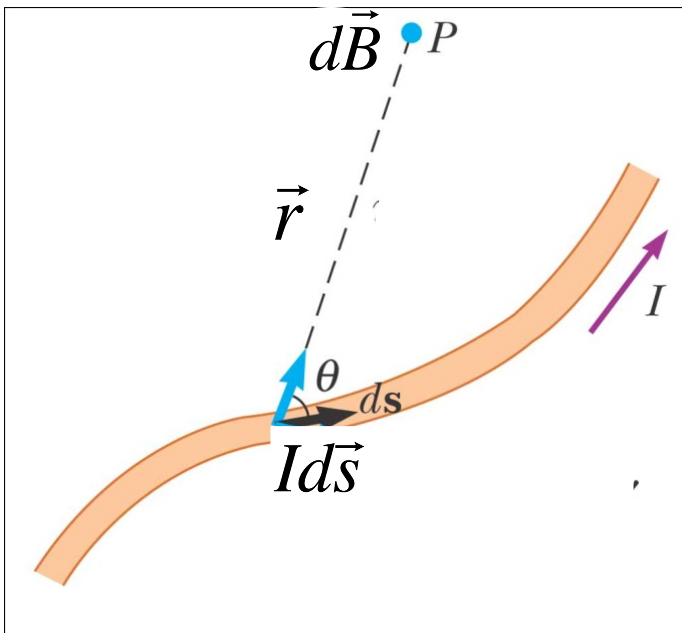
From Coulomb's Law, a point charge dq generates electric field distance $\vec{r}$ away from the source:

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$



From Biot-Savart's Law, a point current $Id\vec{s}$ generates magnetic field distance $\vec{r}$ away from the source:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{s}}{r^2}\times\hat{r}$$



Difference:

1. A current segment $Id\vec{s}$, not a point charge dq, hence a vector.
2. Cross product of two vectors, $d\vec{B}$ is determined by the right-hand rule, not $\vec{r}$.

Total magnetic field

- $d\vec{B}$ is the field created by the current in the wire segment with length $d\vec{s}$, a vector that takes the direction of the current.
- To find the total field, sum up the contributions from all the current elements $Id\vec{s}$

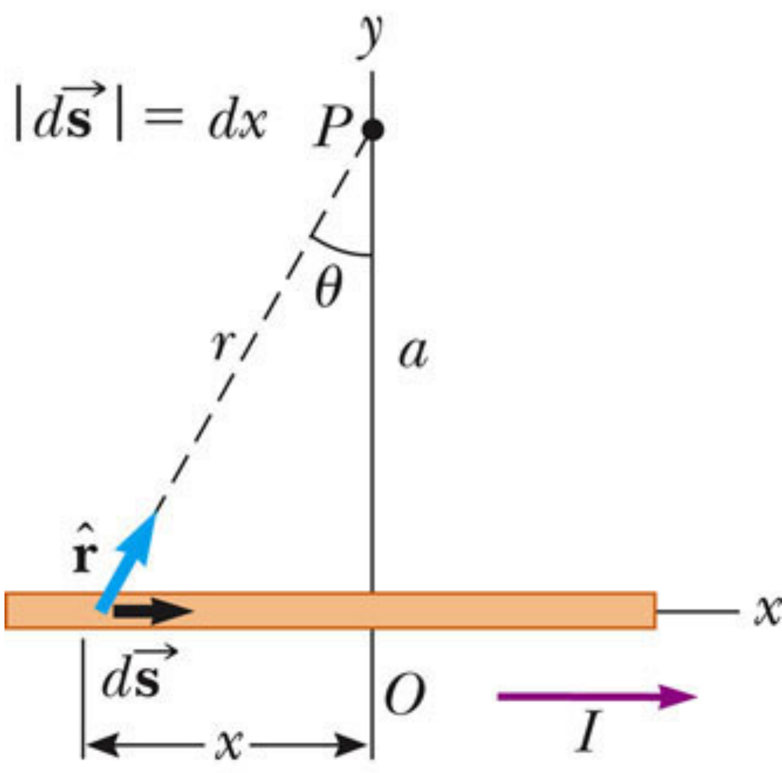
$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{Id\vec{s} \times \hat{r}}{r^2}$$

The integral goes over the entire current distribution

- $\mu_o = 4\pi \times 10^{-7}$ T· m / A , is the constant, the **permeability of free space**

Example: a straight wire

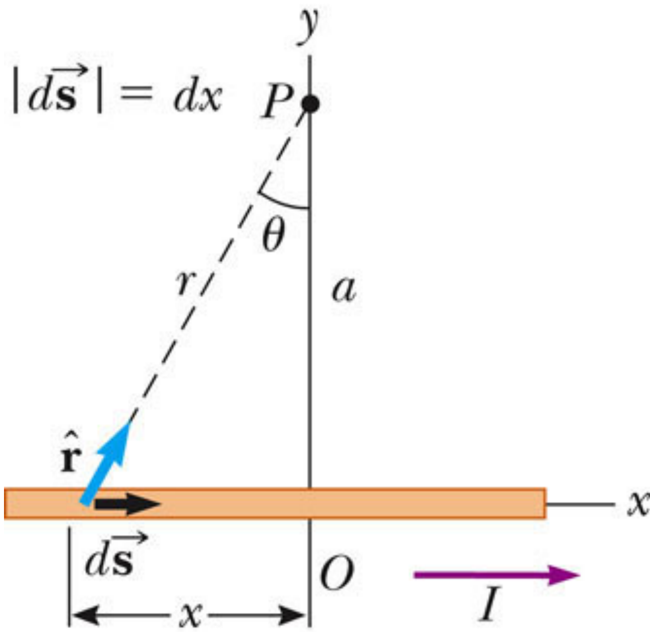
- A thin, straight wire is carrying a constant current I
- Constructing the coordinate system and place the wire along the x-axis, and point P in the x-y plane.
- $d\vec{s} \times \hat{r} = dx \sin\left(\frac{\pi}{2} - \theta\right)\hat{k} = dx\cos(\theta)\hat{k}$
- Integrating over all the current elements gives



$$B = -\frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2}\cos(\theta)d\theta$$

$$= \frac{\mu_0 I}{4\pi a}(\sin\theta_1 - \sin\theta_2), \text{ or } \vec{B} = B\hat{k}$$

The math part



$$d\vec{s} \times \hat{r} = dx \sin\left(\frac{\pi}{2} - \theta\right)\hat{k} = dx \cos\theta \hat{k}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}}{r^2} \times \hat{r} = \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{(a/\cos\theta)^2} dx\hat{k}$$

$$x = -a\tan\theta, \quad dx = -\frac{a d\theta}{\cos^2\theta}$$

$$dB = -\frac{\mu_0 I}{4\pi a}\cos\theta d\theta, \quad B = \int_{\theta_1}^{\theta_2} dB = -\frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\sin\theta_1 - \sin\theta_2)$$

$$\vec{B} = B\hat{k}$$

Now make the wire infinitely long

- The statement "the conductor is an infinitely long, straight wire" translates into:

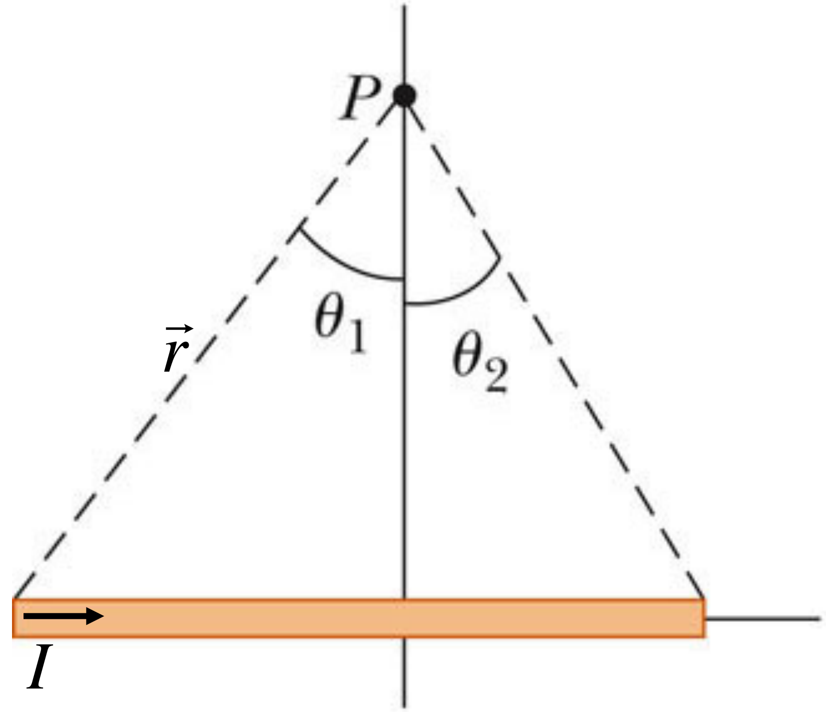
$$\theta_1 = \pi/2,\ \theta_2 = -\pi/2$$

- Then the magnitude of the field becomes

$$B = -\frac{\mu_0 I}{4\pi a}\int_{-\pi/2}^{\pi/2} \cos(\theta)\,d\theta$$

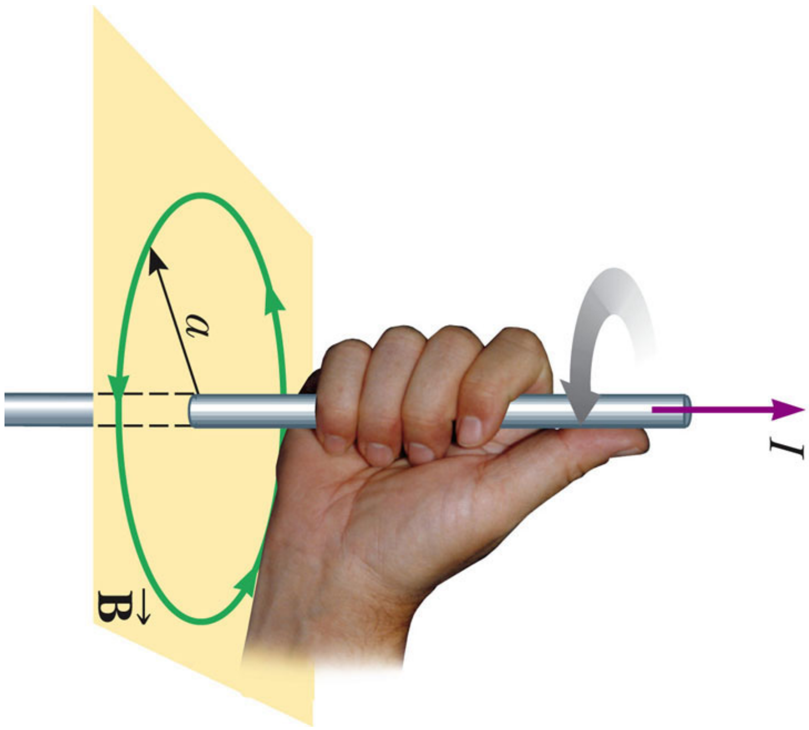
$$= \frac{\mu_0 I}{4\pi a}\left(\sin\frac{\pi}{2} - \sin\frac{-\pi}{2}\right) = \frac{\mu_0 I}{2\pi a}$$

- The direction of the field is determined by the right-hand rule: coming out of the page at the point P.



The magnetic field direction

- The magnetic field lines are circles concentric with the wire.
- The field lines lie in planes perpendicular to to wire
- The magnitude of the field is constant on any circle of a radius a.
- A different and more convenient right-hand rule for determining the direction of the field is shown.

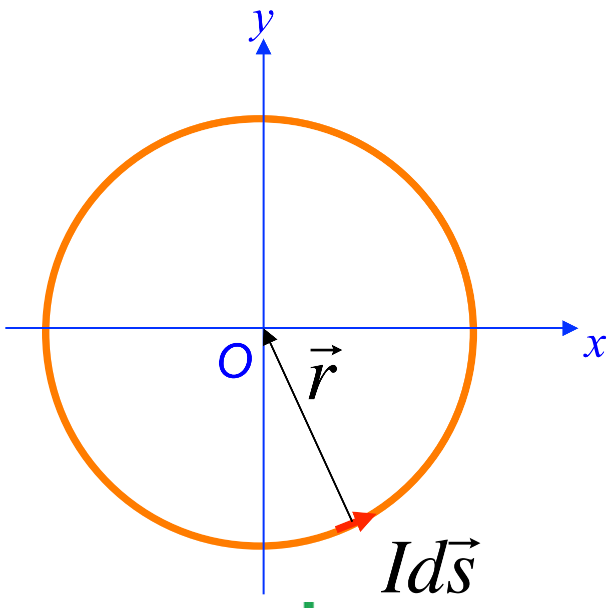


Example: a circle of wire

- From Biot-Savart Law, the field at O from $I d\vec{s}$ is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{s}}{r^2} \times \hat{r} = \frac{\mu_0}{4\pi}\frac{I ds}{r^2}\vec{k},$$

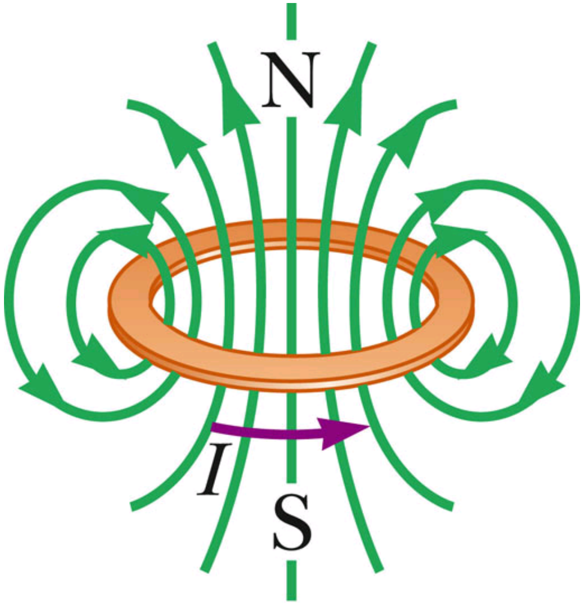
$$B = \frac{\mu_0}{4\pi}\frac{I}{r^2}\int_{\text{full circle}} ds = \frac{\mu_0}{4\pi}\frac{I}{r^2}2\pi r = \frac{\mu_0 I}{2r}$$



- This is the field at the *center* of the loop

$$B = \frac{\mu_0 I}{2a}, \text{ or } \vec{B} = B\hat{k}$$

Off center points of a single loop are not so easy to calculate.



How about a stack of loops?

Along the axis, yes, the formula is

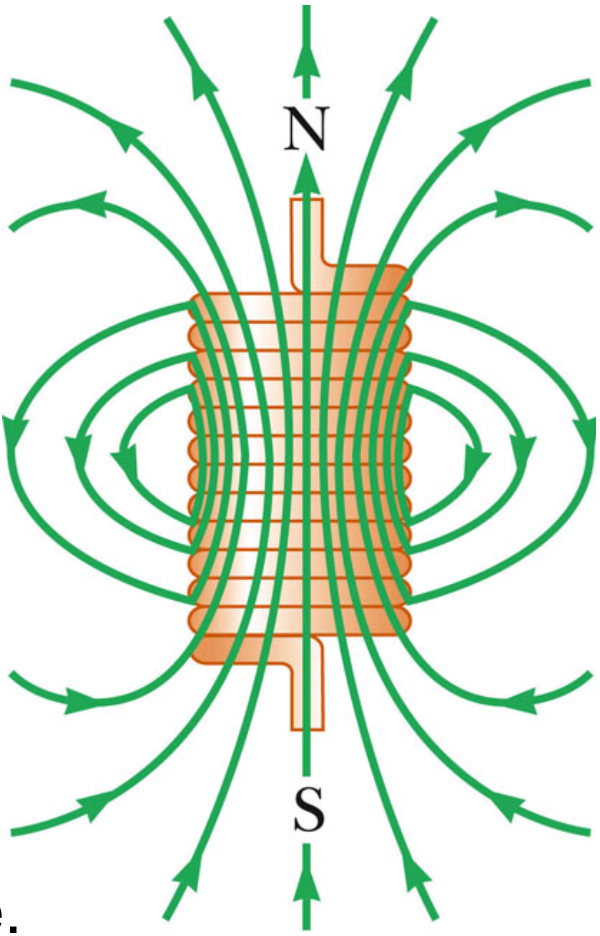
$$B = N\frac{\mu_0 I}{2r}, \text{ or } \vec{B} = B\hat{k}$$

N is the number of turns, r the radius.

When the loop is sufficiently long, what can we say about the field inside the body of this electric magnet?

We need Gauss's Law, Oops, a typo.

We need Ampere's Law. Sorry, Mr. Ampere.

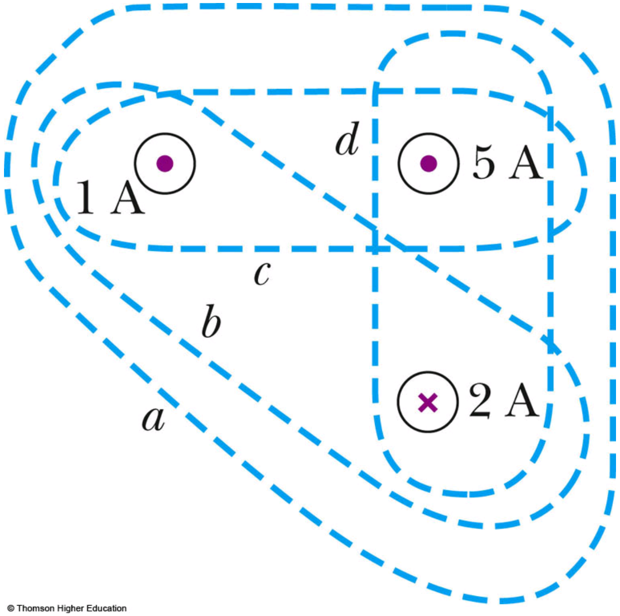


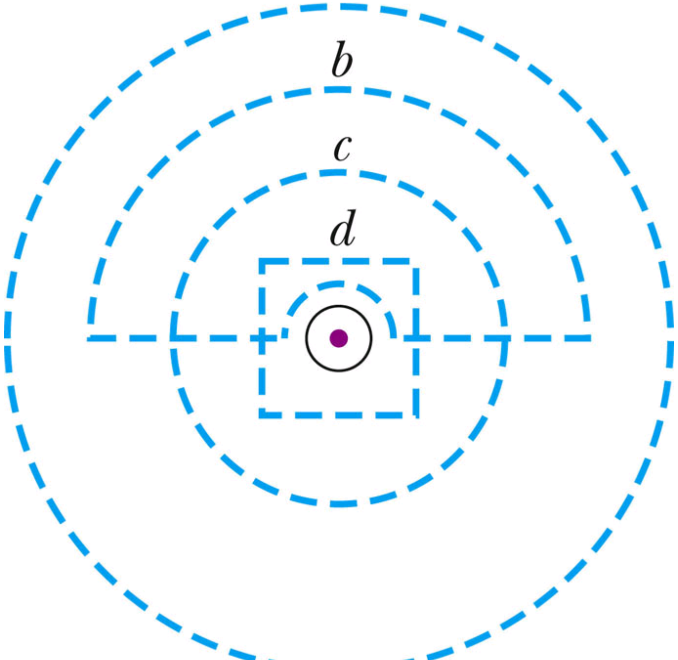
Ampere's Law connects B with I

- Ampere's law states that the line integral of $\vec{B}\cdot d\vec{s}$ around any closed path equals $\mu_o I$ where I is the total steady current passing through any surface bounded by the closed path:

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I$$

This is a line integral over a vector field





Use Ampere's Law: a straight wire

Choose the Gauss's Surface, oops, not again!

Choose the Ampere's loop, as a circle with radius a.

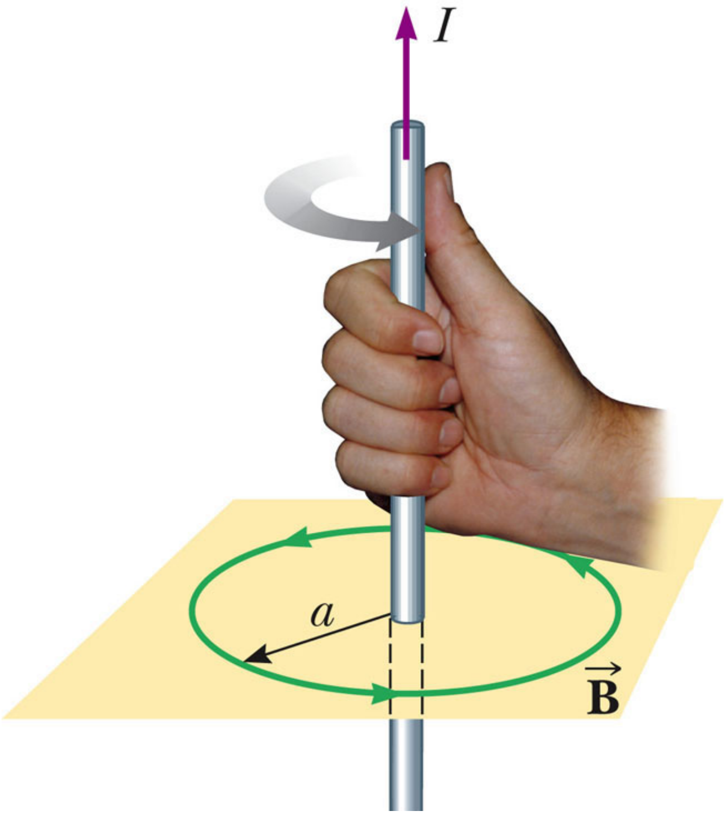
Ampere's Law says

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

$\vec{B}$ is parallel with $d\vec{s}$, so

$$B \oint ds = B 2\pi a = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi a}$$



When the wire becomes a rod with radius R

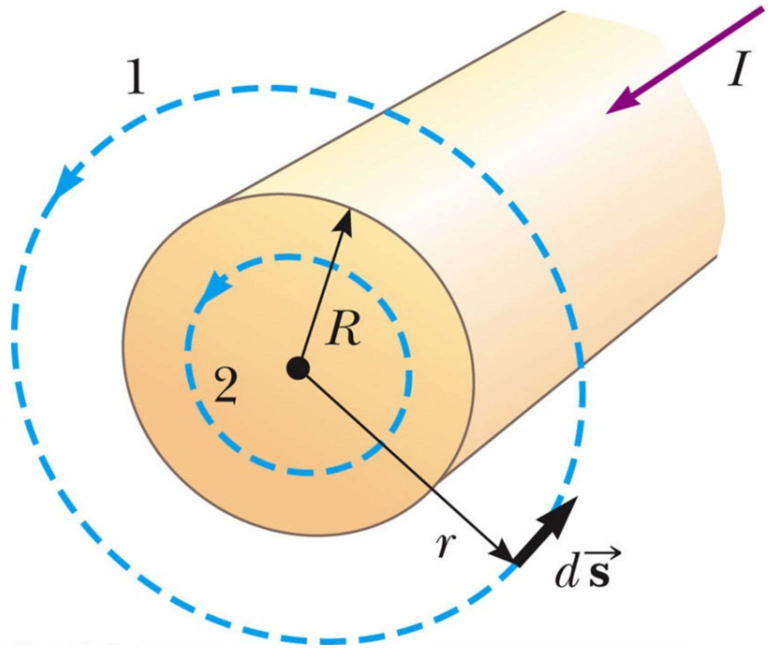
- Outside of the wire, $r > R$

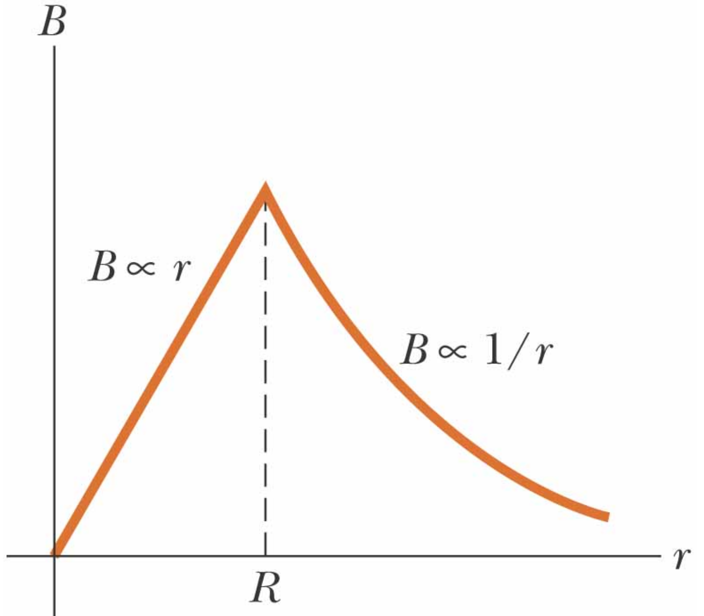
$$\oint \vec{B}\cdot d\vec{s} = B\cdot 2\pi r = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r}$$

- Inside the wire, we need I', the current inside the ampere's circle

$$\oint \vec{B}\cdot d\vec{s} = B\cdot 2\pi r = \mu_0 I' \rightarrow I' = \frac{r^2}{R^2} I$$

$$B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r$$





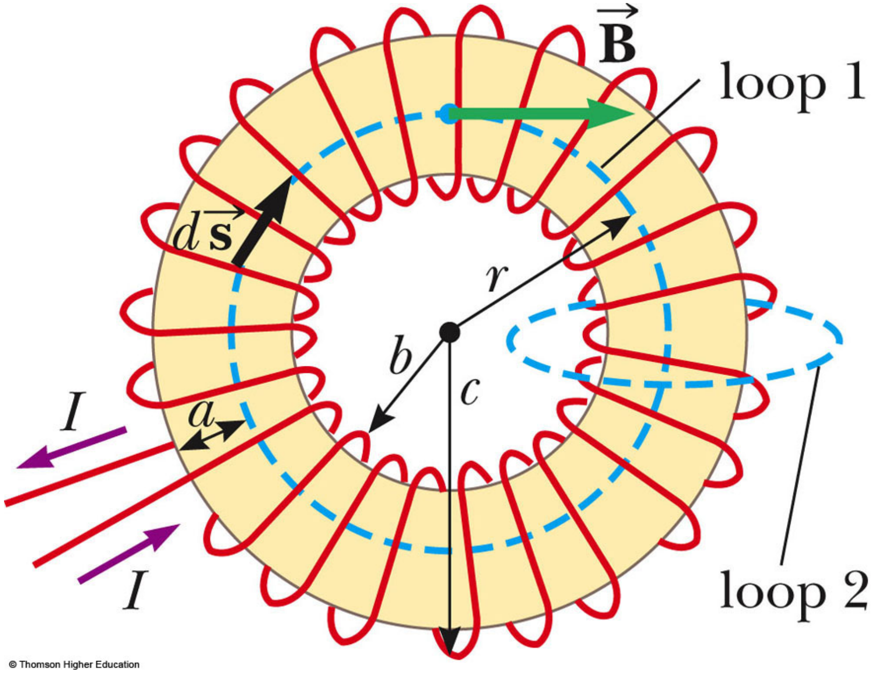
Magnetic field of a toroid

- The toroid has N turns of wire
- Find the field at a point at distance r from the center of the toroid (loop 1)

$$B\oint ds = B2\pi r = \mu_0 NI$$

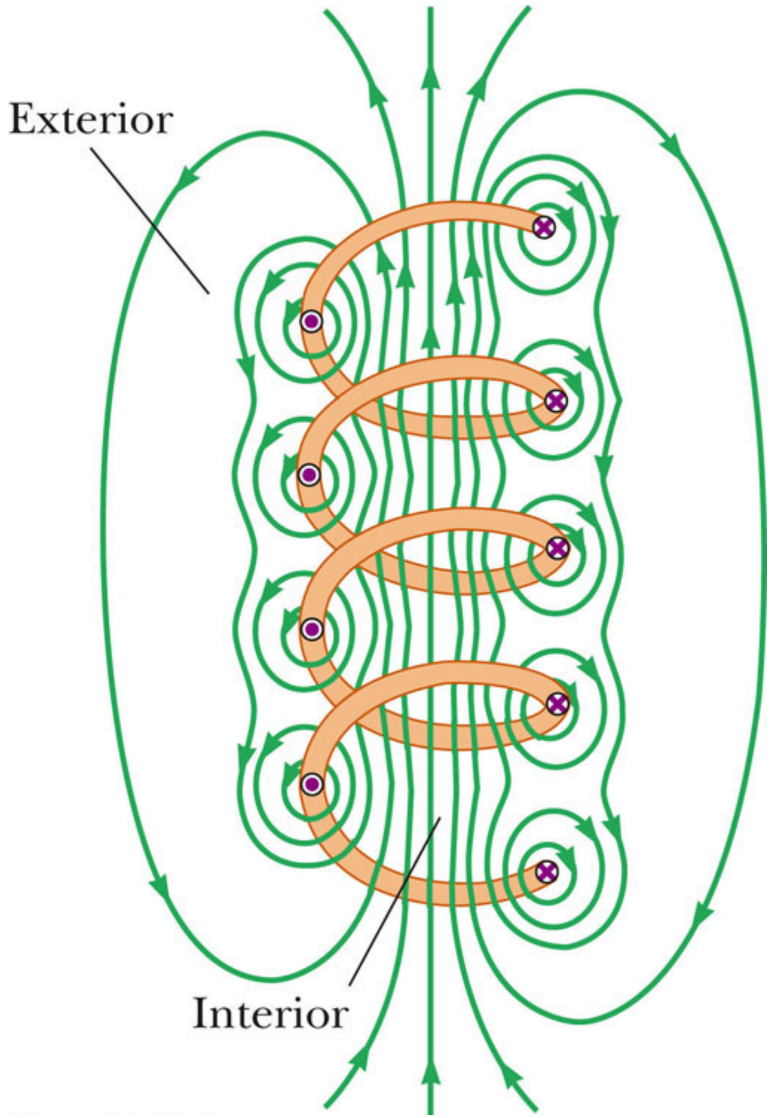
$$B = \frac{\mu_0 NI}{2\pi r}$$

- There is no field outside the coil (see loop 2)



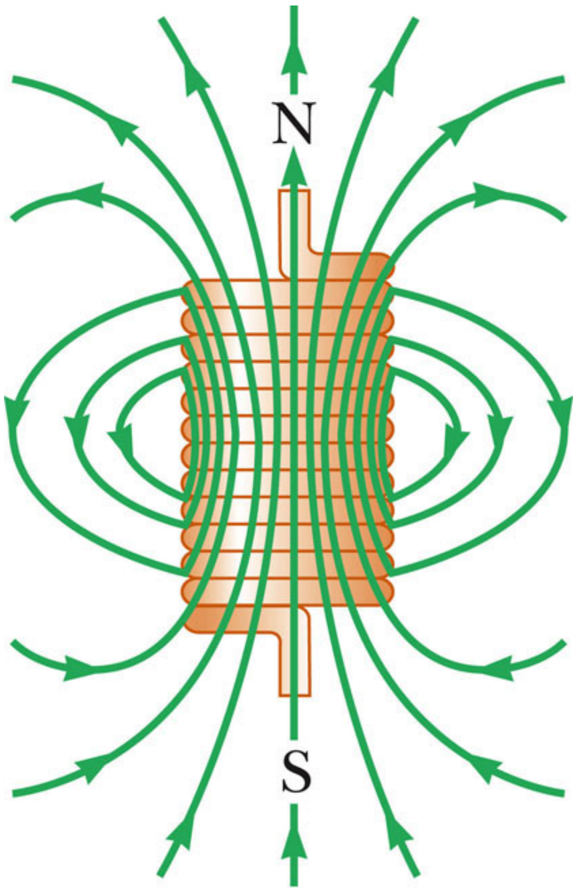
Magnetic field of a solenoid

- A **solenoid** is a long wire wound in the form of a helix.
- A reasonably uniform magnetic field can be produced in the space surrounded by the turns of the wire.
- The field lines in the interior are
  - nearly parallel to each other
  - uniformly distributed
- This is how people generate a uniform magnetic field.



Magnetic field of a tightly wound solenoid

- The field distribution is similar to that of a bar magnet.
- As the length of the solenoid increases:
  - the interior field becomes more uniform.
  - the exterior field becomes weaker.



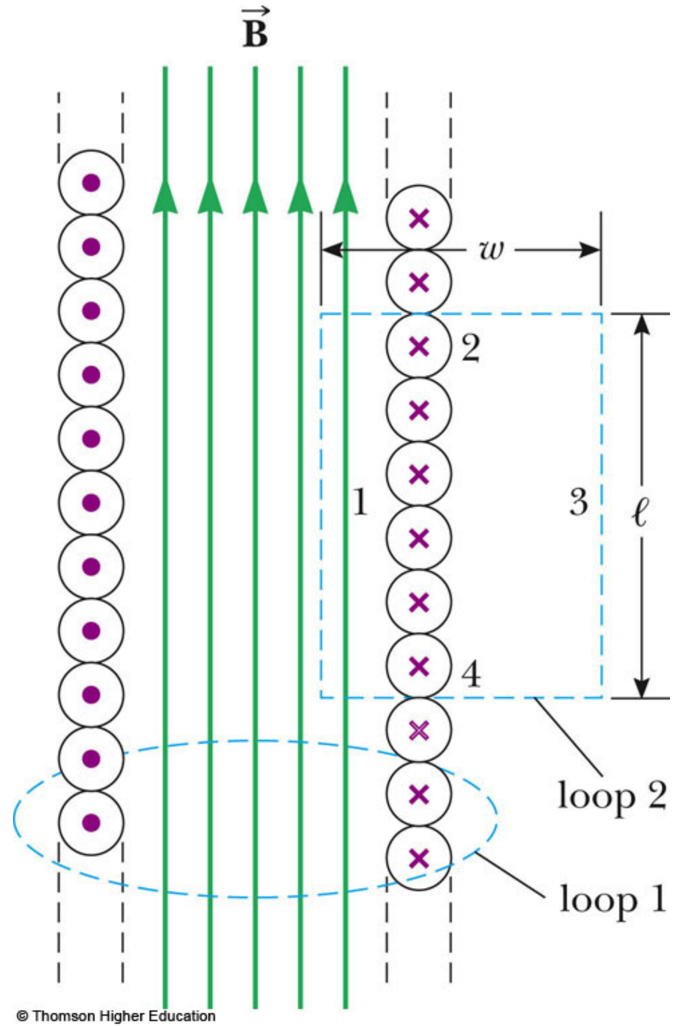
Ideal (infinitely long) solenoid

- An *ideal solenoid* is approached when:
  - the turns are closely spaced
  - the length is much greater than the radius of the turns
- Apply Ampere's Law to loop 2:

$$\oint \vec{B} \cdot d\vec{s} = \int_{path1} \vec{B} \cdot d\vec{s} = B \int_{path1} ds = Bl$$

$$\oint \vec{B} \cdot d\vec{s} = Bl = \mu_0 NI$$

$$B = \mu_0 \frac{N}{l} I = \mu_0 nI$$



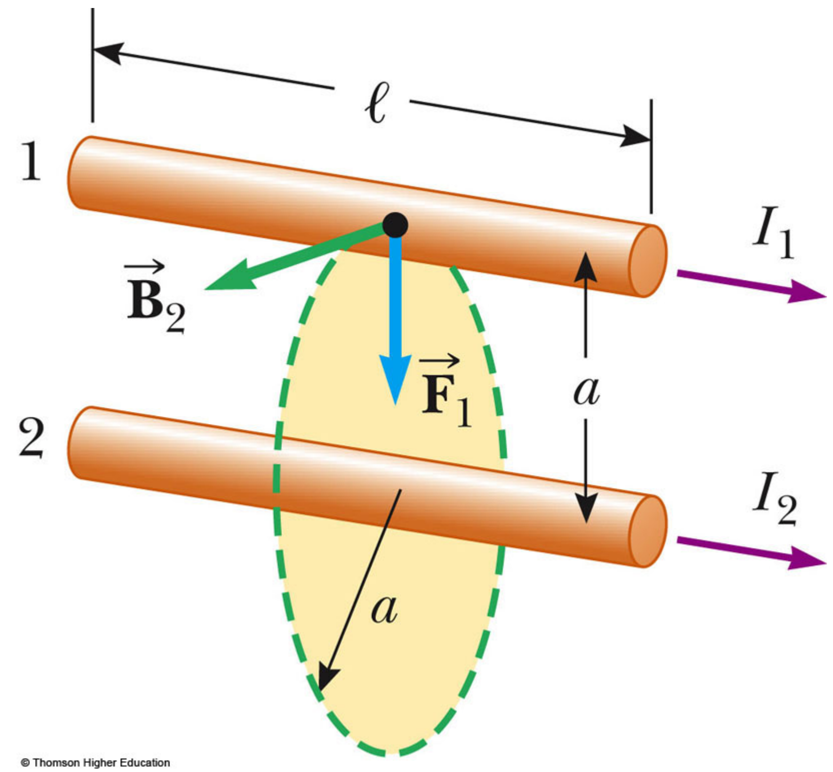
$n = N / \ell$ is the number of turns per unit length

Magnetic force between two parallel wires

- Two parallel wires each carry steady currents
- The field $\vec{B}_2$ due to the current in wire 2 exerts a force on wire 1 of $F_1 = I_1 l B_2$
- Substituting the equation for gives

$$F_1 = \frac{\mu_0 I_1 I_2}{2\pi a} l$$

  - Check with right-hand rule:
    - same direction currents attract each other
    - opposite directions currents repel each other



The force per unit length on the wire is $$\frac{F_B}{l} = \frac{\mu_0 I_1 I_2}{2\pi a}$$

And this formula defines the current unit Ampere.

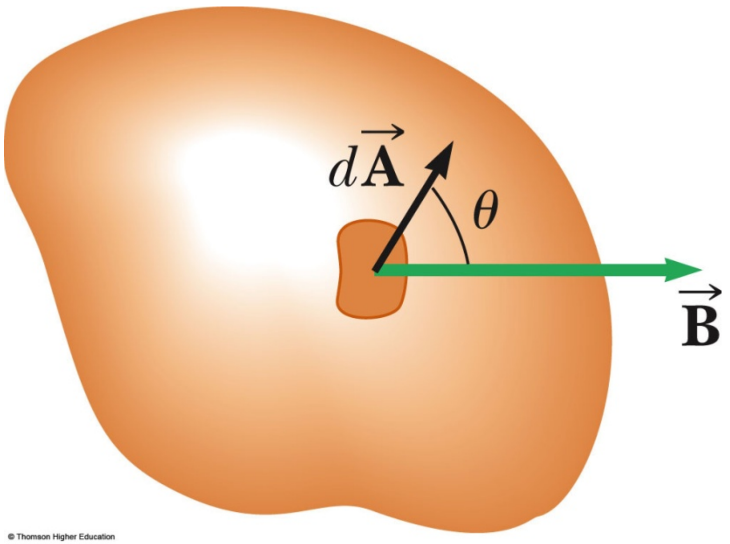
Definition of the Ampere and the Coulomb

- The force between two parallel wires is used to define the ampere.
- When the magnitude of the force per unit length between two long, parallel wires that carry identical currents and are separated by 1 m is 2×10^{-7} N/m, the current in each wire is defined to be 1 A
- The SI unit of charge, the coulomb, is defined in terms of the ampere
- When a conductor carries a steady current of 1 A, the quantity of charge that flows through a cross section of the conductor in 1 second is 1 C

Magnetic Flux

- The magnetic flux over a surface area associated with a magnetic field is defined as

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$



- The unit of magnetic flux is

$\mathrm{T{\cdot}m^2 = Wb}$ (*weber*)

Gauss' Law in Magnetism

- Magnetic fields do not begin or end at any point

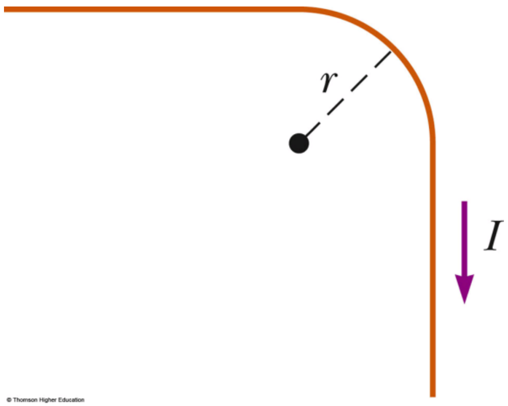
  The number of lines entering a surface equals the number of lines leaving the surface

- **Gauss' law in magnetism** says the magnetic flux through any closed surface is always zero:

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Example problem

A long, straight wire carries current I. A right-angle bend is made in the middle of the wire. The bend forms an arc of a circle of radius r. Determine the magnetic filed at the center of the arc.



Formula to use: Biot-Savart's Law, or more specifically the results from the discussed two examples:

For the straight section $B = \frac{\mu_0 I}{4\pi r}\sin\theta\Big|_0^{\pi/2} = \frac{\mu_0 I}{4\pi r}$

For the arc $B = \frac{\mu_0 I}{4\pi r^2}\int_{\frac{1}{4}\text{ circle}} ds = \frac{\mu_0 I}{4\pi r^2}\frac{2\pi r}{4} = \frac{\mu_0 I}{8r}$

The final answer: magnitude $B = \frac{\mu_0 I}{4\pi r} + \frac{\mu_0 I}{8r} + \frac{\mu_0 I}{4\pi r} = \frac{\mu_0 I}{4r}\left(\frac{2}{\pi} + \frac{1}{2}\right)$

direction pointing into the page.

Reading material and Homework assignment

Please watch this video (about 50 minutes each):
http://videolectures.net/mit802s02_lewin_lec14/

Please check wileyplus webpage for homework assignment.