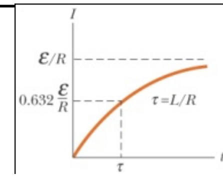
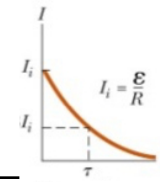
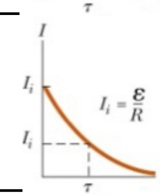
Electromagnetic Oscillations and Alternating Current

1. Electromagnetic oscillations and LC circuit
2. Alternating Current
3. RLC circuit in AC

RL and *RC* circuits

	RL	*RC*
Charging	$I = \frac{emf}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ 	$I = \frac{emf}{R} e^{-\frac{t}{RC}}$ 
Discharging	$I = I_0 e^{-\frac{Rt}{L}}$ 	$I = \frac{Q}{Rc} e^{-\frac{t}{RC}}$ 
Energy	$U_L = \frac{1}{2} L I^2$	$U_C = \frac{Q^2}{2C} = \frac{1}{2} C (\Delta V)^2$

	Magnetic field	Electric field
Energy density	$u_B = \frac{B^2}{2\mu_0}$	$u_B = \frac{\varepsilon_0 E^2}{2}$

LC circuits and oscillations

In RC and RL circuits the charge, current, and potential difference grow and decay exponentially, because the resister R coverts the electric energy into heat and dissipates it.

In an LC circuit, the charge, current, and potential difference vary sinusoidally with period T and angular frequency ω. Energy conserves.

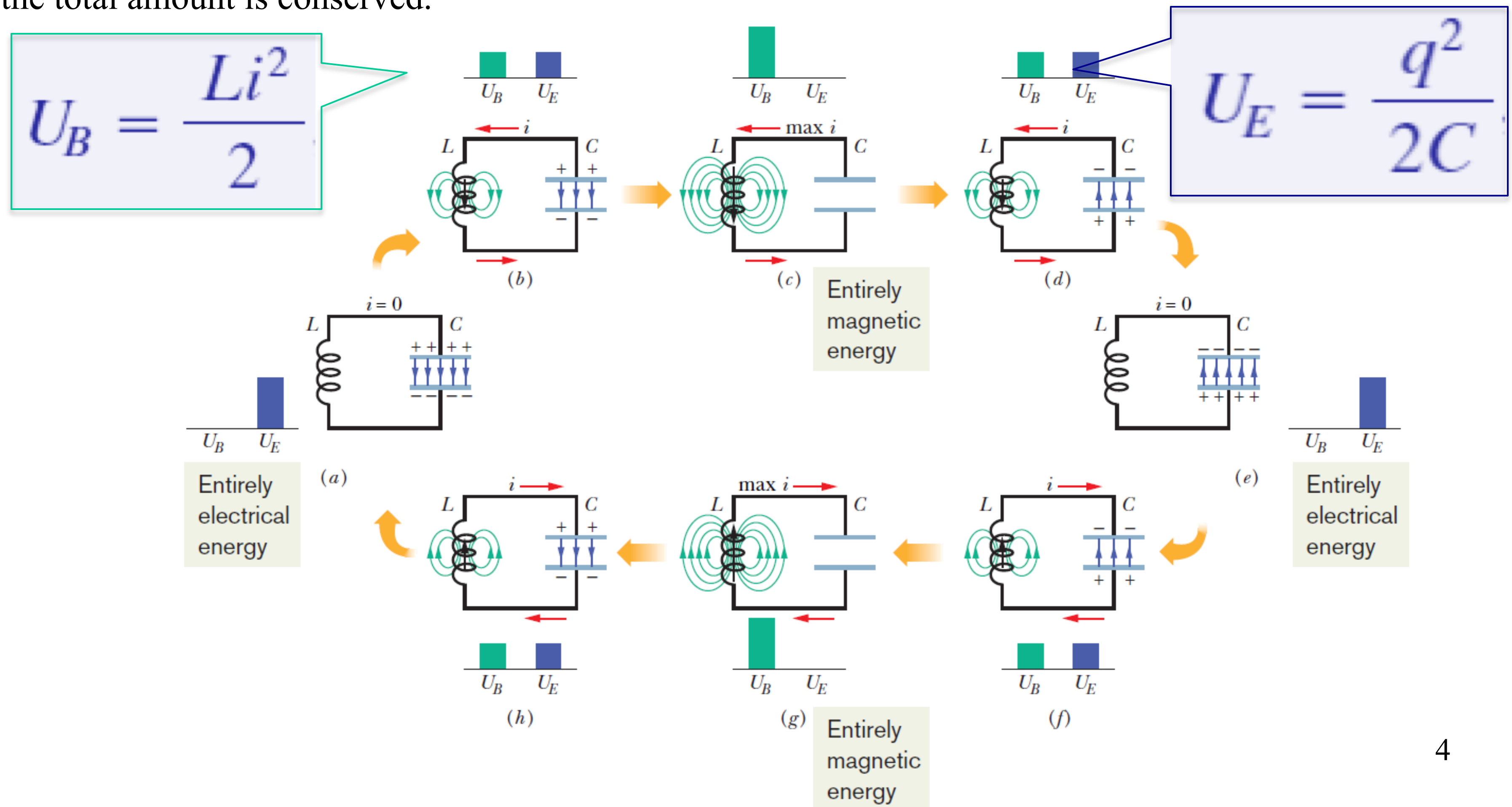
The resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be **electromagnetic oscillations.**

LC circuits and oscillations

The energy stored in the electric field of the capacitor at any time is $U_E = \frac{q^2}{2C}$, where q is the charge on the capacitor at that time.

The energy stored in the magnetic field of the inductor at any time is $U_B = \frac{Li^2}{2}$, where i is the current through the inductor at that time.

As the circuit oscillates, energy shifts back and forth from one type of stored energy to the other, but the total amount is conserved.



LC circuits and oscillations

From energy conservation:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C},$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Li^2}{2} + \frac{q^2}{2C}\right) = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0.$$

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \qquad (LC \text{ oscillations}).$$

$$q = Q\cos(\omega t + \phi) \qquad (\text{charge}), \quad \longrightarrow \quad i = \frac{dq}{dt} = -\omega Q\sin(\omega t + \phi) \qquad (\text{current}).$$

$$I = \omega Q, \quad \longrightarrow \quad i = -I\sin(\omega t + \phi).$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

Angular Frequencies:

$$\omega = \frac{1}{\sqrt{LC}}.$$

$$\frac{d^2q}{dt^2} + \omega^2 q = 0, \quad \omega^2 \equiv \frac{1}{LC}$$

LC circuits and oscillations

The electrical energy stored in the *LC* circuit at time *t* is,

$$U_E = \frac{q^2}{2C} = \frac{Q^2}{2C}\cos^2(\omega t + \phi).$$

The magnetic energy is:

$$U_B = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 Q^2 \sin^2(\omega t + \phi).$$

But

$$\omega = \frac{1}{\sqrt{LC}} \qquad (LC \text{ circuit}).$$

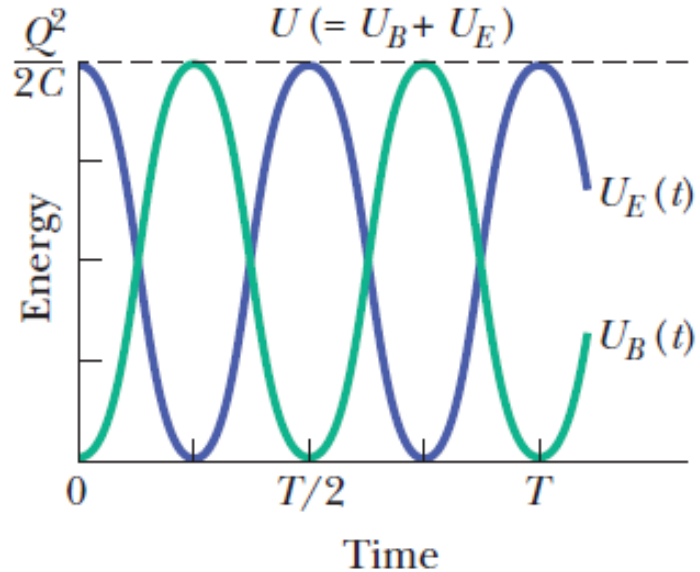
Therefore

$$U_B = \frac{Q^2}{2C}\sin^2(\omega t + \phi).$$

Note that

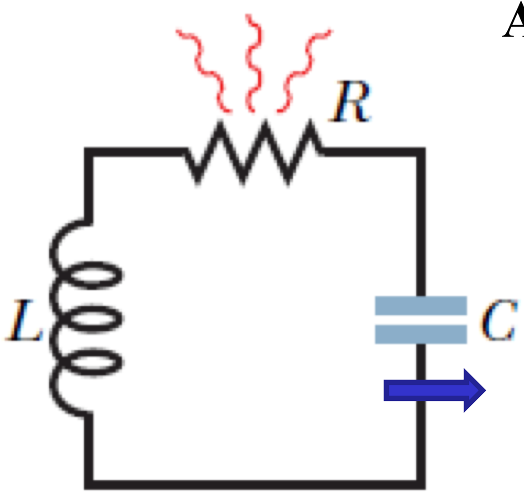
- The maximum values of U_E and U_B are both $Q^2/2C$.
- At any instant the sum of U_E and U_B is equal to $Q^2/2C$, a constant.
- When U_E is maximum, U_B is zero, and conversely.

The electrical and magnetic energies vary but the total is constant.



What is the phase angle ϕ here?

Damped oscillation in an RLC circuit



Analysis:

$$U = U_B + U_E = \frac{Li^2}{2} + \frac{q^2}{2C}$$

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R.$$

This term was 0 in LC circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0 \qquad (RLC \text{ circuit}),$$

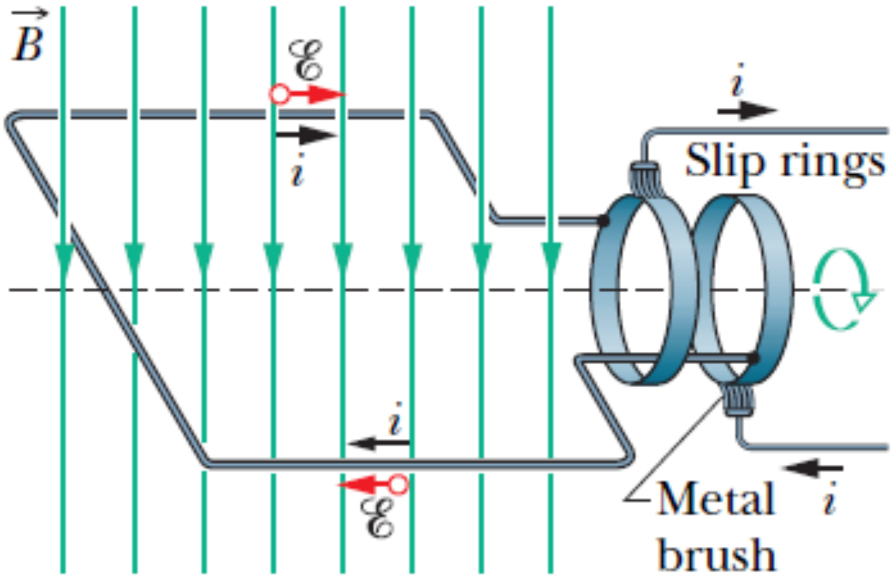
$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi),$$

Where $\omega' = \sqrt{\omega^2 - (R/2L)^2},$

And $\omega = 1/\sqrt{LC}$

$$U_E = \frac{q^2}{2C} = \frac{[Qe^{-Rt/2L}\cos(\omega' t + \phi)]^2}{2C} = \frac{Q^2}{2C}e^{-Rt/L}\cos^2(\omega' t + \phi)$$

Alternating current

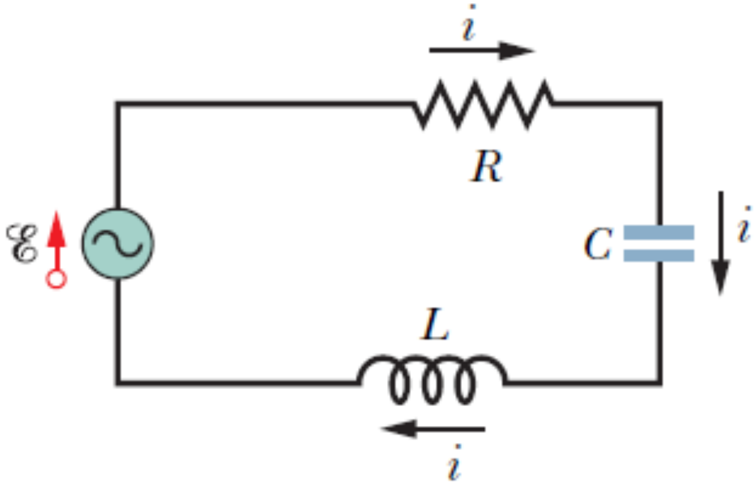


$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$$

$$i = I \sin(\omega_d t - \phi),$$

ω_d is called the driving angular frequency, and I is the amplitude of the driven current.

Forced oscillation

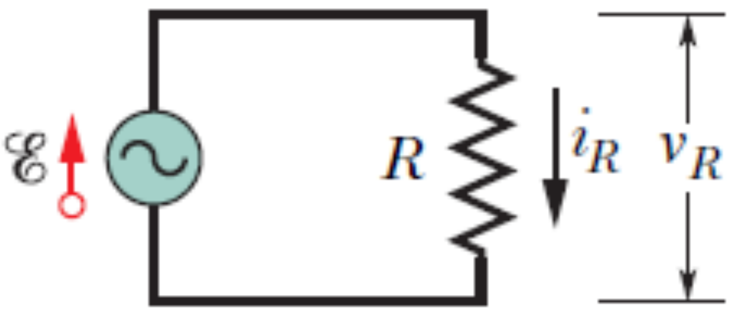


The natural angular frequency ω_0 is determined by the RLC circuit. The driving angular frequency ω_d forces the oscillation to follow.

The question is, given $\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$
What is the current *i* as a function of time.

Exam R, C and L individually

1, only a resistive load



$$\mathscr{E} - v_R = 0.$$

$$v_R = \mathscr{E}_m \sin \omega_d t. = V_R \sin \omega_d t.$$

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega_d t.$$

$$= I_R \sin(\omega_d t - \phi),$$

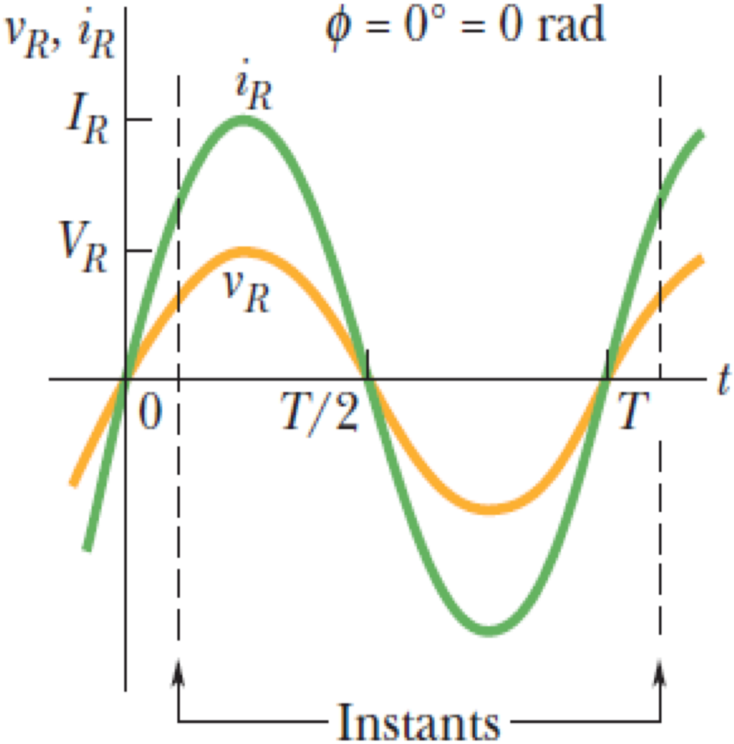
For a purely resistive load the phase constant $\phi = 0°$. So:

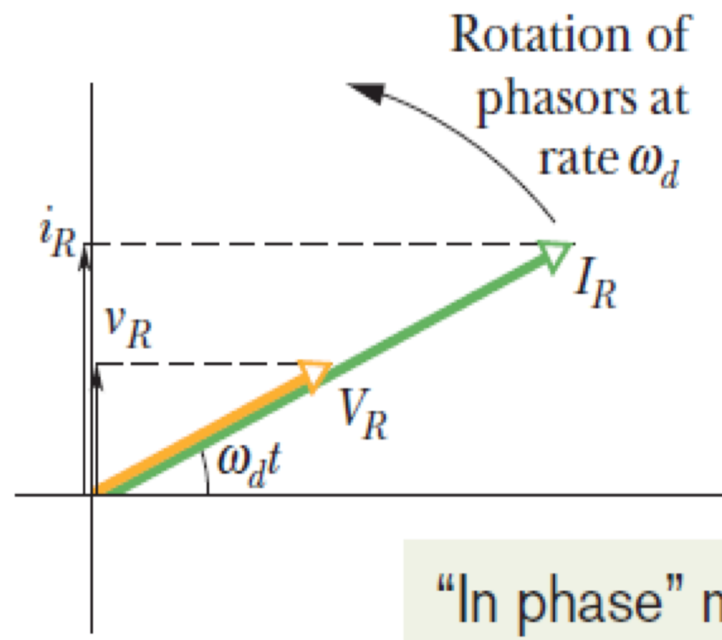
$$i_R = I_R \sin(\omega_d t)$$

Exam R, C and L individually

1, only a resistive load

For a resistive load,
the current and potential
difference are in phase.

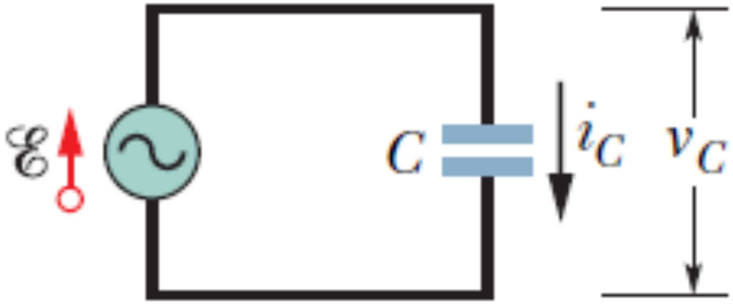




"In phase" means
that they peak at
the same time.

Exam R, C and L individually

2, only a capacitive load



$$v_C = V_C \sin \omega_d t,$$

$$q_C = C v_C = C V_C \sin \omega_d t.$$

$$i_C = \frac{dq_C}{dt} = \omega_d C V_C \cos \omega_d t.$$

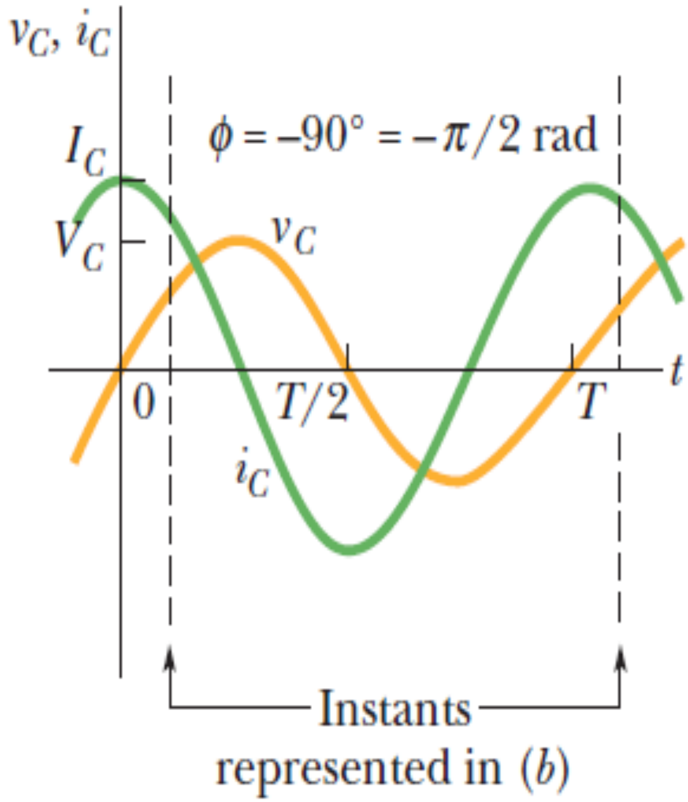
$$X_C = \frac{1}{\omega_d C} \qquad \text{(capacitive reactance).}$$

X_C is called the **capacitive reactance of a capacitor.** The SI unit of X_C is the *ohm*, just as for resistance R.

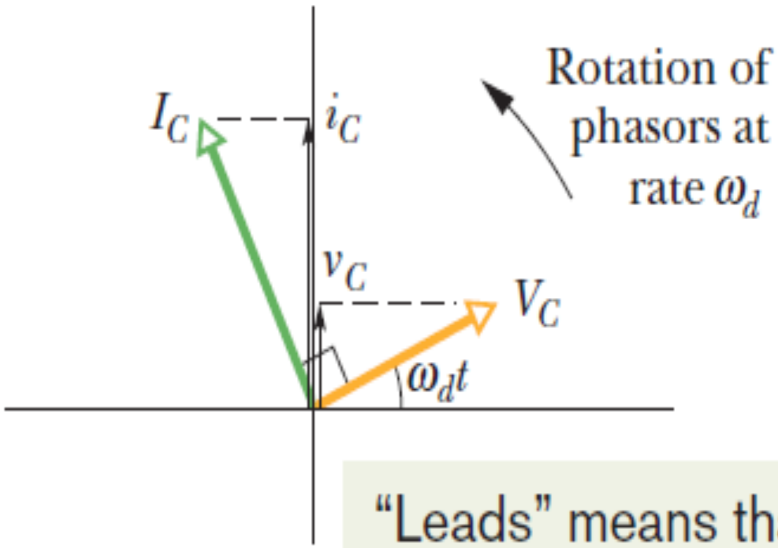
Exam R, C and L individually

2, only a capacitive load

For a capacitive load, the current leads the potential difference by 90°.

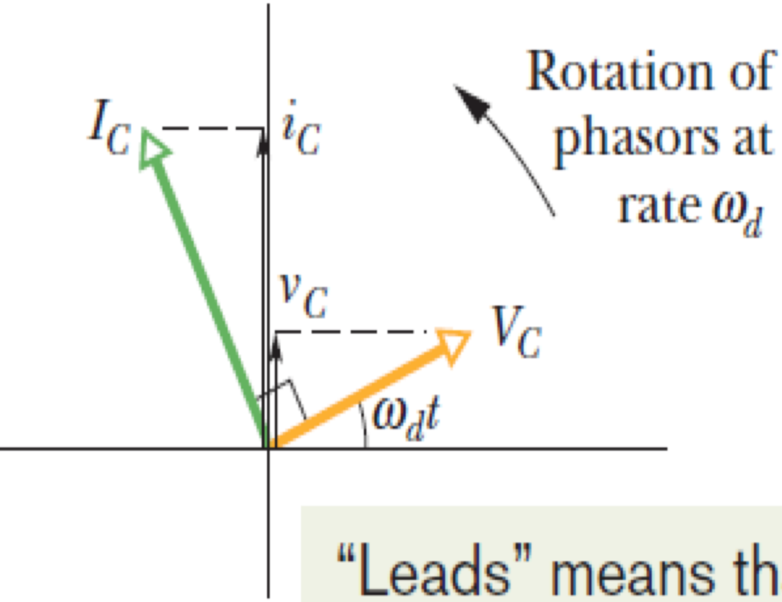


The current in the capacitor leads the voltage by 90°



"Leads" means that the current peaks at an *earlier* time than the potential difference.

Exam R, C and L individually
2, only a capacitive load



"Leads" means that the current peaks at an *earlier* time than the potential difference.

$$\cos\omega_d t = \sin(\omega_d t + 90°).$$

$$i_C = \left(\frac{V_C}{X_C}\right)\sin(\omega_d t + 90°).$$

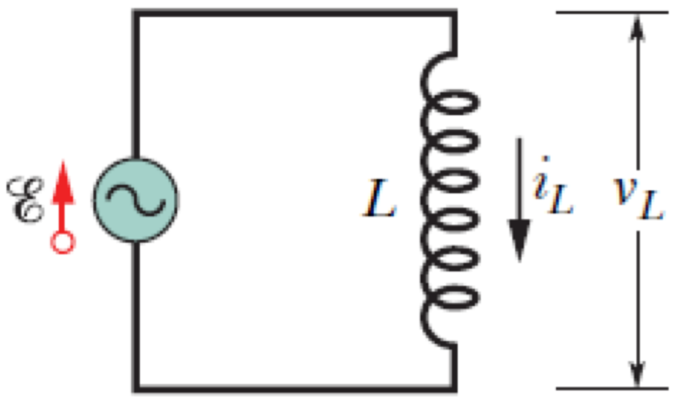
$$i_C = I_C \sin(\omega_d t - \phi),$$

$$V_C = I_C X_C \qquad \text{(capacitor)}.$$

The current in the capacitor leads the voltage by 90°

Exam R, C and L individually

3, only an inductive load



$$v_L = V_L \sin \omega_d t, \qquad v_L = L\frac{di_L}{dt}.$$

$$\frac{di_L}{dt} = \frac{V_L}{L}\sin \omega_d t.$$

$$i_L = \int di_L = \frac{V_L}{L}\int \sin \omega_d t\, dt = -\left(\frac{V_L}{\omega_d L}\right)\cos \omega_d t.$$

$$X_L = \omega_d L \qquad \text{(inductive reactance).}$$

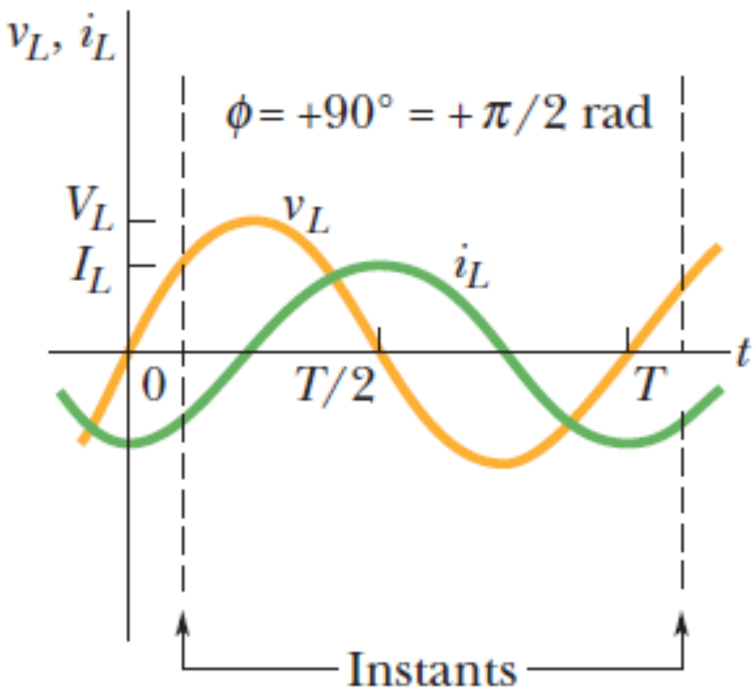
$$i_L = \left(\frac{V_L}{X_L}\right)\sin(\omega_d t - 90°). \qquad i_L = I_L \sin(\omega_d t - \phi),$$

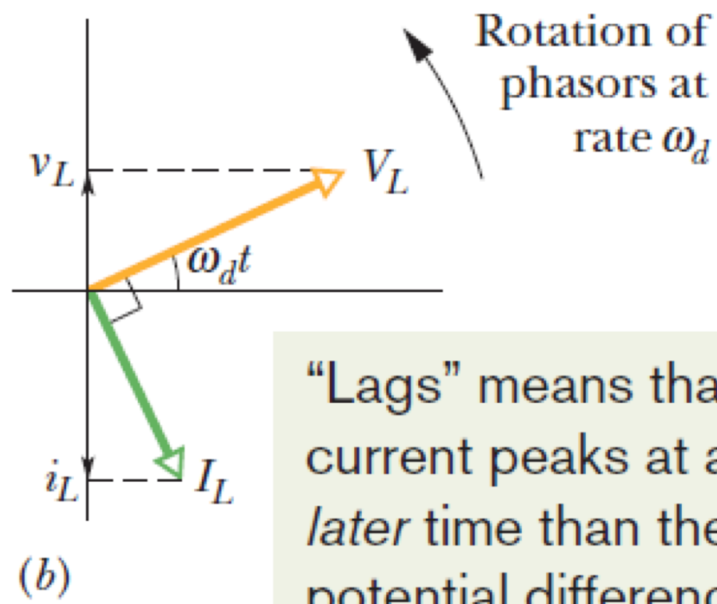
$$V_L = I_L X_L \qquad \text{(inductor).}$$

The value of X_L , **the inductive resistance**, depends on the driving angular frequency ω_d. The unit of the inductive time constant τ_L indicates that the SI unit of X_L is the *ohm*.

Exam R, C and L individually
3, only an inductive load

For an inductive load, the current lags the potential difference by 90°.





"Lags" means that the current peaks at a *later* time than the potential difference.

The current in the inductor lags the voltage by 90°

Exam R, C and L individually

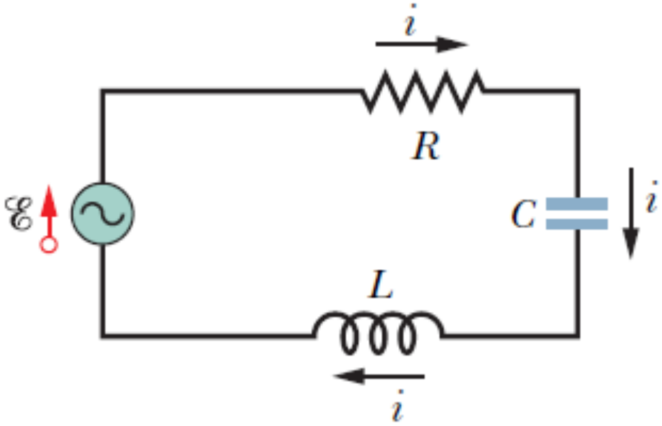
A summary

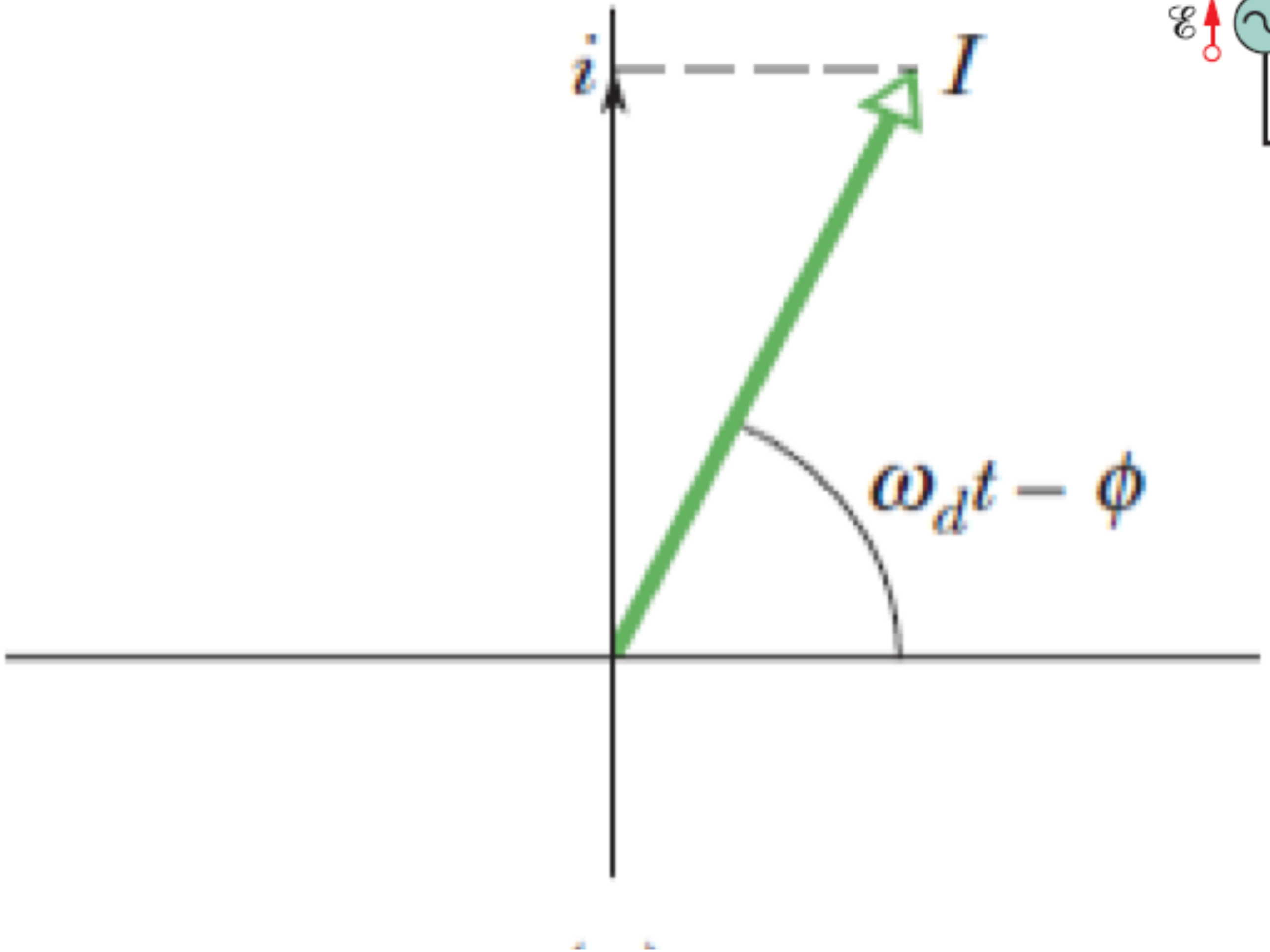
Phase and Amplitude Relations for Alternating Currents and Voltages

Circuit Element	Symbol	Resistance or Reactance	Phase of the Current	Phase Constant (or Angle) ϕ	Amplitude Relation
Resistor	R	R	In phase with v_R	$0°$ (= 0 rad)	$V_R = I_R R$
Capacitor	C	$X_C = 1/\omega_d C$	Leads v_C by $90°$ (= $\pi/2$ rad)	$-90°$ (= $-\pi/2$ rad)	$V_C = I_C X_C$
Inductor	L	$X_L = \omega_d L$	Lags v_L by $90°$ (= $\pi/2$ rad)	$+90°$ (= $+\pi/2$ rad)	$V_L = I_L X_L$

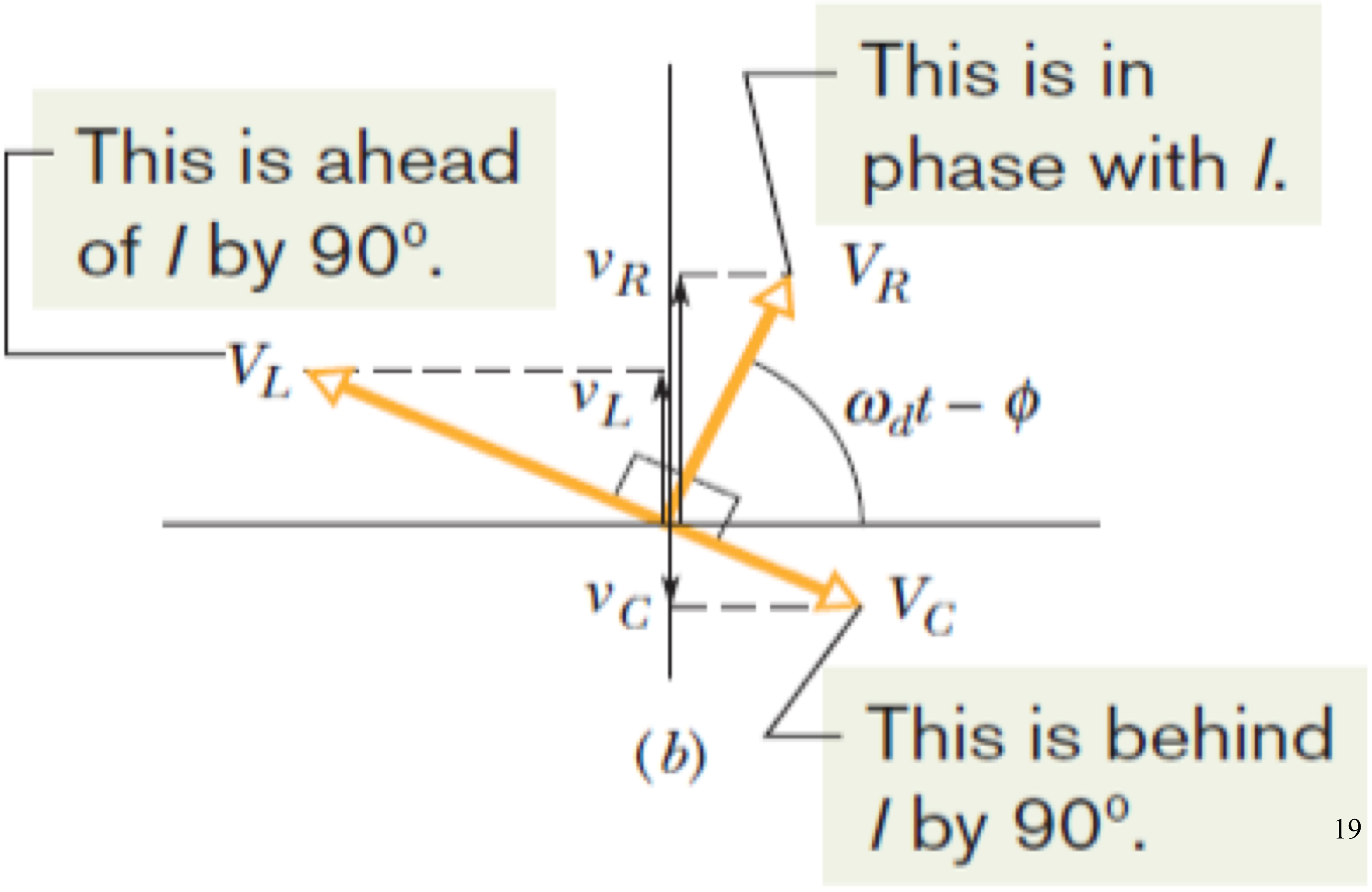
A circuit with RLC in series

In a circuit when components are connected in series, they share the same current i. And this is our starting point.



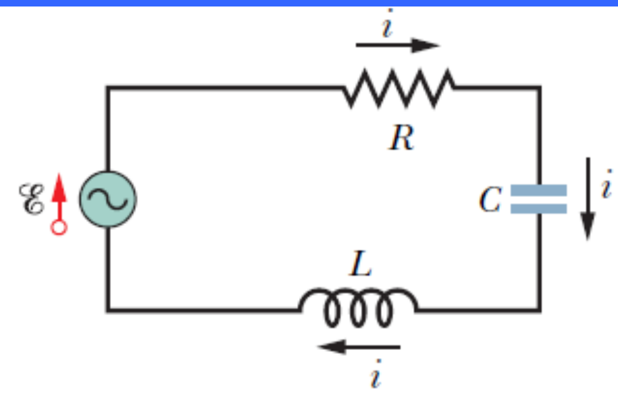


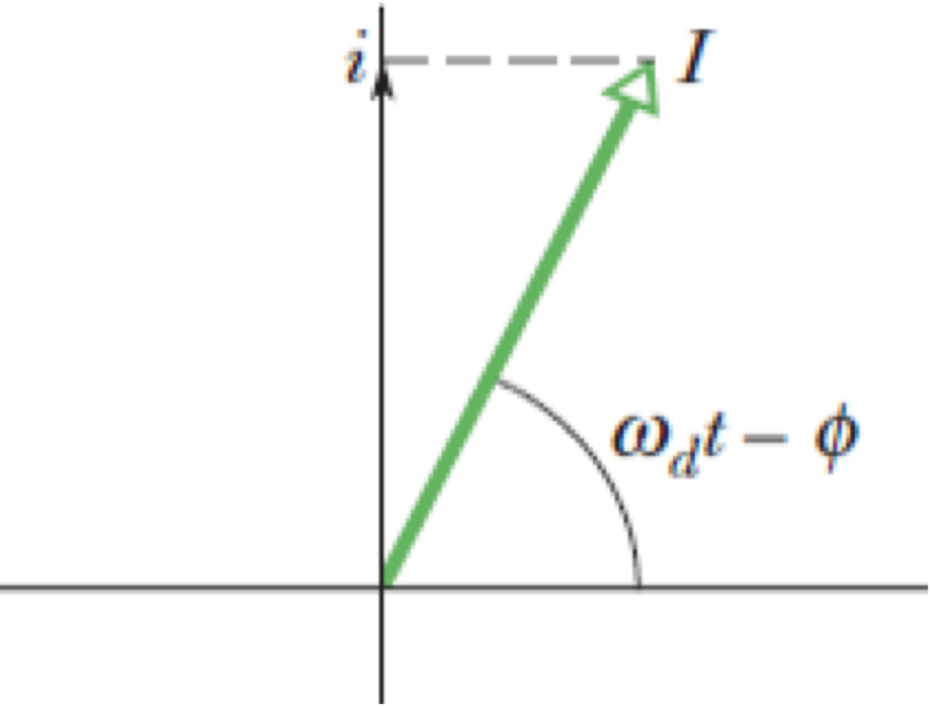
A circuit with RLC in series

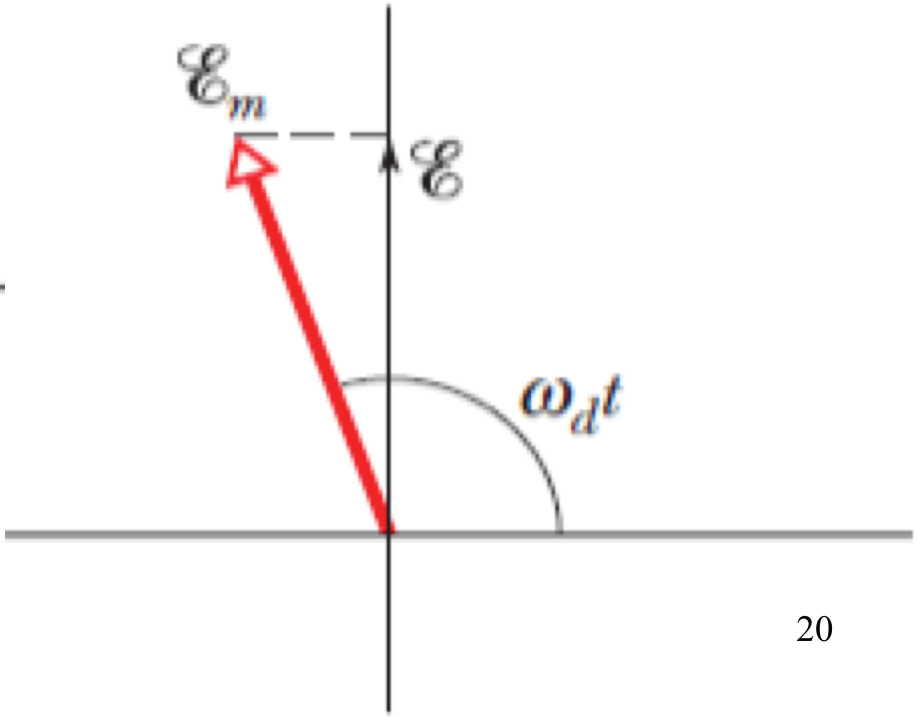


A circuit with RLC in series

Now example the phase between current and driving emf.

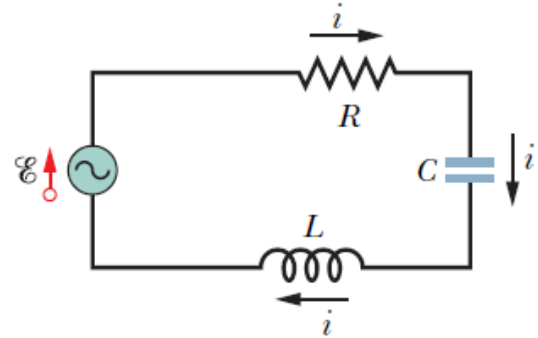




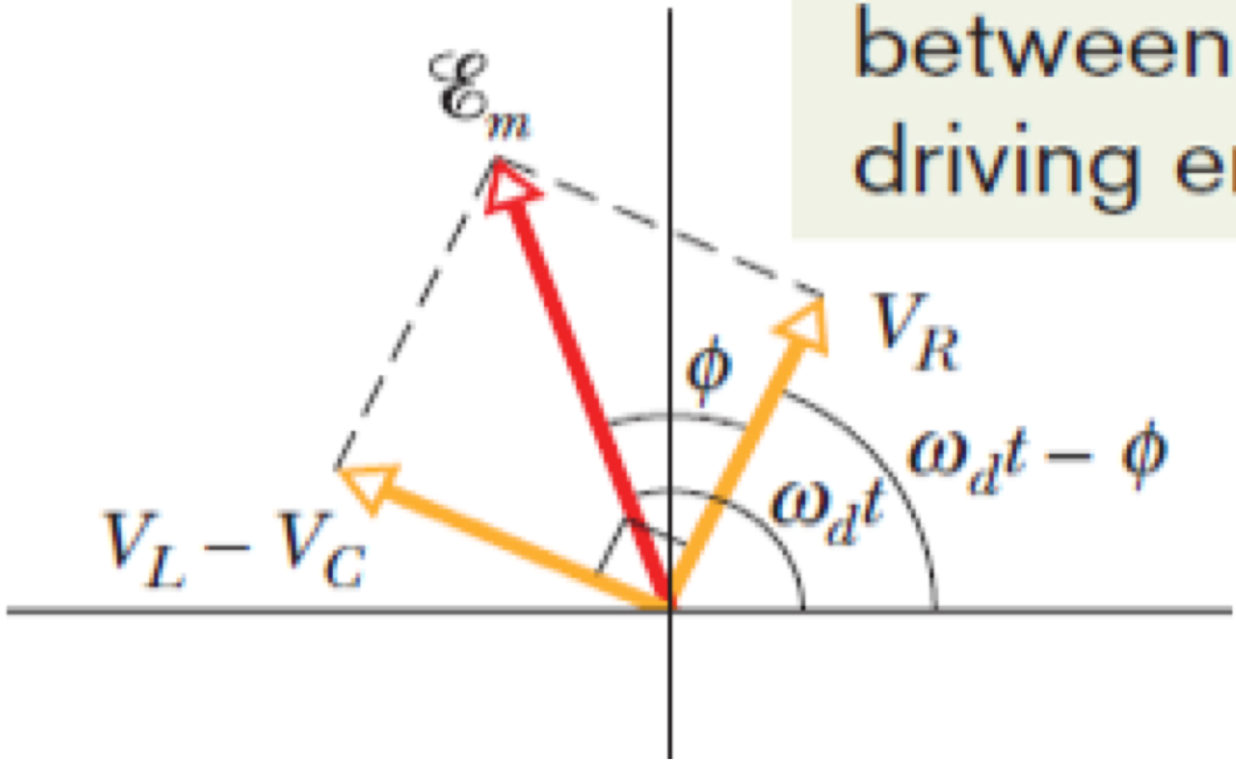


A circuit with RLC in series

Now example the phase between current and driving emf.



This ϕ is the angle between I and the driving emf.



A circuit with RLC in series

$$\mathcal{E} = \mathcal{E}_m \sin \omega_d t$$

$$i = I \sin(\omega_d t - \phi) \equiv \frac{emf_{\max}}{Z} \sin(\omega_d t - \phi)$$

$$\mathcal{E}_m^2 = V_R^2 + (V_L - V_C)^2 = (IR)^2 + (IX_L - IX_C)^2,$$

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}.$$

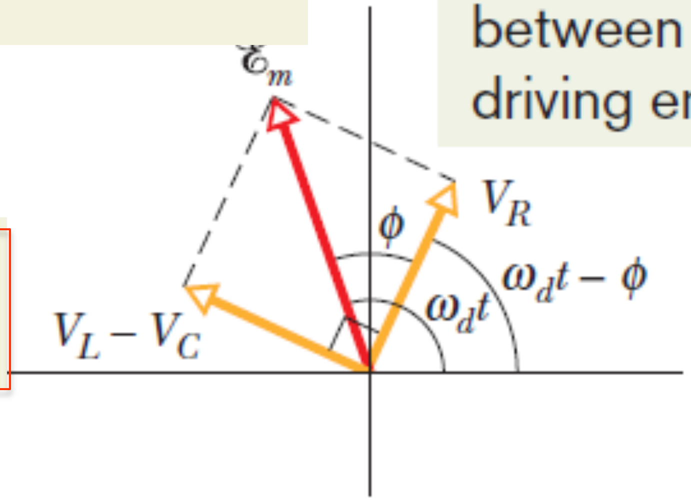
$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(impedance defined).}$$

$$I = \frac{\mathcal{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \quad \text{(current amplitude).}$$

This ϕ is the angle between I and the driving emf.

$$\tan \phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR},$$

$$\tan \phi = \frac{X_L - X_C}{R} \quad \text{(phase constant).}$$

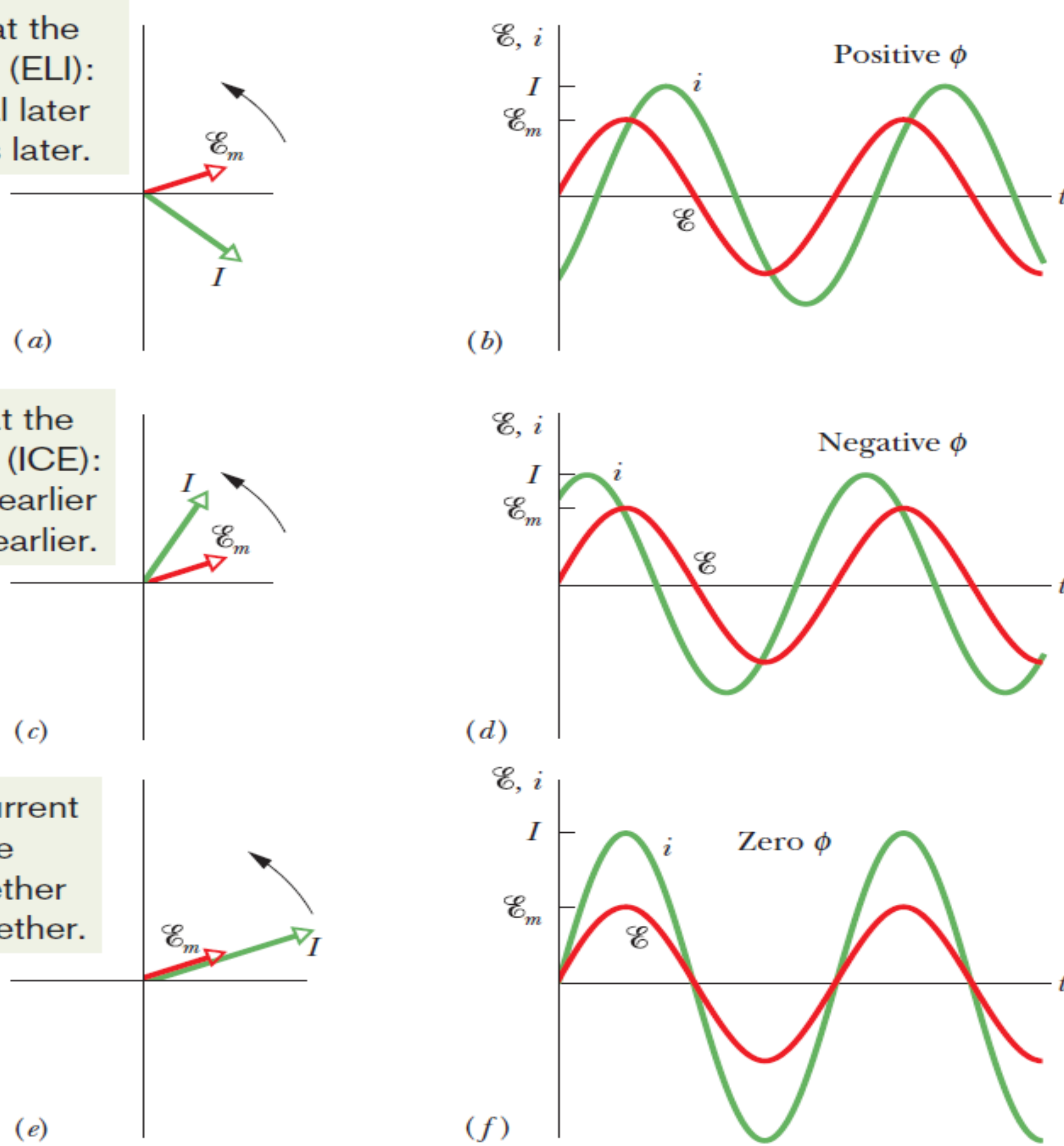


A circuit with RLC in series

Positive ϕ means that the current lags the emf (ELI): the phasor is vertical later and the curve peaks later.

Negative ϕ means that the current leads the emf (ICE): the phasor is vertical earlier and the curve peaks earlier.

Zero ϕ means that the current and emf are in phase: the phasors are vertical together and the curves peak together.



Resonance in a RLC circuit

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}} \qquad \text{(current amplitude).}$$

For a given resistance *R*, that amplitude is a maximum when the quantity $(\omega_d L - 1/\omega_d C)$ in the denominator is zero.

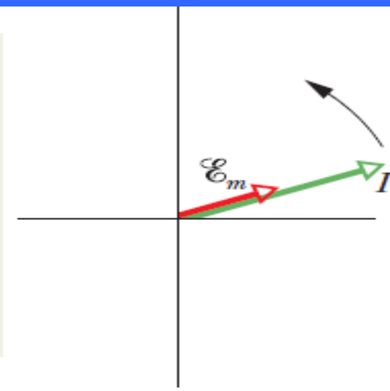
$$\Rightarrow \quad \omega_d L = \frac{1}{\omega_d C} \quad \Rightarrow \quad \omega_d = \frac{1}{\sqrt{LC}} \qquad \text{(maximum } I\text{).}$$

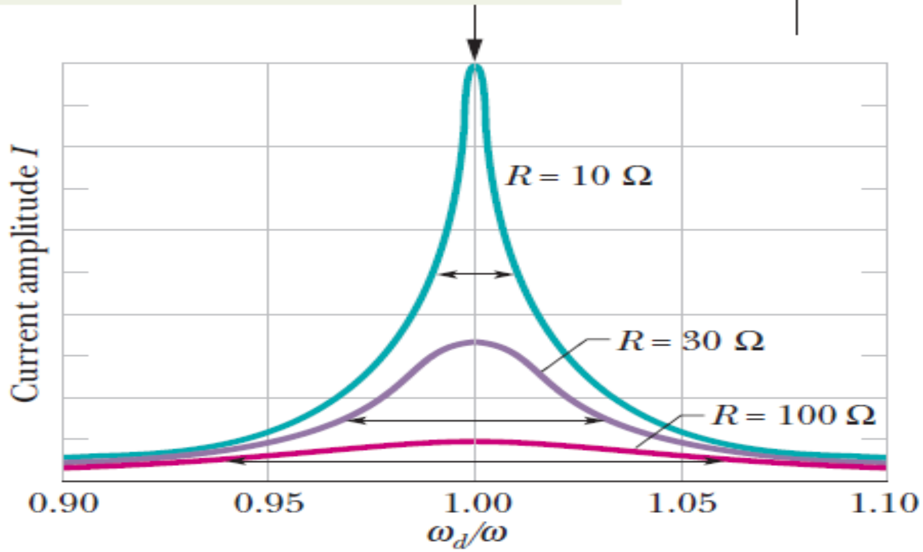
The maximum value of *I* occurs when the driving angular frequency matches the natural angular frequency—that is, at resonance.

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \qquad \text{(resonance).}$$

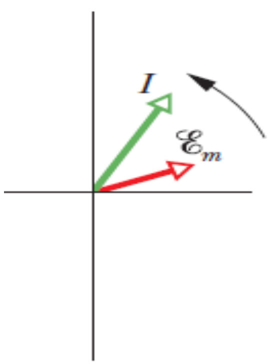
Resonance in a RLC circuit

Driving ω_d equal to natural ω
- high current amplitude
- circuit is in resonance
- equally capacitive and inductive
- X_c equals X_L
- current and emf in phase
- zero ϕ



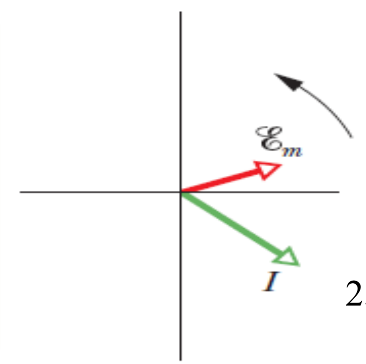


How many of you have built a crystal radio?



Low driving ω_d
- low current amplitude
- ICE side of the curve
- more capacitive
- X_C is greater
- current leads emf
- negative ϕ

High driving ω_d
- low current amplitude
- ELI side of the curve
- more inductive
- X_L is greater
- current lags emf
- positive ϕ



Power in AC circuit

The instantaneous rate at which energy is dissipated in the resistor:

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2 R = I^2 R \sin^2(\omega_d t - \phi).$$

The average rate at which energy is dissipated in the resistor, is the average of this over time:

$$P_{avg} = \frac{I^2 R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R.$$

Since the root mean square of the current is given by:

Similarly, $I_{rms} = \dfrac{I}{\sqrt{2}}$ ➡ $P_{avg} = I_{rms}^2 R$ (average power).

With $V_{rms} = \dfrac{V}{\sqrt{2}}$ and $\mathscr{E}_{rms} = \dfrac{\mathscr{E}_m}{\sqrt{2}}$ (rms voltage; rms emf).

Therefore, $I_{rms} = \dfrac{\mathscr{E}_{rms}}{Z} = \dfrac{\mathscr{E}_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}$,

➡ $$P_{avg} = \frac{\mathscr{E}_{rms}}{Z} I_{rms} R = \mathscr{E}_{rms} I_{rms} \frac{R}{Z}.$$

➡ $P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$ (average power), where $\cos\phi = \dfrac{V_R}{\mathscr{E}_m} = \dfrac{IR}{IZ} = \dfrac{R}{Z}.$

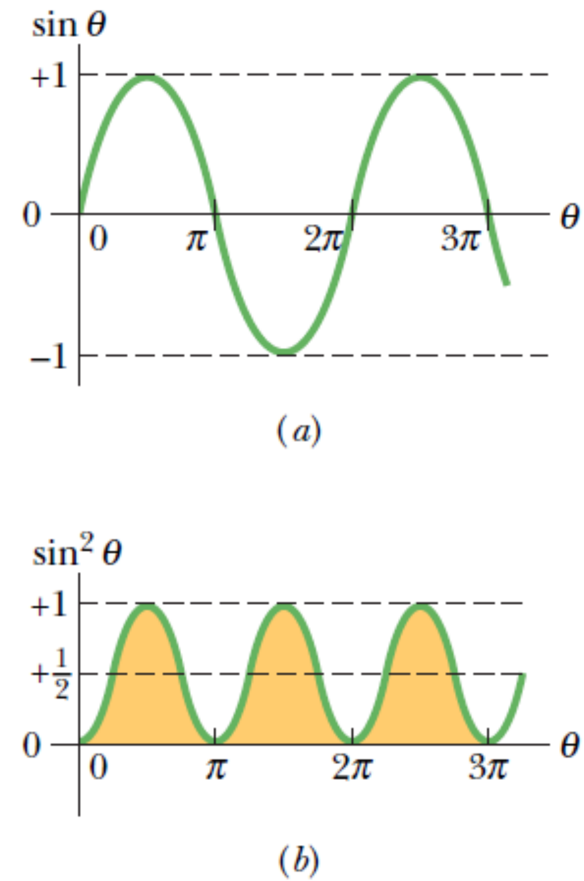


Fig. 31-17 (*a*) A plot of sin *θ* versus *θ*. The average value over one cycle is zero. (*b*) A plot of $\sin^2\theta$ versus *θ*. The average value over one cycle is $\frac{1}{2}$.

Example, Driven RLC circuit:

A series *RLC* circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

Example, Driven RLC circuit:

A series *RLC* circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

$$\cos\phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}.$$

$$\tan\phi = \frac{X_L - X_C}{R} \qquad \text{(phase constant).}$$

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi \qquad \text{(average power),}$$

Transformers

In electrical power distribution systems it is desirable for reasons of safety and for efficient equipment design to deal with relatively low voltages at both the generating end (the electrical power plant) and the receiving end (the home or factory).

Nobody wants an electric toaster or a child's electric train to operate at, say, 10 kV.

On the other hand, in the transmission of electrical energy from the generating plant to the consumer, we want the lowest practical current (hence the largest practical voltage) to minimize I^2R losses (often called ohmic losses) in the transmission line.

Transformers

A device with which we can raise and lower the ac voltage in a circuit, keeping the product current voltage essentially constant, is called the **transformer.**

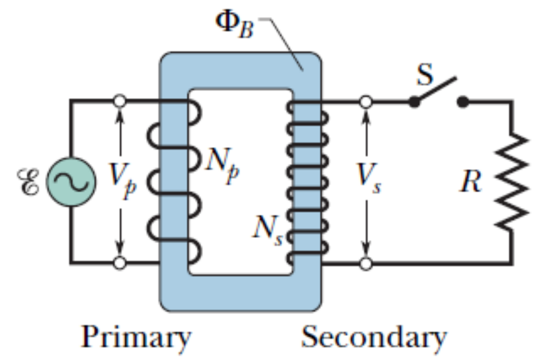


Fig. 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load *R* when switch S is closed.

The ideal transformer consists of two coils, with different numbers of turns, wound around an iron core.

In use, the primary winding, of N_p turns, is connected to an alternating-current generator whose emf at any time *t* is given by

$$\mathscr{E} = \mathscr{E}_m \sin \omega t.$$

The secondary winding, of N_s turns, is connected to load resistance *R,* but its circuit is an open circuit as long as switch S is open.

The small sinusoidally changing primary current I_{mag} produces a sinusoidally changing magnetic flux *B* in the iron core.

Because *B* varies, it induces an emf (*dB/dt)* in each turn of the secondary. This emf per turn is the same in the primary and the secondary. Across the primary, the voltage $V_p = \mathcal{E}_{\text{turn}} N_p$. Similarly, across the secondary the voltage is $V_s = \mathcal{E}_{turn} N_s$.

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage)}$$

Transformers

$$V_s = V_p \frac{N_s}{N_p} \quad \text{(transformation of voltage).}$$

If $N_s > N_p$, the device is a step-up transformer because it steps the primary's voltage V_p up to a higher voltage V_s. Similarly, if $N_s < N_p$, it is a step-down transformer.

If no energy is lost along the way, conservation of energy requires that

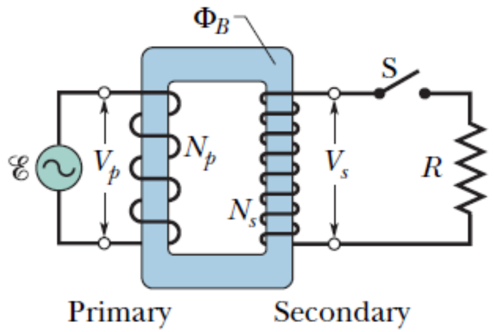


Fig. 31-18 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

$$I_p V_p = I_s V_s. \quad \Rightarrow \quad I_s = I_p \frac{N_p}{N_s} \quad \text{(transformation of currents).}$$

$$\Rightarrow \quad I_p = \frac{1}{R}\left(\frac{N_s}{N_p}\right)^2 V_p. \quad \Rightarrow \quad R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R.$$

Here R_{eq} is the value of the load resistance as "seen" by the generator.

For maximum transfer of energy from an emf device to a resistive load, the resistance of the emf device must equal the resistance of the load. For ac circuits, for the same to be true, the *impedance* (rather than just the resistance) of the generator must equal that of the load.

Reading material and Homework assignment

Please watch this video (about 50 minutes each):
http://videolectures.net/mit802s02_lewin_lec20/ and
http://videolectures.net/mit802s02_lewin_lec25/

Please check wileyplus webpage for homework assignment.