Oscillations

Consider a potential energy (that depends only on \( x \) for simplicity). Taylor expand about a stable equilibrium:

\[
U(x) = U(0) + x \left. \frac{dU}{dx} \right|_{x=0} + \frac{x^2}{2!} \left. \frac{d^2U}{dx^2} \right|_{x=0} + \cdots
\]

Only changes in potential energy are measurable, so \( U(x) \) is only defined up to a constant. Therefore, the first term above \( U(0) \) can be set to zero.

At positions of equilibrium, the force vanishes. The force is \(-\frac{dU}{dx}\) (or \(-\nabla U\) in more than one dimension). So the second term vanishes at equilibrium:

\[
X \left. \frac{dU}{dx} \right|_{x=0} = 0.
\]
The first non-zero term is \( \frac{x^2}{2!} \frac{d^2U}{dx^2} \) or \( \frac{1}{2} k x^2 \) where \( k = \frac{d^2U}{dx^2} \) is positive if the equilibrium is stable (U(x) is concave up).

The force associated with this first approximation to the potential energy is

\[
F = -\frac{d}{dx} \left[ \frac{1}{2} k x^2 \right] = -kx
\]

which you will recognize as the ideal Hooke's law restoring force. A restoring force always brings the system back to equilibrium after displacements away from equilibrium.

Other terms can be added for more precision, but for small oscillations, the \( \frac{1}{2} k x^2 \) term will dominate. The resulting oscillations are called simple harmonic and can be solved exactly.
Why study these oscillations? They occur often in various branches of Physics and the resulting differential equation is linear in \( x \), so there is hope of solving it and the solutions will superpose (linear combinations of solutions will also be a solution).

Mathematician Stanislaw Ulam said that studying non-linear science is like studying non-elephant zoology.

**Newton's 2nd Law:**

\[
\begin{align*}
F &= m \ddot{x} & \ddot{x} + \frac{k}{m} x &= 0 \\
-kx &= m \dddot{x} & \frac{d^2x(t)}{dt^2} + \frac{k}{m} x(t) &= 0
\end{align*}
\]

This is a differential equation. The goal is to find a function \( x(t) \) that makes the equation true for all times.

\( x \) is the function; \( t \) is the variable.

The differential equation is a second order, linear, homogeneous, ordinary.
Second order — the highest derivative is $\ddot{x}$.

linear — the function and its derivatives ($x, \dot{x}, \ddot{x}, \ldots$) occur to at most the first power.

homogeneous — there is no term that does not depend on $x$.

ordinary — there is only one variable, $t$.

If the next term in the Taylor expansion of $U(x)$ were kept

$$\frac{x^3}{3!} \frac{d^3U(x)}{dx^3} \bigg|_{x=0}$$

the force term would be proportional to $x^2$ and the differential equation

$$\ddot{x}(t) + \frac{k}{m}x(t) + cx^2(t) = 0 \quad \text{is non-linear.}$$

How do we solve the linear D.E.?

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0$$

Try an exponential solution: $x(t) = Ae^{rt}$

The first derivative is $\dot{x}(t) = rAe^{rt}$

The second derivative is $\ddot{x}(t) = r^2Ae^{rt}$
\[ x(t) + \frac{k}{m} x(t) = 0 \]

\[ \dot{x}^2 A e^{\lambda t} + \frac{k}{m} A e^{\lambda t} = 0 \]

The guess converts the D.E. into an algebraic equation for \( \lambda \):

\[(\dot{x}^2 + \frac{k}{m}) A e^{\lambda t} = 0 \]

This implies that:

\[ A = 0 \quad \text{or} \quad e^{\lambda t} = 0 \quad \text{or} \]

\[(\lambda^2 + \frac{k}{m}) = 0 \]

\( e^{\lambda t} \) is never zero. If \( A = 0 \), then this is the null solution \( x(t) = 0 \) for all time. Well, that solves the D.E., but it's not interesting. So \( \dot{x}^2 + \frac{k}{m} = 0 \) implies \( \lambda = \pm i \sqrt{\frac{k}{m}} \)

With the definition \( A e^{\lambda t} = \sqrt{\frac{k}{m}} \), the two solutions are \( A e^{i\omega t} \) and \( A e^{-i\omega t} \).

We expect two linearly independent solutions because the D.E. is second order. The arbitrary constants \( A_1 \) and \( A_2 \) are constants of integration (you have to integrate twice to get from \( x \) to \( x \)) and these are used to satisfy the boundary conditions; for example:

\[ x(0), v(0) \quad \text{or} \quad x(0), x(t_0) \quad \text{or} \quad v(0), v(t_0) \]
So the complete solution to \( \ddot{x}(t) + \frac{k}{m} \dot{x}(t) + \frac{k}{m} x(t) = 0 \) is

\[ x(t) = A e^{i\omega t} + A e^{-i\omega t} \]

Other equivalent forms are

\[ x(t) = A \cos(\omega t) + B \sin(\omega t) \]
\[ x(t) = C \cos(\omega t + \theta) \]
\[ x(t) = D \sin(\omega t + \theta) \]

These can all be transformed into the others. Remember:

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \]
\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \]
\[ e^{i\theta} = \cos \theta + i \sin \theta \]

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**Damped Oscillations**

Assume a linear resistive force, like viscous drag

\[ F_{visc} = -bv = -b\dot{x} \quad b \text{ is positive} \]

\[ \begin{array}{c}
\begin{array}{c}
\overbrace{\text{m}}^{\text{(m/s)}} \\
-\begin{array}{c}
b \\
-kx - bx = ma \\
-k\ddot{x} - b\dot{x} = m\dddot{x}
\end{array}
\end{array}
\end{array} \]

\[ \ddot{x}(t) + \frac{k}{m} \dot{x}(t) + \frac{k}{m} x(t) = 0 \]
\[ \dddot{x}(t) + 2\beta \ddot{x}(t) + \omega_0^2 x(t) = 0 \]

\[ \omega_0^2 = \frac{k}{m} \]
\[ \beta = \frac{b}{2m} \]
Try an exponential solution again: \( x(t) = Ae^{rt} \)

The first derivative is \( \dot{x}(t) = rAe^{rt} \)

The second derivative is \( \ddot{x}(t) = r^2 Ae^{rt} \)

\[ \ddot{x}(t) + \alpha \beta \dot{x}(t) + \omega_0^2 x(t) = 0 \]
\[ (r^2 + \alpha \beta) Ae^{rt} = 0 \]

As before, we get an algebraic equation for \( r \)

\[ r^2 + \alpha \beta + \omega_0^2 = 0 \]
has solutions \( r = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \)

So, the complete solution to the D.E. is

\[ x(t) = A_e^{(-\beta + \sqrt{\beta^2 - \omega_0^2})t} + A_e^{(-\beta - \sqrt{\beta^2 - \omega_0^2})t} \]

or

\[ x(t) = e^{-\beta t} \left[ A_+ e^{\sqrt{\beta^2 - \omega_0^2} t} + A_- e^{-\sqrt{\beta^2 - \omega_0^2} t} \right] \]

\[ x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \]
There are three cases to consider:

\[ \omega_0 > \beta \quad \text{so} \quad \sqrt{\beta^2 - \omega_0^2} \text{ is imaginary; underdamped} \]

\[ \omega_0 < \beta \quad \text{so} \quad \sqrt{\beta^2 - \omega_0^2} \text{ is real; overdamped} \]

\[ \omega_0 = \beta \quad \text{so} \quad \sqrt{\beta^2 - \omega_0^2} \text{ is zero; critically damped} \]
Phase space $x$ vs. $v$.

1-dim H.O.:

$x(t) = A \cos(\omega t + \delta)$

$\dot{v}(t) = -\omega A \sin(\omega t + \delta)$

eliminate $t$

\[ v(x) = \pm \sqrt{\frac{A^2 - x^2}{A^2}} = \pm \omega \sqrt{\frac{A^2 - x^2}{A^2}} \]

\[ \frac{x^2}{A^2} + \frac{v^2}{A^2 \omega^2} = 1 \]

ellipse

$\frac{x}{A} + \frac{v}{A \omega} = 1$

$\frac{v}{A \omega}$

$\frac{x}{A}$

$\exists 0 \leq t \leq \frac{2\pi}{\omega}$

A point in P.S. represents a possible state of the system (e.g., initial conditions).

As time evolves, the curve is traced out clockwise.

Phase paths never cross $x(t_0, v_0)$ which may well evolve.

And under D.E., a solution is univocal.

The system cannot exist in states off the phase plane, in particular $x = 0, v = 0$ is disallowed.

$X(t_1) = \frac{A}{2}$ $V(t_1) = \omega$ impossible at same time

Different magnitudes $A$ give different ellipses.

Area of ellipse:

$S = \pi ab = \pi (A/2)(A\omega) = \frac{\pi A^2 \omega}{2}$ km

Area proportional to energy.

Closed curves in P.S. $\Rightarrow$ periodic motion is conserved.

Do not confuse with ellipse jaw figures! It is only 2-d rotation paths can cross

P.S. would be 4-dim.
Undriven oscillations with damping (linear in velocity)

\[ m\ddot{x} = -kx - bx \]

\[ \Rightarrow \dot{x} + \beta \dot{x} + \omega_0^2 x = 0 \]

Try \( x(t) = A e^{\alpha t} \)

\[ \Rightarrow \gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \]

Case 1: \( \omega_0 > \beta \) underdamped

Define \( \omega_i^2 = \omega_0^2 - \beta^2 > 0 \)

\[ \sqrt{\beta^2 - \omega_0^2} = \pm i \omega_i \]

\( \omega_i < \omega_0 \)

\[ x(t) = A_1 e^{-\beta t} + A_2 e^{-\beta t} = e^{-\beta t} \left[ A_1 e^{i\omega_i t} + A_2 e^{-i\omega_i t} \right] \]

\[ = e^{-\beta t} A \cos(\omega_i t + \phi) \]

\[ \frac{\omega_i}{\omega_0} \] is the time between adjacent maxima (minima)

\[ \frac{\omega_i}{\omega_0} > \frac{\omega_i}{\omega_0} = T \text{ period for undamped oscillation} \]

Phase Diagram

\[ \text{infinite number of oscillations} \]

\[ \text{Mechanical energy is lost, } \rightarrow \text{ heat} \]

\[ \text{radiated away to do } \text{ its work.} \]
\[ \text{Case 2: } \omega_0 < \beta \quad \text{over damped} \]

Define \( \omega_2^2 = \beta^2 - \omega_0^2 > 0 \)

\[ x(t) = A_1 e^{\beta t} + A_2 e^{-\alpha t} = e^{-\beta t} \left[ A_1 e^{\alpha t} + A_2 e^{-\alpha t} \right] \]

\[ v(t) = -\beta x + (\beta \omega_0) v \]

Phase diagram:

- One crossing
- Most initial conditions follow this dashed line (\( A \) term dominates for \( t \to \infty \)) unless \( A_1 = 0 \)
- Quadratic decreases to origin (no \( x > 0 \) crossing)
\[ c_2 \beta \quad \omega_0 = \beta \quad \text{critical damping} \]

\[ \gamma = -\beta = \rho \quad \text{degenerate roots of auxiliary equation} \]

\[ A_1 e^{\gamma t} \quad \text{and} \quad A_2 e^{\gamma t} \quad \text{are no longer} \]

\[ \text{independent solutions}. \quad \text{Multiply by powers of} \ t \]

\[ \text{why} \ e^{\sqrt{\rho^2 - \omega_0^2} t} \approx \left( 1 + \sqrt{\rho^2 - \omega_0^2} \right) e^{\rho t} \quad \text{small} \ t \beta \quad \text{can} \]

\[ x(t) = A_1 e^{\gamma t} + tA_2 e^{\gamma t} = e^{(A_1 + tA_2) t} \]

\[ \text{verify and solution} \quad (A_1 e^{-\beta t} \ x \ of \ the \ form \ we \ guessed = \}

\[ \text{guaranteed} \ x_0 \ \text{with) \}

\[ x = e^{-\beta t} tA_2 \]

\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0 \]

\[ \dot{x} = e^{-\beta t} A_2 (1 - \beta t) \]

\[ \ddot{x} + 2\beta \dot{x} + \beta^2 x = 0 \]

\[ \dot{x} = e^{-\beta t} A_2 (-2\beta + \beta^2 t) \]

\[ e^{-\beta t} A_2 (-2\beta + \beta^2 t) + \frac{e^{-\beta t}}{A_2} 2\beta (1 - \beta t) + e^{-\beta t} t^2 A_2 \beta^2 = 0 \]

\[ -2\beta + \beta^2 t + 2\beta - 2\beta^2 t + t^2 \beta^2 = 0 \]

\[ \text{critically damped motion} \]

\[ \text{approaches} \ x = 0 \ \text{fast or} \]

\[ \text{then either under damped or} \]

\[ \text{over damped motion} \]

\[ \text{does not cross} \ x = 0 \ \text{an \ \textit{infinite}} \]

\[ \text{times} \]

\[ \text{if} \ \beta \ \text{were any \ \textit{less}} \ \text{it would} \]

\[ \text{cross} \ x = 0 \ \text{an \ \textit{infinite} \ \textit{times}} \]
\[ m \ddot{x} = -kx - bx + F_0 \cos(\omega t) \] arbitrary driving frequency

\[ \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = A \cos(\omega t) \]

\[ D[x] = A \cos(\omega t) \]

solution = complementary function + particular solution

(solution to homogeneous eqn)

\[ BA + 0 \]

transients - decay exp.

\[ x(t) = x_c(t) + x_p(t) \]

\[ D[x_c(t)] = 0 \quad D[x_p(t)] = A \cos(\omega t) \]

We already know

\[ x_c(t) = e^{-\beta t} \left[ A e^{-\sqrt{\beta^2 - \omega_0^2} t} + A e^{\sqrt{\beta^2 - \omega_0^2} t} \right] \]

under critically, or over-damped.
\[
\ddot{X} + 2\beta \dot{X} + \omega_0^2 X = \frac{F_0}{m} \cos(\omega t) + \phi
\]

\[
X(t) = X_e(t) + X_p(t) \quad \dddot{X}_e + 2\beta \ddot{X}_e + \omega_0^2 X_e = 0
\]

\[
X_e(t) = e^{-\beta t} \int_{-\beta t}^{0} e^{\beta \tau} \left( A_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + A_2 e^{-\sqrt{\beta^2 - \omega_0^2} t} \right) dt
\]

Guess a form for the particular solution:

\[
X_p(t) = D \cos(\omega t - \delta)
\]

Response to forcing at frequency \(\omega\) is oscillation at frequency \(\omega\), possibly phase shift.

\(D\) and \(\delta\) are not arbitrary — they will be determined completely. Only \(A_1\) and \(A_2\) in complementary solution are up to you.

\[
\dot{X}_p(t) = -\omega D \sin(\omega t - \delta)
\]

\[
\ddot{X}_p(t) = -\omega^2 D \cos(\omega t - \delta)
\]

\[
\dddot{X}_p + 2\beta \ddot{X}_p + \omega_0^2 X_p = \frac{F_0}{m} \cos(\omega t)
\]

\[
D \left[ -\omega^2 \cos(\omega t - \delta) - 2\beta \omega \sin(\omega t - \delta) + \omega_0^2 \cos(\omega t - \delta) \right] = \frac{F_0}{m} \cos(\omega t)
\]

\[
\sin(\omega t) \cos(\delta) - \cos(\omega t) \sin(\delta)
\]

\[
\cos(\omega t) \cos(\delta) + \sin(\omega t) \sin(\delta)
\]

\[
\left\{ \frac{F_0}{m} - D \left[ (\omega_0^2 - \omega^2) \cos \delta + 2\omega \beta \sin \delta \right] \right\} \cos(\omega t)
\]

\[
- D \left[ (\omega_0^2 - \omega^2) \sin \delta - 2\omega \beta \cos \delta \right] \sin(\omega t) = 0 \quad \text{for all} \ t
\]

but \(\sin(\omega t)\) and \(\cos(\omega t)\) are linearly independent functions.

\[
\frac{F_0}{m} - D \left[ (\omega_0^2 - \omega^2) \cos \delta + 2\omega \beta \sin \delta \right] = 0
\]

\[
D \left[ (\omega_0^2 - \omega^2) \sin \delta - 2\omega \beta \cos \delta \right] = 0 \quad \leftrightarrow \quad D = 0
\]
\[ \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{2\omega \beta}{\omega_0^2 - \omega^2} \]

\[ \sin \phi = \frac{2\omega \beta}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \]

\[ \cos \phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \]

1st eq. \[ D = \frac{F_{0,m}}{(\omega_0^2 - \omega^2) \cos \phi + 2\omega \beta \sin \phi} \]

\[ D = \frac{F_{0,m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \]

no dumping \[ f = 0 \]

\[ D \rightarrow \infty \ \text{as} \ \omega \rightarrow \omega_0 \]

Particular solution:

\[ x_p(t) = \frac{F_{0,m}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \cos \left[ \omega t - \tan^{-1}\left(\frac{2\omega \beta}{\omega_0^2 - \omega^2}\right) \right] \]

no free parameters.

\( S \) is the phase shift between input \( F_0 \cos(\omega t) \) and output (response) \( x_p(t) = D \cos(\omega t - S) \)

\[ \frac{S}{\pi} + \frac{1}{2\pi} \]

\[ \frac{\pi}{2} \]

\[ \pi \]

\[ \omega_0 \]

\( S = 0 \) no phase shift at \( \omega = 0 \) (DC) \( \rightarrow \) response follows input

\( S = \frac{\pi}{2} \) at \( \omega = \omega_0 \) (natural frequency without damping)

\( S = \pi \) at \( \omega > \omega_0 \) \( \rightarrow \) response opposes input
Resonance

\[ \frac{F_0}{m} = \frac{F_0}{k} \]

D.C. amplitude

\[ \frac{dD}{d\omega} \bigg|_{\omega = \omega_p} = 0 \quad \Rightarrow \quad \omega_p = \sqrt{\omega_0^2 - \beta^2} \]

where \( \omega_p \) is imaginary.

For \( \beta > \frac{\omega_p}{\sqrt{2}} \)

there is no resonance (\( \omega_p \) is imaginary)

\[ D \uparrow \]

\[ \frac{F_0}{k} \uparrow \quad \Rightarrow \quad \omega \]

Quality factor \( Q = \frac{\omega_p}{\omega_0} \)

Foucault pendulum - many oscillations over several days
high Q oscillator
facing fork - 204

Not all physical quantities peak at the same frequency.

Amplitude resonance frequency = \( \omega_p = \sqrt{\omega_0^2 - \beta^2} \)

Potential energy resonance freq. \( \omega_p \) depends on \( \beta \)

Kinetic energy resonance freq. \( \omega_\tau = \omega_0 \)

Velocity resonance freq. \( \omega_v = ? \)

Breit-Wigner?
Simple Harmonic Oscillator (SHO)

Case 1: some spring constant in both directions

\[
\begin{align*}
\vec{F} &= -k \vec{r} \\
F_x &= m \ddot{x} = -k x \\
F_y &= m \ddot{y} = -k y
\end{align*}
\]

\[
x(t) = A \cos(\omega_0 t + \theta) \\
y(t) = B \cos(\omega_0 t + \phi)
\]

Shape of the curve $y(x)$?

eliminate time $t$

\[
t = \left( \frac{x^2}{A} \right) \frac{A}{\omega_0}
\]

sub into $y(t)$

\[
y(x) = B \cos \left[ \omega_0 \frac{\cos^{-1}(\frac{x}{A})}{\omega_0} + \phi \right] = B \cos \left[ \cos^{-1}(\frac{x}{A}) + \phi - \alpha \right]
\]

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
y'(x) = B \cos \left[ \cos^{-1}(\frac{x}{A}) \right] \cos(\beta - \alpha) - B \sin \left[ \cos^{-1}(\frac{x}{A}) \right] \sin(\beta - \alpha)
\]

\[
\cos \left[ \cos^{-1}(\frac{x}{A}) \right] = \frac{x}{A}
\]

\[
\theta = \cos^{-1}(\frac{x}{A})
\]

\[
\sin \left[ \cos^{-1}(\frac{x}{A}) \right] = \sin(\theta) = \frac{y}{y_0} = \frac{\sqrt{A^2 - x^2}}{x}
\]

\[
y(x) = \frac{B}{A} x \cos \left( \frac{\pi y}{y_0} \right) - \frac{B}{A} \sqrt{A^2 - x^2} \sin \left( \frac{\pi y}{y_0} \right)
\]

\[
y'(x) = \frac{B}{A} x \cos \frac{\pi y}{y_0} - \frac{B}{A} \sqrt{A^2 - x^2} \sin \frac{\pi y}{y_0}
\]

Square both sides

\[
B^2 x^2 - 2AB xy \cos \delta + A^2 y^2 = A^2 B^2 \sin^2 \delta
\]
ellipses: circle \( x^2/a^2 + y^2/b^2 = 1 \)

original line is a degenerate ellipse

\[ \delta = \frac{\pi}{2} \]

\[ \delta = \frac{3\pi}{2} \] Amplitude when \( \delta = \pi/2 \)

**Case 2 - different spring constant for \( x \) and \( y \) directions**

\[ F_x = m\ddot{x} = -k_x x \]
\[ F_y = m\ddot{y} = -k_y y \]

\[ x(t) = A \cos(\omega t + \phi) \]
\[ y(t) = B \cos(\omega t + \phi) \]

\[ \omega_1 = \sqrt{\frac{k_x}{m}} \]
\[ \omega_2 = \sqrt{\frac{k_y}{m}} \]

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**Resonant figures.**

**Effect of amplitude, frequency ratio, phase**

\[ \omega_x = 2\omega_y \]
\[ \omega_x = 3\omega_y \]
Simple Pendulum (all mass in point)

\[ \tau = I_0 \alpha \]

Free-body

\[ T \cdot 0 - W L \sin \theta = mL^2 \ddot{\theta} \]

\[ \ddot{\theta} = -\frac{g}{L} \sin \theta \]

not SHM

but for small angles \( \sin \theta \approx \theta \)

\[ \ddot{\theta} \approx -\frac{g}{L} \theta \]

looks like

\[ \ddot{x} = -\omega^2 x \]

\[ \Rightarrow \omega_0 = \sqrt{\frac{g}{L}} \]

Hooke's law spring is SH for both small and large amplitudes

Pendulum only small for small \( \theta \)

\[ \sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \ldots \] (in valid)

whenever \( \frac{\theta^3}{6} \ll \theta \)

Aside: the simple pendulum can be solved without the approximation

Physical Pendulum

\[ \tau = I_0 \alpha \]

\[ p_0 - W L \sin \theta = I_0 \alpha \]

\[ -mgL \sin \theta = I_0 \ddot{\theta} \]

\[ \ddot{\theta} = -\frac{mg}{I_0} \theta \]

\[ \Rightarrow \omega_0 = \sqrt{\frac{mg}{I_0}} \]