Dispersion Relation $\omega(k)$

- Light in vacuum
  \[ \omega = c \kappa \quad \text{c speed of light} \]

- Light in a material with index of refraction $n(\omega)$
  \[ \omega = \frac{c}{n(\omega)} \kappa \]

$\text{index of refraction } n \geq 1$

$\text{vacuum } n = 1$

\[ \text{white} \]

\[ \text{red} \quad \text{green} \quad \text{blue} \]
O.R. \[ w = 2\sqrt{\frac{\varepsilon}{m}} \left| \sin \left( \frac{ka}{2} \right) \right| \]

Suppose we send a wave packet into the crystal.

Two velocities:
- Phase velocity \[ U_p = \frac{\omega}{k} \]
- Group velocity \[ U_g = \frac{\text{d}w}{\text{d}k} \]
\[ \nu_g = \frac{d\nu}{dk} = \frac{d}{dk}\left[ 2\sqrt{\frac{E}{m}} \sin \left( \frac{ka}{2} \right) \right] = a\sqrt{\frac{E}{m}} \cos \left( \frac{ka}{2} \right) \]

\[ \nu_g = 0 \quad \text{at } k = \pm \frac{\pi}{a} \]

\[ \nu_g \text{ is tangent slope} \]

first Brillouin zone

At the zone boundaries \( k = \pm \frac{\pi}{a} \)

standing wave \( \nu_g = 0 \)
\[
\frac{U_{s+1}}{U_s} = \frac{A e^{ika(s+1)}}{A e^{ika s}} = e^{ika}
\]
At zone boundary \( ka = \pm \pi \)
\[
\frac{U_{s+1}}{U_s} = e^{\pm i\pi} = -1
\]

Only wave vectors \( k \) in the first Brillouin zone have physical meaning

\[
k = \frac{2 \pi n}{\lambda}
\]
For \( k \)’s outside the Brillouin zone

\[ |k| > \frac{\pi a}{2} \]

\[ U_s = A e^{i k x} = A e^{i\pi (k_1 + \alpha n)} = A e^{i\pi k_1} \]

**Long wavelength limit (small \( k \) limit)**

D.R. \( \omega^2 = \frac{2e}{\mu} \left[ 1 - \cos(ka) \right] \)

\( \lambda \gg a, \quad ka \ll 1 \)
\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots
\]

\[
\omega^2 = \frac{2\varepsilon}{\mu} \left[1 - (1 - \frac{\lambda^2}{\varepsilon})\right] + \ldots
\]

\[
\omega^2 = \frac{2\varepsilon}{\mu} \frac{\lambda^2}{\varepsilon}
\]

\[
\omega = \alpha \sqrt{\frac{\varepsilon}{\mu}} \quad \text{no dispersion}
\]

\[
\text{speed of sound (long A's)}
\]

\[
\text{no dispersion}: \quad \mathbf{U}_g = \mathbf{U}_g
\]

\[
\mathbf{U}_g = \frac{\omega}{k} = \alpha \sqrt{\frac{\varepsilon}{\mu}} = \mathbf{U}_g = \frac{d\mathbf{w}}{dk}
\]
Longitudinal waves more than nearest neighbor interactions

\[ \omega^2 = \frac{2C}{m} \left[ 1 - \cos(ka) \right] \]

\[ \omega^2 = \frac{2}{m} \sum_{p=1}^{\infty} \xi_p \left[ 1 - \cos(kp\alpha) \right] \]
Transverse waves

No contribution from nearest neighbour interactions
If these springs are under tension (rest length $a$), there is a restoring force proportional to displacement.

$F \propto Cu^e

\text{Our problem}

\frac{cu}{e}

\text{Rest length of springs = } a

F \propto u^3

\frac{cu}{a}

\text{new length is } \sqrt{a^2 + u^2}

\text{amount spring stretched is } \sqrt{a^2 + u^2} - a \approx a
\[ a \sqrt{1 + \frac{u^2}{a^2}} - a \]

\[ a (1 + \frac{u^2}{a^2})^{\frac{1}{2}} - a \]

\[ a \left[ 1 + \frac{u^2}{2a^2} + \ldots \right] - a \]

\[ \frac{u^2}{a} \]

\[
\text{Force } \quad F = C \frac{u^2}{a} \\
F_y = F \sin \theta \approx F \tan \theta \approx F \frac{u}{a} \\
2F_y = F_{net} = C \frac{u^3}{a^2} + u^3
\]