\[ \Phi(\hat{r}) = \frac{kQ}{r} \left[ \frac{d}{r} \cos \theta + O\left(\frac{d^2}{r^2}\right) \right] \]

\text{dipole term no monopole term this time}

Suppose we want all but the dipole term to vanish. This will be a "point dipole."

Take the limits; \( q \to \infty \) such that \( d \to 0 \)

\[ qd = \alpha \text{ remains fixed.} \]

\[ \Phi(\hat{r}) = \frac{kQ \cos \theta}{r^2} = \frac{k \mu \cdot \hat{r}}{r^2} \]

\text{dipole}

The next term is proportional to

\[ \frac{kQ}{r} \frac{d^2}{r^2} = \frac{k \theta d^2}{r^4} = \frac{k \mu d^2}{r^4} \to 0 \text{ as } d \to 0 \]

\( \hat{\mu} \) is the dipole moment vector \( \text{RESERVE} \)

\( \hat{\mu} \) points from (-) to (+) charge.
Other multipoles:
monopole, dipole, quadrupole, octupole, \ldots 2^n\text{-}pole
\begin{align*}
  n=0 & \quad n=1 & \quad n=2 & \quad n=3 \\
\end{align*}

You can now solve problem #3.

Mechanical Analogues:
Monopole \rightarrow total charge \leftrightarrow total mass of system
Dipole \leftrightarrow Center of Mass vector
Quadrupole \leftrightarrow Moment of Inertia Tensor

If you knew all of the multipole moments, you could reconstruct the charge distribution exactly.
Let's generalize the results for our specific examples. We seek a multipole expansion for the potential \( \Phi(\mathbf{r}) \).

Each term is composed of two factors:

1. "Field Factor" - depends only on the coordinates of the point at which the potential is calculated, the unprimed field coordinates.
   
   \[ \text{e.g.} \quad \frac{\mu}{r^3} \quad \text{is the field factor in the point dipole potential.} \]

2. "Source Factor" - depends only on the distribution of charge in the source.
   
   \[ \text{e.g.} \quad \hat{\mu} \quad \text{is the source factor in the point dipole potential.} \]

**RESERVE**
For the general analysis, consider a charge distribution that is localized
\[ \rho(\vec{r}) = 0 \quad \text{for} \quad |\vec{r}| > R \]
We will calculate \( \Phi(\vec{r}) \) only for points outside a sphere of radius \( R \).
\[ |\vec{r}| > R > |\vec{r}_0| \]
\[ \uparrow \quad \text{Field} \quad \uparrow \quad \text{Source} \]
We will expand \( \frac{1}{|\vec{r} - \vec{r}_0|} \), which occurs in the expression for the potential \( \Phi(\vec{r}) \), in a 3-dimensional Taylor series.

Reminder: In 1-dimension, if \( f \) is differentiable to all orders (analytic), then
\[ f(x) = f(a) + (x-a) f'(a) + \frac{1}{2!} (x-a)^2 f''(a) + \ldots \]

\[ \text{RESERVE} \]
But this is not the Taylor series we will use. We want the "increment" form:

\[ f(x+a) = f(x) + a f'(x) + \frac{1}{2!} a^2 f''(x) + \ldots \]

Now consider a scalar function of 3-d coordinates:

\[ f(\vec{r}+\vec{a}) = f(\vec{r}) + \vec{a} \cdot \vec{\nabla} f(\vec{r}) + \frac{1}{2!} (\vec{a} \cdot \vec{\nabla})^2 f(\vec{r}) + \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sum_{i=1}^{3} i_i = 1}^{3} a_{i_1} \ldots a_{i_n} \frac{\partial^n f(\vec{r})}{\partial x_{i_1} \ldots \partial x_{i_n}} \]

where \( \vec{a} \cdot \vec{\nabla} = \sum_{i=1}^{3} a_{i} \frac{\partial}{\partial x_{i}} \)

and \( (\vec{a} \cdot \vec{\nabla})^2 = \sum_{i_1, i_2 = 1}^{3} a_{i_1} a_{i_2} \frac{\partial^2}{\partial x_{i_1} \partial x_{i_2}} \), etc.

for the multiple expansion:

\[ \vec{a} = -\nabla f \quad \text{and} \quad f(\vec{r}-\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} \]

\[ \text{RESERVE} \]
The Taylor series converges for $|\mathbf{\cdot}| > |\mathbf{\cdot}|$

$$
\frac{1}{(\mathbf{\cdot})^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n = 1}^{3} \xi_{i_1} \cdots \xi_{i_n} \frac{\partial^n}{\partial \xi_{i_1} \cdots \partial \xi_{i_n}} \left( \frac{1}{\mathbf{\cdot}} \right)
$$

So

$$
\overline{\mathbf{\Phi}}(\mathbf{\cdot}) = \int dV \left[ \frac{k \rho(\mathbf{\cdot})}{(\mathbf{\cdot})^2} \right]
$$

$$
= k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1, \ldots, i_n = 1}^{3} \overline{Q}_{i_1 \cdots i_n} \frac{\partial^n}{\partial \xi_{i_1} \cdots \partial \xi_{i_n}} \left( \frac{1}{\mathbf{\cdot}} \right)
$$

where $\overline{Q}_{i_1 \cdots i_n} = \int dV \rho(\mathbf{\cdot}) \xi_{i_1} \cdots \xi_{i_n}$

This is the $2^n$-pole moment tensor.

It is a tensor of rank $n$ and by its definition can be seen to be symmetric under the interchange of any two indices.

Some examples to clarify the notation!  

`RESERVE`
\[ n = 0 \quad \bar{\rho} = \int dV \rho(\vec{r}') = \text{monopole moment (total charge) scalar} \]

\[ n = 1 \quad \bar{\rho}_i = \int dV \rho(\vec{r}') \vec{x}_i = \text{dipole moment vector} \]

(this is what we called \( \bar{\rho} \) previously)

\[ n = 2 \quad \bar{\rho}_{ij} = \int dV \rho(\vec{r}') \vec{x}_i \vec{x}_j = \text{quadrupole moment tensor} \]

\[ n = 3 \quad \bar{\rho}_{ijk} = \int dV \rho(\vec{r}') \vec{x}_i \vec{x}_j \vec{x}_k = \text{octupole moment tensor} \]

Notice that only the primed source coordinates (\( \vec{r}' \)) appear in \( \bar{\rho} \).

You are ready for problem #4.
Warning! The definition of multipole tensors varies from author to author. The definition of \( \bar{Q}_i \ldots \bar{Q}_m \) arises in a natural way through a Taylor series expansion in Cartesian coordinates. Later, we will expand in spherical coordinates and these are the multipole moments used by Jackson.

Why the bar over the \( \bar{Q}_i \ldots \bar{Q}_m \)?

It is customary (but not necessary) to redefine the Cartesian multipole moment tensors.

Right now, we have

\[
\Phi (\mathbf{r}) = k \left( \frac{\bar{Q}}{r} - \bar{Q} \cdot \mathbf{\nabla} \left( \frac{1}{r} \right) + \frac{1}{2!} \sum_{i,j} \bar{Q}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) + \cdots \right)
\]

Note: the 2\textsuperscript{nd} order term falls off \( \frac{1}{r^3} \). This is what makes the expansion useful, especially at large \( r \).
In problem #2 we saw that
\[ \nabla \left( \frac{1}{r} \right) = -\frac{\hat{r}}{r^3} \]

we can similarly show that
\[ \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) = \frac{1}{r^5} \left( 3 x_i x_j - r^2 \delta_{ij} \right) \]

so we can write the potential as
\[ \Phi(r) = k \frac{Q}{r} + \frac{k}{r^3} \cdot \vec{r} + \frac{1}{2} \sum_{ij} \bar{Q}_{ij} \cdot \frac{1}{r^5} (3 x_i x_j - r^2 \delta_{ij}) \]

Consider the tensor: \( 3 x_i x_j - r^2 \delta_{ij} \)

its trace is \( \sum_{ij} (3 x_i x_j - r^2 \delta_{ij}) \delta_{ij} \)

\[ \sum_{i} (3 x_i x_i - r^2 \delta_{ii}) = (3 r^2 - 3 r^2) = 0 \]

Therefore, we may add to \( \bar{Q}_{ij} \) any multiple of the unit tensor \( \delta_{ij} \) without changing the electrostatic potential.

\text{RESERVE}
Define: \( Q_{ij} = \overline{Q}_{ij} + A \delta_{ij} \)

where \( A \) is arbitrary.

The new quadrupole piece of \( I(r) \) is

\[
\frac{1}{2} \sum_{ij} (3x_i x_j - r^2 \delta_{ij}) Q_{ij} = \frac{1}{2} \sum_{ij} (3x_i x_j - r^2 \delta_{ij}) (\overline{Q}_{ij} + A \delta_{ij})
\]

\[
= \frac{1}{2} \sum_{ij} (3x_i x_j - r^2 \delta_{ij}) \overline{Q}_{ij} + A (0)
\]

same as the old quadrupole piece.

It is customary to choose \( A = \frac{1}{3} \sum_k \overline{Q}_{kk} \)

that is \((-\frac{1}{3})\) times the trace of the old tensor.

\[ Q_{ij} = \overline{Q}_{ij} - \frac{1}{3} (\sum_k \overline{Q}_{kk}) \delta_{ij} \]

Why do this?

Because \( Q_{ij} \) is now traceless.
\[
\text{Tr}[Q_{ij}] = \sum_i Q_{ii} = \sum_i \left( \overline{Q}_{ii} - \frac{1}{3} \sum_k \overline{Q}_{kk} \delta_{ii} \right) = \sum_i \overline{Q}_{ii} - \frac{1}{3} \sum_k \overline{Q}_{kk} = 0
\]

In 3 dimensions, a 2nd rank tensor (matrix) has 9 components

- general - 9 independent elements
- symmetric - 6 independent elements

symmetric + traceless - 5 independent elements

since \( Q_{11} + Q_{22} + Q_{33} = 0 \)

A traceless, symmetric rank n tensor has \( 2n+1 \) independent components.

\( Q_{i\ldots m} \) are called "irreducible" tensors because all of their elements are independent.
Translation Dictionary:

\[ \overline{Q} = \overline{Q} \quad \text{no change in the total charge}, \]

\[ Q_i = \overline{Q}_i \quad \text{no change in the dipole moment vector}, \]

\[ Q_{ij} = \overline{Q}_{ij} - \frac{1}{2} \left( \sum_k \overline{Q}_{kk} \right) \delta_{ij} \]

the connection between \( Q_{i\ldots i_n} \) and \( \overline{Q}_{i\ldots i_n} \) is more complicated for \( n > 2 \).

\[
\Phi(r) = k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i\ldots i_n=1}^3 Q_{i\ldots i_n} \frac{2^n}{2X_i \ldots 2X_i} \left( \frac{1}{r} \right)
\]

is the same potential as before.

The spherical tensors used by Jackson which we will meet later are automatically irreducible. These are also used extensively in Quantum Mechanics.

End Lecture #2 RESERVE