Until now, we have dealt with a collection of point monopoles. We could also consider point multipoles, the most important example of which is a distribution of Point Dipoles.

One point dipole at the origin

\[ \mathbf{p}(\mathbf{r}) = \frac{\mathbf{p}}{r^3} \]

One point dipole at position \( \mathbf{r}' \)

\[ \mathbf{p}(\mathbf{r}) = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

A collection of point dipoles \( \mathbf{p}_i \) located at points \( \mathbf{r}_i \):

\[ \mathbf{p}(\mathbf{r}) = \sum_{i=1}^{N} \frac{\mathbf{p}_i \cdot (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} \]

It's a short step from the sum over a discrete dipole distribution to the integral over a continuous volume distribution of dipoles.

Let \( \mathbf{P}(\mathbf{r}') \) be the volume dipole moment density.

Then

\[ \mathbf{p}(\mathbf{r}) = \int d\mathbf{V}' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]
This will come up again later when we study dielectric media.

For a continuous surface distribution of dipoles

$$\vec{D}(\vec{r}) = \int dS' \frac{\vec{D}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Let's consider a special case in which all of the dipoles are aligned normal to the surface $S$. Then

$$dS' \vec{D}(\vec{r}') = dS' D(\vec{r}') \hat{n}$$

where $\hat{n}$ is a unit vector normal to the surface.

$$\vec{D}(\vec{r}) = \int dS' \frac{D(\vec{r}') \hat{n} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$\hat{n} \cdot (\vec{r} - \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} \cos \alpha$$

$$\vec{D}(\vec{r}) = \int dS' \frac{\cos \alpha}{|\vec{r} - \vec{r}'|^2} D(\vec{r}')$$

**RESERVE**
The definition of infinitesimal solid angle is

\[ d\Omega' = \frac{dS' \cos \alpha}{|\vec{r} - \vec{r}'|^2} \]

This is the solid angle subtended by a piece of surface area \(dS'\) located at \(\vec{r}'\) as seen by an observer at \(\vec{r}\).

\[ E(\vec{r}) = \int d\Omega' \cdot D(\vec{r}') \]

Now let's specify the problem to the case where \(D(\vec{r}')\) is constant over the entire surface:

\[ E(\vec{r}) = D \int d\Omega' = D \Omega \]

This says that the potential is equal to the constant surface dipole moment density \(D\) times the solid angle subtended by the whole surface, regardless of its shape!
The change in potential across a surface dipole moment density is

$$\Delta \Phi = 4\pi D$$

Very close to the surface, the solid angle subtended by the surface is half of all space, or $2\pi$ steradians. Below the surface, on the "tail" side of $\hat{n}$, the solid angle is $-2\pi$ steradians because of the way we have defined $d\Omega$ in terms of the angle between $\hat{n}$ and $(\hat{r} - \hat{r}')$.

So $\vec{E}$ is **discontinuous** across a surface distribution of dipole moment.

We will see shortly that this is analogous to the discontinuity in $\vec{E}$ across a surface charge density, $\tau$ (another word for charge is "monopole").
B) Differential and Integral Theorems of 

Electrostatics.

So far, we know that

$$\Phi(\vec{r}) = \int_{\text{All Space}} \frac{\Phi(\vec{r})}{|\vec{r} - \vec{r}'|} \, dV'$$

and

$$\vec{E}(\vec{r}) = -\nabla \Phi(\vec{r})$$

A vector calculus identity assures us that

$$\nabla \times (\nabla \Phi) = 0 \quad \forall \Phi(\vec{r})$$

that is, for any scalar function \( \Phi \),

but \( \vec{E}(\vec{r}) \) is the gradient of a scalar function \( \Phi(\vec{r}) \)

so

$$\nabla \times \vec{E} = 0$$

Remember: this is true only for electrostatics.

What does this equation mean physically?

Electrostatic field lines never close on themselves.
Proof: Suppose a field line did close

Apply Stoke's Theorem

\[ \oint \vec{E} \cdot d\vec{l} = \iint_S \nabla \times \vec{E} \cdot \hat{n} \, dS \]

Closed field line

\[ = 0 \]

S is any surface (open surface) with the field line as its boundary.

but on the field line \( \oint \vec{E} = \oint d\vec{l} \vec{E} \)

since \( d\vec{l} \) and \( \vec{E} \) point in the same direction.

\[ \oint \vec{E} \cdot d\vec{l} = 0 \]

field line

Since the integrand is positive semi-definite, the integral can only vanish if \( \vec{E} = 0 \) everywhere along the field line.

This is certainly not true in general.

Electrostatic field lines do not close.
Electrostatic field lines can diverge from and converge to points:

\[ \text{source} \quad \rightarrow \text{sink} \]

Now, we will discuss the divergence of \( \vec{E} \), but first, we need some mathematical results:

\[ \text{Note: } \nabla \cdot \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad \text{from homework} \]

\[ \nabla \cdot \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\nabla \cdot \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \]

So, we have

\[ \nabla \cdot \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -\nabla \cdot \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \]

We need one more result, namely

\[ \nabla \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = \nabla \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = -\nabla \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) \]

RESERVE
Now we're ready to tackle the field $\mathbf{E}(\mathbf{r})$

$$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r}) = -\nabla \int dV' \phi(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\int dV' \phi(\mathbf{r}') \nabla_r \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$$= +\int dV' \phi(\mathbf{r}') \nabla_r \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

Now take the divergence of both sides

$$\nabla_r \cdot \mathbf{E}(\mathbf{r}) = +\int dV' \phi(\mathbf{r}') \nabla_r \cdot \left[ \nabla_r \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]$$

$$= -\int dV' \phi(\mathbf{r}') \nabla_r \cdot \left[ \nabla_r \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right]$$

$$= -\int dV' \phi(\mathbf{r}') \nabla_r^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$

$\nabla_r^2$ is the Laplacian. In Cartesian coordinates it is

$$\nabla_r^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}$$
$\nabla_\mu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right)$ is a very peculiar function.

To see this, integrate over a sphere of radius $R$ centered at $\vec{r}'$:

$$\int dV' \nabla_\mu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right)$$

$$= \int dV' \nabla_\mu' \cdot [\nabla_\nu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right)]$$

use the divergence theorem

$$= \int \int_{S^2} \nabla_\mu' \cdot [\nabla_\nu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right)]$$

on the surface of the sphere:

$$\hat{n} = \frac{\vec{n} - \vec{r}'}{1 - \vec{r}' \cdot \vec{r}'}$$

and $1 - \vec{r}' \cdot \vec{r}' = R$

$$\nabla_\nu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right) = -\frac{\vec{r}'}{(1 - \vec{r}' \cdot \vec{r}')^3}$$

and $\hat{n} \cdot \nabla_\nu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right) = -\frac{R^2}{R^4} = -\frac{1}{R^2}$

$$\int \int_{S^2} dS' \nabla_\mu', \left( \frac{1}{1 - \vec{r}' \cdot \vec{r}'} \right) = \int \int_{S^2} dS' \left( -\frac{1}{R^2} \right) = 4\pi R^2 \left( -\frac{1}{R^2} \right) = -4\pi$$

RESERVE
This result is independent of the radius of the sphere. The only way that can be true is it
\[ \nabla_{\mu} \left( \frac{1}{r - r'} \right) = 0 \quad \text{for} \quad r \neq r' \]
(See homework problem #?)

But the volume integral of the peculiar function \( \nabla_{\mu} \left( \frac{1}{r - r'} \right) \) does not vanish so the integrand cannot vanish everywhere. It must be infinite at \( r = r' \).

\[ \nabla_{\mu} \left( \frac{1}{r - r'} \right) = C \delta^3(r - r') \]

Let us determine the constant \( C \): \( \int dV' \nabla_{\mu} \left( \frac{1}{r - r'} \right) = -4\pi = \int dV' C \delta^3(r - r') = C \)

\[ \text{RESERVED} \quad C = -4\pi \]
Divergence of $\vec{E}(\vec{r})$

$$\nabla \cdot \vec{E}(\vec{r}) = - \int dV' \rho(\vec{r}') \nabla' \cdot \left( \frac{1}{(\vec{r} - \vec{r}')^2} \right)$$

$$= - \int dV' \rho(\vec{r}') \left[ -4\pi \delta^3(\vec{r} - \vec{r}') \right]$$

$$= + 4\pi \rho(\vec{r})$$

\[ \nabla \cdot \vec{E}(\vec{r}) = 4\pi \rho(\vec{r}) \]  

**differential form of Gauss' Law**

If we substitute the definition $\vec{E}(\vec{r}) = -\nabla \phi(\vec{r})$

$$\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})$$  

is called the **Poisson equation**

If the region we are considering is charge-free, that is $\rho(\vec{r}) = 0$

then

$$\nabla^2 \phi(\vec{r}) = 0$$  

is the **Laplace Equation**

**RESERVE**
Integral form of Gauss' law:

Start with \( \nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi} \rho(\vec{r}) \)

Integrate both sides over a Volume \( V \)

\[
\int dV \ \nabla \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi} \int dV \rho(\vec{r}) = \frac{1}{4\pi} Q \text{ in } V
\]

\( \nabla \) divergence theorem

\[
\int_{S} ds \ \vec{n} \cdot \vec{E} = \frac{1}{4\pi} Q \text{ enclosed in } S
\]

You are ready for problem # 8

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Next Time:

What is needed to specify \( \vec{E}(\vec{r}) \) in a Finite region if the charge density \( \rho(\vec{r}) \) is known only in that region (not everywhere)?

Look at Green's Identities

RESERVE

--- End Lecture #3 ---