2) Suppose the region of interest is between two full infinite cylinders. The solution \( \Phi (s, \phi) \) must still be periodic in \( \phi \Rightarrow c_1 = 0 = c_2 = c_3 \Rightarrow \alpha = \pi \\

This time, the \( s = 0 \) axis is excluded, so the most general solution is

\[
\Phi (s, \phi) = A_0 + A'_0 \log s + \sum_{n=1}^{\infty} s^n \left[ A_n \cos (n\phi) + B_n \sin (n\phi) \right] \\
+ \sum_{n=1}^{\infty} s^{-n} \left[ A'_n \cos (n\phi) + B'_n \sin (n\phi) \right]
\]

The expansion coefficients \( A_0, A'_0, A_n, B_n, A'_n, B'_n \) are determined by Fourier analyzing the two bounding surfaces \( s = a, s = b \). Dirichlet boundary conditions might be:

\[
\begin{align*}
\Phi (a, \phi) &= f_1 (\phi) \\
\Phi (b, \phi) &= f_2 (\phi)
\end{align*}
\]
(2) A sector of an infinitely long circular cylinder with the following Dirichlet boundary conditions:

\[ \Phi(\varphi, \beta) = \begin{cases} \text{We want a complete set of functions of } \varphi \text{ to reproduce } f(\varphi) \text{ on the boundary. } \Rightarrow \text{Type (2) solution} \\
\end{cases} \]

The boundary conditions imply: \( C_2 = 0 \) and \( \omega = \frac{\pi}{\beta} \)

\[ \Phi(\varphi, \beta) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi \varphi}{\beta} \right) \frac{\sin \left( \frac{n\pi \beta}{\beta} \right)}{\sin \left( \frac{n\pi}{\beta} \right)} \]

(4) A sector, but with different boundary conditions:

\[ \Phi(\varphi, \beta) = g(\varphi) \]

Now we want a complete set of functions of \( \varphi \) to reproduce \( g(\varphi) \) on the boundary \( \Rightarrow \) type (3) solution

These boundary conditions imply: \( C_2 = 0 \), \( D_2 = 0 \)

\[ \Phi(\varphi, \beta) = \int_0^\beta d\alpha B(\alpha) \sinh(\alpha \varphi) \sin \left( \alpha \ln \left( \frac{\beta}{a} \right) \right) \]

This is a Fourier transform rather than a Fourier series because there is no longer a restriction that \( \alpha \) must be integral. \( \alpha \) assumes all real values.
iii) Boundary conditions depend on all 3 coordinates.

Look for factorizable solutions \( F(s, q, z) = R(s) F(q) Z(z) \).

\[
\frac{\nabla^2 F}{F} = 0 = \frac{(8R')'}{8R} + \frac{1}{8^2} \frac{F''}{F} + \frac{Z''}{Z} = 0
\]

\( f(s, q) \quad g(z) \)

At first glance it does not appear that we have succeeded in separating the variables since \( s \) and \( q \) are still entangled, but notice that the first two terms together are a function of \( s \) and \( q \) alone and the last term is a function of \( z \) alone. This can only hold true if both functions are constant.

Call the first separation constant \( C \).

\[
\frac{(8R')'}{8R} + \frac{1}{8^2} \frac{F''}{F} = C \quad \frac{Z''}{Z} = -C
\]

The first equation can be rewritten as:

\[
\frac{8(8R')'}{R} - s^2 C + \frac{F''}{F} = 0
\]

\( h_1(s) \quad h_2(q) \)
The first two terms form a function of $z$ alone and the third term is a function of $\psi$ alone. Call the second separation constant $K$. The fully separated equations are:

\[ Z'' + CZ = 0 \]
\[ F'' + KF = 0 \]
\[ \frac{s(\theta k_1)}{R} - \beta^2 C = K \]

In general, the two constants of separation are arbitrary real numbers — positive, negative, or zero. There are too many special case geometries to deal with in detail, so we will confine our discussion to one particular problem — a full circular cylinder of radius $a$ and length $L$.

Since the full range of $\psi$ is included in the problem, the solution to Laplace's equation, $\tilde{F}(\psi, \phi, z)$, must be periodic in $\psi$ with period $2\pi$.

For this special case $K = n^2$ where $n = 0, \pm 1, \pm 2, \ldots$

\[ F'' = -n^2 F(\psi) \implies F(\psi) = a_1 \cos(n\psi) + a_2 \sin(n\psi) \]
\[ a_2 = a_3 e^{i\psi} \]

\[ \text{RESERVE} \]
The remaining differential equations are:

\[ Z'' + C Z = 0 \quad \text{and} \quad \frac{S(R')}{}R - S^2C = n^2 \]

There are still three sub-cases to consider: the first separation constant, \( C \), can be positive, negative or zero:

1) \( C = 0 \)

\[ Z'' = 0 \quad \Rightarrow \quad Z(x) = b_1x + b_2 \]

\[ S(R') - n^2R = 0 \quad \Rightarrow \quad \begin{cases} n = 0, \quad R(x) = d_1x + d_2 \\ n \neq 0, \quad R(x) = c_1 \sinh(nx) + c_2 \cosh(nx) \end{cases} \]

2) \( C = -k^2 \quad \text{Kreal, positive} \)

\[ Z'' - k^2Z = 0 \quad \Rightarrow \quad Z(x) = b_1e^{kt} + b_2e^{-kt} \]

\[ \phi = \beta_1 \sinh(kx) + \beta_2 \cosh(kx) \]

\[ \frac{1}{S} (S(R'))' + (k^2 - \frac{n^2}{S^2})R = 0 \quad \Rightarrow \quad R(x) = d_1'J_n(kx) + d_2'N_n(kx) \]

where \( J_n(u) \) is the Bessel function of integer order \( n \), and \( N_n(u) \) is the Neumann function of integer order \( n \).

\( J_n \) and \( N_n \) are complete. Any function of \( S \) can be expanded in \( J_n \) and \( N_n \).
$J_n(u)$ is also called the Bessel function of the first type. $N_n(u)$ is also called the Bessel function of the second type, or the Weber function and the symbol is sometimes written $Y_n(u)$. In Mathematica, they are denoted BesselJ and BesselY, respectively.

These functions are defined for negative integer order by:

$$J_{-n}(u) = (-1)^n J_n(u)$$

$$N_{-n}(u) = (-1)^n N_n(u)$$

$J_n(u)$ and $N_n(u)$ are oscillatory functions. Think of them as cylindrical coordinate versions of sines and cosines. They each have an infinite number of zeroes.

$$J_n(u_{ns}) = 0 \quad N_n(u_{ns}) = 0 \quad s = 1, 2, ...$$

The zeroes are labeled by $s$. These zeroes are tabulated or available from Mathematica.
The derivatives of these functions also oscillate and have an infinite number of zeroes,

\[ J_n'(u_0) = 0 \quad N_n'(u_0) = 0 \quad n = 1, 2, \ldots \]

Asymptotically,

\[ J_n(u) \xrightarrow{u \to \infty} \sqrt{\frac{2}{\pi u}} \cos \left( u - \frac{n\pi}{2} - \frac{\pi}{4} \right) \quad n \geq 0 \]

\[ N_n(u) \xrightarrow{u \to \infty} -\sqrt{\frac{2}{\pi u}} \sin \left( u - \frac{n\pi}{2} - \frac{\pi}{4} \right) \quad n \geq 0 \]

The Bessel's functions of the first type are well-behaved at the origin:

\[ J_n(u) \xrightarrow{u \to 0} \frac{1}{n!} \frac{u^n}{a^n} \quad n \geq 0 \]

The Neumann functions diverge at the origin:

\[ N_0(u) \xrightarrow{u \to 0} -\frac{2}{\pi} \ln \left( \frac{u}{2} \right) + \frac{3}{4} y_0^2 \]

\[ N_n(u) \xrightarrow{u \to 0} -\frac{1}{\pi} (n-1)! \left( \frac{3}{u} \right)^n \quad n \geq 1 \]

Hence, when the \( p=0 \) axis is included in the physical region, we must exclude the Neumann functions, \( N_n(u) \).
For \( n \neq 0 \), all the \( J_n(u) \) vanish at the origin; but the origin is not counted as one of the zeroes \( u_n \).
\( N_0(u) \) vs. \( u \)

\( N_1(u) \) vs. \( u \)
\[ \text{In}[20]:= \quad \text{tmp}[u_] = (D[\text{BesselJ}[0,u], u]) \]
\[ \text{Out}[20]= \quad \frac{\text{BesselJ}[-1, u] - \text{BesselJ}[1, u]}{2} \]

\[ \text{In}[21]:= \quad \text{Plot}[\text{tmp}[u], \{u, 0.1, 30\}] \]

\[ \frac{\text{Out}[21]}{} \quad \text{Graphics} \]

\[ \text{In}[22]:= \quad \text{tmpq}[u_] = (D[\text{BesselJ}[1,u], u]) \]
\[ \text{Out}[22]= \quad \frac{\text{BesselJ}[0, u] - \text{BesselJ}[2, u]}{2} \]

\[ \text{In}[23]:= \quad \text{Plot}[\text{tmpq}[u], \{u, 0.1, 30\}] \]

\[ \frac{\text{Out}[23]}{} \quad \text{Graphics} \]
There are enough relations among the Bessel functions to fill several texts and many courses. We will only need the following orthogonality relation:

$$ \int_0^a s \, ds \, J_n \left( U_n s \frac{\theta}{a} \right) J_n \left( U_n \frac{\theta}{a} \right) = \frac{a^2}{2} \left[ J_{n+1} \left( U_n \frac{\theta}{a} \right) \right]^2 $$

Don't forget the "s" in the measure!

This is used to extract the expansion coefficients by the cylindrical analogue of Fourier's trick for sines and cosines.

--- End Lecture #12 ---