C) Spherical Boundaries

The Laplacian is

\[ \nabla^2 \Phi(r, \theta, \phi) = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \]

the first term can also be written as

\[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r \Phi) \]

i) boundary conditions depend only on \( r \) \( \Rightarrow \) the potential can depend only on \( r \) \( \Rightarrow \) \( \Phi = \Phi(r) \)

This case describes uniform spheres held at a constant potential

\[ \nabla^2 \Phi(r) = 0 = \frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \Phi(r) \right] = 0 \]

\[ \Rightarrow r^2 \frac{d}{dr} \Phi(r) = \text{constant} = -C_i \]

\[ \Phi(r) = \frac{C_i}{r} + C_2 \]

If the origin \( (r=0) \) is included in \( V \), then \( C_i = 0 \).

Suppose \( V \) is the region between two spheres of radii \( a \) and \( b \).

\[ \Phi(a) = \frac{C_i}{a} + C_2 = \Phi_a \]
\[ \Phi(b) = \frac{C_i}{b} + C_2 = \Phi_b \]

\( \Rightarrow \) determine \( C_i \) and \( C_2 \)

in \( V \): \( \Phi(r) = \frac{C_i}{r} + C_2 \)

RESERVE
We have already considered the case in which the boundary conditions depend on $\eta$ only. The geometry describes infinite wedges, which we considered in cylindrical coordinates.

We will not treat in detail the case in which the boundary conditions depend only on $\theta$. The geometry describes infinite cones, each held at a constant potential.

\[ \Phi(r, \theta) = \frac{U(r)}{r} T(\theta) \]

\[ \nabla^2 \Phi(r, \theta) = 0 \]

\[ \frac{r^2 \nabla^2 \Phi(r, \theta)}{\Phi(r, \theta)} = 0 = \frac{r^2 U''(r)}{U(r)} + \frac{1}{T(\theta)} \sin \theta \left[ \sin \theta T''(\theta) \right] \]

**RESERVE**

In anticipation of later results, we will call the separation constant $\lambda (\ell + 1)$. 

14-2
The separated differential equations are:

\[ U''(r) - \frac{L(L+1)}{r^2} U(r) = 0 \Rightarrow U(r) = A r^{L+1} + B r^{-L} \]

\[ \frac{d}{d\theta} \left[ \sin \theta \frac{d}{d\theta} T(\theta) \right] + L(L+1) \sin \theta T(\theta) = 0 \]

It is convenient to change variables:

\[ x = \cos \theta \quad \text{and} \quad x = d(\cos \theta) = -\sin \theta \, d\theta \]

\[ \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P(x) \right] + L(L+1) P(x) = 0 \]

This is the Legendre Differential Eq.

There are two types of solutions:

1) Legendre Functions (of the first kind): \( P_L(\cos \theta) \)

These are oscillatory, like sines and cosines or like the Bessel functions \( J_n \) and \( N_n \).

2) Legendre Functions of the second kind: \( Q_L(\cos \theta) \)

These are like exponentials (sineh's and cosineh's) or like the modified Bessel functions \( I_n \) and \( K_n \).

The \( Q_L(\cos \theta) \) functions diverge at \( \theta = 0, \pi \) so if the region of interest is the full sphere including the polar axis, we must exclude the \( Q_L(\cos \theta) \) solutions.

RESERV

14-3
The Taylor series expansion of $P_0(\cos \theta)$ terminates, that is it has a finite number of terms, so the Legendre functions (of the first kind) are actually Legendre polynomials.

The first few are:

\begin{align*}
P_0(\cos \theta) & = 1 \\
P_1(\cos \theta) & = \cos \theta \\
P_2(\cos \theta) & = \frac{1}{2}(3\cos^2 \theta - 1)
\end{align*}

These are normalized such that $P_0(1) = 1$.

The functions are orthogonal and complete on the interval $-1 \leq x \leq 1$ or equivalently $0 \leq \theta \leq \pi$.

\underline{Orthogonality}

\[ \int_{-1}^{1} \! dx \: P_n(x) P_m(x) = \frac{2}{2n+1} \delta_{nm} = \frac{\pi}{\sin \theta} \int_0^{\pi} \! d\theta \: P_n(\cos \theta) P_m(\cos \theta) \]

\underline{Completeness}

Any "nice" function $f(x)$ can be expanded as

\[ f(x) = \sum_{k=0}^{\infty} c_k \ P_k(x) \quad \text{on the interval} \quad -1 \leq x \leq 1 \]
One of the many identities involving Legendre polynomials is:

\[
\frac{d}{dx} P_{n+1}(x) - \frac{d}{dx} P_n(x) = (2n+1) P_n(x)
\]

So for a full sphere with azimuthal symmetry

\[
\Phi(r,\theta) = \frac{U(r)}{r} T(\theta) = \sum_{\ell=0}^{\infty} \left[ A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}} \right] P_\ell(\cos \theta)
\]

If \( r = 0 \) is included \( \Rightarrow B_0 = 0 \)

If \( r \to \infty \) is included \( \Rightarrow A_0 = 0 \)

**Motivation**

We have actually met the Legendre polynomials before. Consider the potential of a point charge \( q \) at position \( \vec{a} \) from the origin. Choose the \( z \)-axis along \( \vec{a} \).

\[
\Phi(R) = \frac{q}{|R-\vec{a}|} = \frac{q}{\sqrt{(R-\vec{a}) \cdot (R-\vec{a})}}
\]

\[
= \frac{q}{\sqrt{R^2 - 2R \cdot \vec{a} + \vec{a} \cdot \vec{a}}}
\]

Now perform a multipole expansion of this potential. In order for the series to converge, we distinguish two regions.
1) \( n < a \)

\[
\overline{\Phi}(\tilde{r}) = \frac{\tilde{r}}{a} \frac{1}{\sqrt{1 - 2\left(\frac{n}{a}\right)\cos \theta + \frac{n^2}{a^2}}} 
\]

Use the binomial expansion with expansion parameter \( \frac{n}{a} < 1 \)

\[
= \frac{\tilde{r}}{a} \left[ 1 + \left(\frac{n}{a}\right)\cos \theta + \left(\frac{n}{a}\right)^2 \left(\frac{3}{8} \cos^2 \theta - \frac{1}{2}\right) + \ldots \right]
\]

These are the Legendre Polynomials!

\[
= \frac{\tilde{r}}{a} \sum_{k=0}^{\infty} \left(\frac{n}{a}\right)^k \mathcal{P}_k(\cos \theta)
\]

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2) \( n \geq a \)

\[
\overline{\Phi}(\tilde{r}) = \frac{\tilde{r}}{n} \frac{1}{\sqrt{1 - 2\left(\frac{n}{a}\right)\cos \theta + \frac{n^2}{a^2}}} 
\]

Expansion parameter \( \frac{a}{n} < 1 \)

\[
= \frac{\tilde{r}}{n} \left[ 1 + \left(\frac{n}{a}\right)\cos \theta + \left(\frac{n}{a}\right)^2 \left(\frac{3}{8} \cos^2 \theta - \frac{1}{2}\right) + \ldots \right]
\]

\[
= \frac{\tilde{r}}{n} \sum_{k=0}^{\infty} \left(\frac{n}{a}\right)^k \mathcal{P}_k(\cos \theta)
\]

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We can describe an algorithm for isolating the \( n^{th} \) Legendre Polynomial from the series above;

Differentiate \( L \) times with respect to the expansion parameter, divide by \( L! \), and finally set the expansion parameter to zero.
Symbolically:

\[ P_\ell (\cos \theta) = \frac{1}{\ell!} \left[ \frac{\partial^\ell}{\partial \ell^\ell} G(t, \cos \theta) \right]_{t=0} \]

where

\[ G(t, \cos \theta) = \frac{1}{\sqrt{1 - 2t \cos \theta + t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell (\cos \theta) \]

is the Legendre polynomial generating function.

Now, a remarkable statement:

If we know the potential on the polar axis \( \theta = 0 \), we can determine the potential for all space. This is an incredible saving in time and effort!

Suppose that, by any means whatsoever, you determine the potential along a line to be \( f(r) \), where \( r \) is the distance from some origin. Then choose the polar axis to coincide with the line, and

\[ f(r, 0) = \sum_{\ell=0}^{\infty} \left[ A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] = f(r) \]

we used: \( P_\ell (\cos 0) = P_\ell (1) = 1 \)

The sum over \( \ell \) is called a "Laurent Expansion" of the function \( f(r) \).
A Laurent Expansion is simply a Taylor series with both positive and negative powers of \( r \).

We can read off the coefficients \( A_0 \) and \( B_0 \) from the Laurent expansion of \( f(r) \). Then it is trivial to "take the solution off axis." We simply put the \( B_0 \cos \theta \) back in the sum!

An example will illustrate the power and beauty of this technique:

Suppose we have a ring of charge \( q \) and radius \( a \).

For a point a distance \( r \) from the origin (at the center of the ring), we have

\[
\Phi(r) = \frac{q}{\sqrt{r^2+a^2}} = f(r)
\]

Potential along the dashed line

\[
\Phi(r) = \begin{cases} 
\frac{q}{a} \frac{1}{\sqrt{1+\frac{r^2}{a^2}}} & , r < a \\
\frac{q}{r} \frac{1}{\sqrt{1+\frac{a^2}{r^2}}} & , r > a
\end{cases}
\]

\( 14-8 \)
The Laurent expansion of

\[ \frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} (-1)^n C_n x^n \]

(convergent if \( x^2 < 1 \).

\[ C_0 = 1, \quad C_n = \frac{(2n-1)!!}{2^n n!}, \quad n \geq 1 \]

where \( m!! = m (m-2)(m-4) \ldots 1 \)

Now we can go off axis:

\[ \Phi(n, \theta) = \begin{cases} \frac{\varphi}{a} \sum_{n=0}^{\infty} (-1)^n C_n \left( \frac{\varphi}{a} \right)^{2n} P_n (\cos \theta), & n \leq a \\ \frac{\varphi}{r} \sum_{n=0}^{\infty} (-1)^n C_n \left( \frac{\varphi}{r} \right)^{2n} P_n (\cos \theta), & n > a \end{cases} \]

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End Lecture #14

RESERVE