Now suppose we did experiments to determine $S$
Since $S_{ii}$ are real, $S$ must be of the form

$$S = \begin{pmatrix} a & b + ic \\ b - ic & 1 - a \end{pmatrix} \quad S^+ = S \quad \text{hermitian}$$

$$\text{Tr}(S) = 1$$

So we need to perform 3 experiments. Suppose the results of the experiments imply that $\text{Tr}(S^2) = \text{Tr}(S)$.
We now show that this condition implies that the radiation is polarized. To show this, we note that since $S$ is hermitian it can be brought into diagonal form by a unitary transformation. If the eigenvalues of $S$ are distinct (non-degenerate) then the normalized eigenvectors must be orthogonal.

$$U^\dagger (i) U (j) = \delta_{ij} \quad (i, j = 1, 2)$$

$\times$ dot (scalar) product

The eigenvalues are defined as $S_i$ with

$$S U (i) = S_i U (i)$$

Since the eigenvectors are complete, we have

$$\sum_{i=1}^{2} U (i) U^\dagger (i) = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\bigotimes$ outer product
Then right multiplying the eigenvector equation by $U_i^+$ we get

$$ S U(i) U_i^+ = S_i U(i) U_i^+ $$

now sum over $i$:

$$ S \sum_{i=1}^{2} U_i U_i^+ = \sum_{i=1}^{2} S_i U(i) U_i^+ $$

Above decomposition

$$ S = \left( \sum_{i=1}^{2} S_i U(i) U_i^+ \right) \left( \sum_{j=1}^{2} S_j U(j) U_j^+ \right) $$

$$ = \sum_{i=1}^{2} \sum_{j=1}^{2} S_i S_j U(i) U(i)^+ U(j) U(j)^+ $$

$$ = \sum_{i=1}^{2} S_i S_i U(i) U(i)^+ $$

In fact $S^n = \sum_{i=1}^{2} S_i (A_n U(i))^+$

Note also $\text{Tr} \left[ U(i) U_i^+ \right] = \text{Tr} \left[ U_i^+ U(i) \right] = U_i^+ U(i) = 1$

Scalar product

So $\text{Tr}(S^2) = \text{Tr}(S)$ implies

$$ \sum_{i=1}^{2} S_i^2 = \sum_{i=1}^{2} S_i $$

or $\sum_i (S_i^2 - S_i) = 0$ or $\sum_i S_i (1 - S_i) = 0$
or \( S_1 (1-S_1) + S_2 (1-S_2) = 0 \)

Since \( 0 \leq S_i \leq 1 \) then each term is positive or zero and hence each term must be zero.

Thus \( S_1 (1-S_1) = 0 \) and \( S_2 (1-S_2) = 0 \)

But the sum \( S_1 + S_2 \) must be 1 so either \( S_1 = 1 \) and \( S_2 = 0 \) or \( S_1 = 0 \) and \( S_2 = 1 \).

This means \( \mathbf{S} = \mathbf{U}_1 \mathbf{U}_1^+ \) or \( \mathbf{S} = \mathbf{U}_2 \mathbf{U}_2^+ \)

But this is exactly the form of a polarized wave with \( \mathbf{U} = \frac{\mathbf{E}_0}{\sqrt{\mathbf{E}_0^+ \mathbf{E}_0}} \).

Polarized radiation \( \iff \text{Tr} (\mathbf{S}^2) = 1 \)

\( \iff \text{Tr} (\mathbf{J}^2) = \left( \text{Tr} (\mathbf{J}) \right)^2 \)
If the radiation is not polarized, then

$$\hat{S} = \hat{S}_1 \mathbf{U}_1(1) \mathbf{U}_1^+ + \hat{S}_2 \mathbf{U}_2(2) \mathbf{U}_2^+$$

If \( \hat{S}_1 \neq \hat{S}_2 \) (eigenvalues are not degenerate)
then \( \hat{S}_2 = 1 - \hat{S}_1 \) and \( \mathbf{U}_i(i) \) is unique up to a phase factor. Since \( \hat{S} \) must contain 3 independent parameters the eigenvectors \( \mathbf{U}_i(i) \) must contain 2 independent parameters. The phase does not count as an independent parameter because it contributes nothing to \( \hat{S} \).

To parametrize \( \mathbf{U}_i(i) \) write

$$\mathbf{U}_i(i) = \hat{R} \cdot \mathbf{U}_i(i)$$

where \( \hat{R} \) is the 2x2 real orthogonal matrix

$$\hat{R} = \begin{pmatrix} \cos \Phi & -\sin \Phi \\ \sin \Phi & \cos \Phi \end{pmatrix}$$

and

$$\mathbf{U}_i(1) = \frac{1}{\sqrt{2-\varepsilon^2}} \begin{pmatrix} 1 \\ i\sqrt{1-\varepsilon^2} \end{pmatrix}, \quad \mathbf{U}_i(2) = \frac{1}{\sqrt{2-\varepsilon^2}} \begin{pmatrix} i\sqrt{1-\varepsilon^2} \\ 1 \end{pmatrix}$$

\( \hat{R} \) generates a rotation about the direction of propagation \( \hat{k} \) through an angle \( \Phi \) counterclockwise.
The vectors \( \vec{V}_{(1)} \) and \( \vec{V}_{(2)} \) describe ellipses.

To see this, define

\[
\vec{V}_{(i)} = X_{(i)} + iY_{(i)}
\]

with \( X_{(i)} = \begin{pmatrix} X_{(i)}^1 \\ X_{(i)}^2 \end{pmatrix} \) and \( Y_{(i)} = \begin{pmatrix} Y_{(i)}^1 \\ Y_{(i)}^2 \end{pmatrix} \) real.

If, for general phase angle \( \phi_{(i)} \), we put in the expressions above for \( \vec{V}_{(i)} \) we will find curves in the \( X_{(i)}^1 - X_{(i)}^2 \) plane (and also in the \( Y_{(i)}^1 - Y_{(i)}^2 \) plane but these give nothing new). The curves turn out to be ellipses.

Homework

Show that the column vector \( \vec{V}_{(1)} \) describes an ellipse with major axis along \( \xi_{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) minor axis along \( \xi_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and eccentricity \( \varepsilon \).

Show that \( \vec{V}_{(2)} \) has major axis along \( \xi_{(2)} \), minor axis along \( \xi_{(1)} \), and eccentricity \( \varepsilon \).

Show that \( \vec{V}_{(i)}^+ \vec{V}_{(j)} = \delta_{ij} \) and that \( R^+R = I_2 \) and hence that \( \vec{U}_{(i)}^+ \vec{U}_{(j)} = \delta_{ij} \). What geometric figures do the vectors \( \vec{U}_{(i)} \) describe?