The fields in the incident medium are

\[ \mathbf{E}_i(\mathbf{n},t) = E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + E_0' e^{i(\mathbf{k}' \cdot \mathbf{r} - \omega t)} \]

\[ \mathbf{B}_i(\mathbf{n},t) = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \frac{\mathbf{k}' \times \mathbf{E}_0'}{\omega} e^{i(\mathbf{k}' \cdot \mathbf{r} - \omega t)} \]

The fields in the refracted medium are

\[ \mathbf{E}_r(\mathbf{n},t) = E'_0 e^{i(\mathbf{k}' \cdot \mathbf{r} - \omega t)} \]

\[ \mathbf{B}_r(\mathbf{n},t) = \frac{\mathbf{k} \times \mathbf{E}_0'}{\omega} e^{i(\mathbf{k}' \cdot \mathbf{r} - \omega t)} \]

When the field point \( \mathbf{r} \) is on the boundary, we designate it by \( \mathbf{R} \). The first important result is obtained by noting that the boundary conditions can only be satisfied if all the wave phase factors are the same for all \( \mathbf{R} \) and \( t \):

\[ e^{i(\mathbf{k} \cdot \mathbf{R} - \omega t)} = e^{i(\mathbf{k}' \cdot \mathbf{R} - \omega t)} = e^{i(\mathbf{k}'' \cdot \mathbf{R} - \omega t)} \]

The time aspect has already been dealt with (it resulted in the same frequency \( \omega \) being used in each part of the wave). The special aspects require

\[ k \mathbf{R} = k' \mathbf{R} = k'' \mathbf{R} \]
If we choose the origin on the interface, then we can write the previous equation as
\[ \vec{k} \times \nabla = \vec{k}' \times \nabla = \vec{k}'' \times \nabla \quad (\nabla \text{ is the normal}) \]
or as
\[ k \sin(\theta_i) = k' \sin(\theta_b) = k'' \sin(\theta_r) \]
But \( k = \frac{\omega}{v} = k'' \) and \( k' = \frac{\omega}{v'} \)
where \( v = \frac{1}{\sqrt{\varepsilon \mu}} \) and \( v' = \frac{1}{\sqrt{\varepsilon' \mu'}} \)

Thus \( \theta_i = \theta_r \) angle of incidence equals angle of reflection

and since \( n = \frac{c}{v} = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \) index of refraction of the incident medium
\( n' = \frac{c}{v'} = \sqrt{\frac{\varepsilon' \mu'}{\varepsilon_0 \mu_0}} \) index of refraction of the refracted medium

\[ n \sin(\theta_i) = n' \sin(\theta_o) \quad \text{Snell's Law (or Snell's)} \]

Note that the medium with the larger index of refraction has the smaller relevant angle.
Now look again at the boundary conditions:

\[ \hat{n} \cdot \Delta \vec{D} = 0 \Rightarrow \hat{n} \cdot \left[ \varepsilon (\vec{E}_0 + \vec{E}_0') - \varepsilon' \vec{E}_0' \right] = 0 \]

\[ \hat{n} \times \Delta \vec{E} = 0 \Rightarrow \hat{n} \times \left[ \vec{E}_0 + \vec{E}_0'' - \vec{E}_0' \right] = 0 \]

\[ \hat{n} \cdot \Delta \vec{B} = 0 \Rightarrow \hat{n} \cdot \left[ \vec{k} \times \vec{E}_0 + \vec{k} \times \vec{E}_0'' - \vec{k} \times \vec{E}_0' \right] = 0 \]

\[ \hat{n} \times \Delta \vec{H} = 0 \Rightarrow \hat{n} \times \left[ \frac{\vec{k} \times \vec{E}_0}{\mu} + \frac{\vec{k}'' \times \vec{E}_0''}{\mu} - \frac{\vec{k} \times \vec{E}_0'}{\mu''} \right] = 0 \]

We now define the plane of incidence as the plane containing the vectors \( \vec{k} \) and \( \hat{n} \) — in our diagram, it is the plane of the paper. We also need two polarization vectors \( \vec{\varepsilon}_1 \) and \( \vec{\varepsilon}_2 \):

\[ \vec{\varepsilon}_1 = \vec{\varepsilon}_1' = \vec{\varepsilon}_1'' \text{ (into page)} \]

\[ \vec{\varepsilon}_1 \times \vec{\varepsilon}_2 = \vec{k} \]

\[ \vec{\varepsilon}_1' \times \vec{\varepsilon}_2' = \vec{k}' \]
\( \hat{n} \cdot \vec{e}_1 = 0 \)  
\( \hat{n} \cdot \vec{e}_2 = \sin(\theta_i) = \hat{n} \cdot \vec{e}_2' \)  
\( \hat{n} \cdot \vec{e}_2'' = \sin(\theta_b) \)

\( \hat{n} \times \vec{e}_2 = \vec{e}_1 \cos(\theta_i) \)  
\( \hat{n} \times \vec{e}_2' = -\vec{e}_1 \cos(\theta_i) \)  
\( \hat{n} \times \vec{e}_2'' = -\vec{e}_1 \cos(\theta_i) \)

\( \vec{e}_1 \) is chosen to be perpendicular to the plane of incidence. Decompose the electric fields of the waves:

\[
\vec{E}_0 = E_{01} \vec{e}_1 + E_{02} \vec{e}_2
\]

where e.g., \( E_{01} = \vec{E}_0 \cdot \vec{e}_1 \)

\[
\vec{E}_0' = E_{01}' \vec{e}_1' + E_{02}' \vec{e}_2'
\]

\[
\vec{E}_0'' = E_{01}'' \vec{e}_1'' + E_{02}'' \vec{e}_2''
\]

Back to the boundary conditions at the interface:

\[ \hat{n} \cdot \vec{E}_0 = 0 \Rightarrow \varepsilon (E_{02} + E_{02}'') \sin(\theta_i) - \varepsilon' E_{02}' \sin(\theta_b) = 0 \]

\[ \hat{n} \times \vec{E}_0 = 0 \Rightarrow (E_{01} + E_{01}' - E_{01}') (\hat{e}_1 \times \hat{n}) \]

\[
- \left[ (E_{02} - E_{02}'') \cos(\theta_i) - E_{02}' \cos(\theta_b) \right] \hat{e}_1 = 0
\]

This is a vector equation and both perpendicular components are zero separately. That is

\[
E_{01} + E_{01}'' - E_{01}' = 0
\]

and

\[
(E_{02} - E_{02}'') \cos(\theta_i) - E_{02}' \cos(\theta_b) = 0
\]
\[ \hat{n} \cdot \Delta \mathbf{B} = 0 \Rightarrow (k E_0 + k'' E_0') \sin(\theta_i) = k' E_0' \sin(\theta_b) = 0 \]

Using \( k = \frac{c_0}{c} = k'' \) and \( k' = \frac{c_0'}{c} \)

With Snell's law \( n \sin(\theta_i) = n' \sin(\theta_b) \)

This equation becomes

\[ E_0 + E_0'' - E_0' = 0 \]

but we knew this already from the previous page, so nothing new here.

\[ \hat{n} \times \Delta \mathbf{H} = 0 \Rightarrow -\left[ \frac{1}{\mu} (k E_0 - k'' E_0') \cos(\theta_i) - \frac{1}{\mu} k' E_0' \cos(\theta_b) \right] \hat{\mathbf{e}}_i \]

\[ + \left[ \frac{1}{\mu} (k E_0 + k'' E_0') - \frac{1}{\mu'} k' E_0' \right] \hat{\mathbf{e}}_i \times \hat{n} = 0 \]

This is another vector equation so both terms = 0.

\[ \frac{n}{\mu} (E_0 - E_0'') \cos(\theta_i) - \frac{n'}{\mu'} E_0' \cos(\theta_b) = 0 \]

and

\[ \frac{n}{\mu} (E_0 - E_0'') - \frac{n'}{\mu'} E_0' = 0 \]

Now we have 5 equations (in the boxes) and only 4 unknowns \( \frac{E_0'}{E_0}, \frac{E_0''}{E_0}, \frac{E_0'}{E_0'}, \frac{E_0''}{E_0''} \) so the equations can not be linearly independent. (The incident amplitudes \( E_0 \) and \( E_0' \) are arbitrary, so ratios are the unknowns.)