Now examine two cases:

\( E_{0z} = 0 \) (incident wave linearly polarized perpendicular to the plane of incidence)

The equations on the previous page \( \Rightarrow E_{0z}' = 0 \) and \( E_{0z}'' = 0 \)
The reflected and refracted waves are also polarized perpendicular to the plane of incidence.
The equations involving the \( x \) components

\[
E_{0x} + E_{0z}'' - E_{0x}' = 0
\]

\[
\frac{n}{\mu} \cos(\theta_i) (E_{0x} - E_{0x}'') - \frac{n'}{\mu'} \cos(\theta_i) E_{0x}' = 0
\]

can be solved for the ratios: (Use Snell's law to eliminate \( \theta_a \))

\[
\frac{E_{0x}'}{E_{0x}} = \frac{2n \cos(\theta_i)}{n \cos(\theta_i) + \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}
\]

\[
\frac{E_{0x}''}{E_{0x}} = \frac{n \cos(\theta_i) - \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}{n \cos(\theta_i) + \frac{\mu}{\mu'} \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}
\]

Second case: \( E_{0x} = 0 \) (incident wave linearly polarized in the plane of incidence).

This time \( E_{0x}' = 0 \) and \( E_{0x}'' = 0 \). That is, the reflected and refracted waves are also polarized in the plane of incidence.
Two of the three equations involving the 2 components can be solved for the ratios (remember they are not all linearly independent).

\[
(E_0 - E_0') \cos(\theta_i) - E_0' \cos(\theta_b) = 0
\]

\[
\frac{n}{\mu} (E_0 + E_0'') - \frac{n'}{\mu'} E_0' = 0
\]

\[
\frac{E_0'}{E_0} = \frac{2nn' \cos(\theta_i)}{\mu' n' \cos(\theta_i) + n \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}
\]

\[
\frac{E_0''}{E_0} = \frac{\mu n \cos(\theta_i) - n \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}{\mu' n' \cos(\theta_i) + n \sqrt{n'^2 - n^2 \sin^2(\theta_i)}}
\]

Look at some special cases: Usually, we may set \( \mu = \mu' \). (Recall that for static fields in most materials \( \mu = \mu_0 \) except in ferromagnets.)

If we also have normal incidence \( \theta_i = 0 \), then

\[
\frac{E_0'}{E_0} = \frac{2n}{n+n'} \quad \frac{E_0''}{E_0} = \frac{n-n'}{n+n'}
\]

\[
\frac{E_0'}{E_0} = \frac{2n}{n+n'} \quad \frac{E_0''}{E_0} = \frac{n-n'}{n+n'}
\]
In both polarization directions for \( n' > n \), there is a phase reversal upon reflection. That is, \( \vec{E}_0 \) and \( \vec{E}_0'' \) are oppositely directed.

This is similar to what happens when a wave on a rope made of a light piece and a heavy piece travels from the light piece to the heavy piece.

Polarization by Reflection (\( n = n' \))

Q: Is it possible to have the reflected wave amplitude vanish?

For \( \vec{E}_i \) polarization, \( \vec{E}_0'' = 0 \) \( \Rightarrow \) \( n \cos(\theta_i) = \sqrt{n'^2 - n^2 \sin^2(\theta_i)} \)

or \( n^2 \cos^2(\theta_i) = n'^2 - n^2 \sin^2(\theta_i) \)

\( n^2 \left[ \cos^2(\theta_i) + \sin^2(\theta_i) \right] = n'^2 \Rightarrow n^2 = n'^2 \Rightarrow n = n' \)

Well, if \( n = n' \) there certainly is no reflected wave, but there is also no interface!
For \(\hat{z}\) polarization, \(E_{0z} = 0 \Rightarrow n_i^2 \cos(\theta_i) = n \sqrt{n_i^2 - n^2 \sin^2(\theta_i)}\)

\[
\begin{align*}
 n_i^4 \cos^2(\theta_i) &= n_i^2 \left[ n_i^2 - n^2 \sin^2(\theta_i) \right] \\
 n_i^4 \left[ 1 - \sin^2(\theta_i) \right] &= n_i^2 n_i^2 - n_i^4 \sin^2(\theta_i) \\
 n_i^2 (n_i^2 - n^2) &= (n_i^4 - n_i^4) \sin^2(\theta_i) = (n_i^2 + n_i^2)(n_i^2 - n_i^2) \sin^2(\theta_i) \\
 n_i^2 &= (n_i^2 + n_i^2) \sin^2(\theta_i)
\end{align*}
\]

\[
\Rightarrow \quad \sin(\theta_i) = \frac{n_i}{\sqrt{n_i^2 + n_i^2}} \quad \cos(\theta_i) = \frac{n}{\sqrt{n_i^2 + n_i^2}}
\]

\[
\tan(\theta_i) = \frac{\sin(\theta_i)}{\cos(\theta_i)} = \frac{n_i}{n} \quad \Rightarrow \quad \theta_i = \arctan \left( \frac{n_i}{n} \right)
\]

This is Brewster's angle.

Brewster's angle is the angle of incidence at which an incident wave polarized in the plane of incidence is totally transmitted — there is no reflection.

Thus an unpolarized beam of radiation will become polarized perpendicular to the plane of incidence upon reflection when \(\theta_i = \theta_{\text{Brewster}}\).

From Snell's Law: \(n \sin(\theta_i) = n_i' \sin(\theta_i')\), it follows that

\[
\sin(\theta_i) = \cos(\theta_i') \quad \text{Brewster} \quad \text{Brewster}
\]

\[
\Rightarrow \quad \theta_i + \theta_i' = \frac{\pi}{2} \quad \text{at Brewster's angle}
\]
Total Internal Reflection

Consider the case \( n > n' \). (The incident medium is more optically dense than the refracted medium.)

Snell's Law is \( \frac{n}{n'} \sin(\theta_i) = \sin(\theta_b) \)

There is a critical angle \( \theta_{i,\text{crit}} \) for which \( \theta_b = \frac{\pi}{2} \)

\[
\sin(\theta_{i,\text{crit}}) = \frac{n'}{n} < 1 \quad \Rightarrow \quad \theta_{i,\text{crit}} = \arcsin\left(\frac{n'}{n}\right)
\]

However, for \( \theta_i > \theta_{i,\text{crit}} \), we have \( \sin(\theta_b) > 1 \). In this case, the refracted angle becomes complex.

\[
\cos(\theta_b) = \sqrt{1 - \sin^2(\theta_b)} = \sqrt{1 - \frac{n'^2}{n'^2} \sin^2(\theta_i)}
\]

\[
= i \sqrt{\frac{n'^2}{n'^2} \sin^2(\theta_i) - 1}
\]

Thus \( \cos(\theta_b) \) is purely imaginary.

The plane wave factor \( e^{i \vec{k} \cdot \hat{n}} \) for the refracted wave can be written as

\[
e^{i \vec{k} \cdot \hat{n}} = e^{i \vec{k}' \cdot \hat{n}} e^{i \vec{k}_\parallel \cdot \hat{n}}
\]

where \( \vec{k}' \cdot \hat{n} = k' \cos(\theta_b) = i k' \sqrt{\frac{n'^2}{n'^2} \sin^2(\theta_i) - 1} \)

and \( \vec{k}_\parallel \equiv \vec{k}' \times \hat{n} \quad \hat{n}_\parallel \equiv \hat{n} \times \hat{n} \)
\( e^{i \mathbf{k}' \cdot \mathbf{n}} \) is still a phase factor, but
\[
e^{i \mathbf{k}' \cdot \mathbf{n} (\mathbf{n} \cdot \mathbf{n})} = -k'(\mathbf{n} \cdot \mathbf{n}) \sqrt{\frac{n^2}{n'^2} \sin^2(\theta_i) - 1}
\]
is a decaying exponential. Thus the electric and magnetic fields decay exponentially in the refraction medium.

Homework! The wave vector in the refraction medium for \( \theta_i > \theta_i^{\text{crit}} \) is
\[
\mathbf{k}' = k'_{\parallel} + i k' \mathbf{n} \quad \text{where} \quad k'_{\parallel} \cdot \mathbf{n} = 0
\]
and
\[
k' = k \sqrt{\sin^2(\theta_i) - \sin^2(\theta_i^{\text{crit}})}
\]
\[
k, \text{ not } k'
\]
\[
sin(\theta_i^{\text{crit}}) = \frac{n}{n'}
\]
\[
k = \frac{\omega}{v} \quad k' = \frac{\omega}{v'} \quad k' = \frac{v}{v'} k = \frac{n'}{n'} k
\]

Show that the Poynting vector in the refraction medium averaged over one period is
\[
\langle \mathbf{S} \rangle = \frac{1}{2 \omega} \left[ \mathbf{k}' (\mathbf{E}_0^* \cdot \mathbf{E}_0') + 2 \text{Re}(i \mathbf{E}_0^* \cdot \mathbf{n} \cdot \mathbf{E}_0') \right] e^{-2 \mathbf{n} \cdot \mathbf{R}}
\]
Caution! Since \( \mathbf{k}' \) is complex, you must treat it carefully.
It follows from this result that $\langle \hat{n} \cdot \vec{E} \rangle = 0$ since $\vec{E}_0 \cdot \vec{E}_0'$ is real and $\hat{n} \cdot \vec{k}_i = i \hat{z}$ is imaginary. Thus no energy is transmitted into the refracted medium (on the average). There is a component parallel to the interface, but it decays exponentially as you look deeper into the refracted medium with penetration depth $d = \frac{c}{2\omega} \sqrt{n^2 \sin^2(\theta_i) - n'^2}$.

Note that $d \to \infty$ as $\theta_i \to \theta_{i_c}$.

The reflected waves for $\vec{e}_1$ and $\vec{e}_2$ polarizations become, for $\theta_i > \theta_{i_c}$

$$\frac{E'_{01}}{E_{01}} = \frac{n \cos(\theta_i) - i \frac{\mu}{\mu'} \sqrt{n^2 \sin^2(\theta_i) - n'^2}}{n \cos(\theta_i) + i \frac{\mu}{\mu'} \sqrt{n^2 \sin^2(\theta_i) - n'^2}}$$

$$\frac{E'_{02}}{E_{02}} = \frac{\frac{\mu}{\mu'} n'^2 \cos(\theta_i) - i n \sqrt{n'^2 \sin^2(\theta_i) - n'^2}}{\frac{\mu}{\mu'} n'^2 \cos(\theta_i) + i n \sqrt{n'^2 \sin^2(\theta_i) - n'^2}}$$

So $E_{01}'' = e^{i\psi_1} E_{01}$ and $E_{02}'' = e^{i\psi_2} E_{02}$.