Consider an initially linearly polarized incident plane wave with polarization vector
\[ \mathbf{E} = \cos \varphi \, \hat{e}_1 + \sin \varphi \, \hat{e}_2 \]
\[ \mathbf{E}(r,t) = \mathbf{E}_0 \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r} - \omega t \right) \right] \]

So the incident amplitudes are
\[ E_{01} = E_0 \cos \varphi \quad \text{and} \quad E_{02} = E_0 \sin \varphi \]

The reflected wave is then
\[ \mathbf{E}''(r,t) = E_0 \left[ \cos \varphi \, e^{i \frac{\varphi}{2}} \hat{e}_1 + \sin \varphi \, e^{i \frac{\varphi}{2}} \hat{e}_2 \right] \exp \left[ i \left( \mathbf{k}'' \cdot \mathbf{r} - \omega t \right) \right] \]

which is an elliptically polarized wave.

In the analysis above we chose
\[ \cos (\Theta_b) = -i \sqrt{\frac{n''^2 \sin^2 (\Theta_b)}{n^2 \sin^2 (\Theta_b)} - 1} \]

That was correct for a refracted medium which occupies an infinite half-space — that is, we know physically that we should have an exponentially decaying wave as we look further into the medium away from the interface. This is called an evanescent wave.
But consider the problem where the medium with index of refraction \( n' \) is bounded on both sides by index \( n \):

\[
\begin{array}{c}
\text{n} \\
\text{n'} \\
\text{n}
\end{array}
\]

In this case, we will need both an exponentially decaying wave and an exponentially growing wave in the \( n' \) medium in order to match boundary conditions at both interfaces. There will be a propagating wave in both \( n \) media (with diminished amplitude in the upper one if the wave is incident from the lower one). This situation is analogous to quantum mechanical tunneling and is called frustrated total internal reflection.
Absorption and Dispersion

Model for the Dielectric Function

We give a classical model for the dielectric function $\varepsilon(\omega)$. We assume that all charged oscillators are harmonically bound in atoms, $Z$ per atom. We assume that the oscillators are damped to take into account the loss of energy when the atoms radiate. This is not a bad model since even classically the system will have discrete frequencies, just like a quantum system. (The damped oscillators do not change their frequency when the amplitude decreases.) However, the radiation in the model contains harmonics and this is not observed experimentally, so the model isn't too good. We assume that there are many modes of vibration characterized by frequencies $\omega_n$ and damping rates $\gamma_n$. The coordinates of an electron in the $n^{th}$ mode of the $a^{th}$ atom is $\vec{\gamma}_n$ relative to the center of mass of the $a^{th}$ atom which is $\vec{R}_a$. We then have

$$m \ddot{\gamma}_n^{(a)} + m \gamma_n \dot{\gamma}_n^{(a)} + m \omega_n^2 \gamma_n^{(a)} = -e \Re \left[ E_0 e^{i[k \cdot (\vec{R}_n^{(a)} + \vec{R}_n) - \omega t]} \right]$$

charge
for the dynamics of an electron in the presence of an electromagnetic plane wave (we neglect the magnetic force which is $O(\frac{v}{c})$ smaller than the electric force). This differential equation is nonlinear and hence insoluble because of the $\vec{k} \cdot \vec{R}_{an}$ factor in the exponential. Thus we will assume that we have sufficiently long wavelengths so that $\vec{k} \cdot \vec{R}_{an} \ll 1$ and we may neglect $\vec{k} \cdot \vec{R}_{an}$ in the exponential. We then introduce a complex position vector $\vec{Z}_{an}$ which obeys the equation of motion

$$\ddot{\vec{Z}}_{an}(t) + \gamma_n \dot{\vec{Z}}_{an}(t) + \omega_n^2 \vec{Z}_{an}(t) = -\frac{e}{m} \vec{E}_0 e^{i(k \cdot \vec{R}_a - \omega t)}$$

with $\vec{R}_{an} = \text{Re}[\vec{Z}_{an}]$. We look for forced solutions

$$\vec{Z}_{an}(t) = \vec{Z}_{an}^0 e^{i(k \cdot \vec{R}_a - \omega t)}$$

$$\vec{Z}_{an}^0 = \frac{-\frac{e}{m} \vec{E}_0}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$
The dipole moment of the $n^{th}$ atom is then

$$\mathbf{d}_n = \sum_n (-e) f_n \mathbf{R}_n = \Re\left[ \sum_n (-e) f_n \mathbf{Z}_n \right]$$

where $f_n$ is the number of electrons in the $n^{th}$ mode.

$$\mathbf{P} = \Re \left[ \frac{e^2}{m} \sum_n f_n \mathbf{E}_0 \frac{\mathbf{R}_n \cdot \mathbf{R}_n - \omega^2}{\omega_n^2 - \omega^2 - i\gamma_n \omega} \right]$$

Now consider a volume element of dimension $l$, with $kL \ll 1$. Then the factor $i\mathbf{k} \cdot \mathbf{R}_n$ is essentially the same for all atoms in the element. The polarization is then

$$\mathbf{P} = \mathbf{P}_0 e^{i(k \cdot \mathbf{R} - \omega t)}$$

with

$$\mathbf{P}_0 = \frac{N e^2}{m} \sum_n f_n \mathbf{E}_0 \frac{1}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$

$N$ = number of atoms per unit volume. 
$\mathbf{R}$ = location of the center of the volume element.

Now

$$\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0 + \mathbf{P}_0 \equiv \varepsilon(\omega) \mathbf{E}_0$$

with

$$\varepsilon(\omega) = \varepsilon_0 + \frac{N e^2}{m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$
For $\omega \ll (\omega_n)_{\text{min}}$ we have

$$\varepsilon(\omega) \approx \varepsilon(0) = \varepsilon_0 + \frac{N e^2}{m} \sum \frac{f_n}{\omega_n^2}$$

If $(\omega_n)_{\text{max}}$ is the highest occupied frequency then

for $\omega \gg (\omega_n)_{\text{max}}$ we have

$$\varepsilon(\omega) \approx \varepsilon_0 - \frac{N e^2}{m} \sum \frac{f_n}{\omega^2} = \varepsilon_0 - \frac{N e^2 Z}{m \omega^2}$$

where we used $\sum \frac{f_n}{\omega_n^2} = Z = \text{number of electrons per atom}$

The expression above for the dielectric function also arises in a quantum mechanical treatment provided we make the following interpretation:

(a) $\omega_n = \frac{E_n - E_0}{\hbar}$ where $E_n = \text{energy of the } n^{th} \text{ atomic energy level and } E_0 = \text{ground state energy. This assumes the atom is in its ground state (we neglect thermal excitation).}$ The sums in the formulae above then go over only the excited states.
(b) \( \gamma_n = \frac{1}{\tau_n} \) where \( \tau_n \) is the lifetime of the excited state of the atom. The excited state emits radiation spontaneously and does not live indefinitely.

(c) \( f_n \) = "oscillator strength" which involves matrix elements of the electric dipole moment operator. The result \( \sum f_n = \mathcal{Z} \) still holds and is called the "f-sum rule."

We recall that the connection between the wave number \( k \) and frequency \( \omega \) is

\[
k^2 = \frac{\omega^2}{c^2} = \omega^2 \varepsilon(\omega) \quad \text{where we take } \mu(\omega) = \mu_0.
\]

Since \( \varepsilon(\omega) \) is generally complex, then so is \( k^2 \) and so is \( k \). We write

\[
\varepsilon(\omega) = \varepsilon_R(\omega) + i \varepsilon_I(\omega) \quad \text{and} \quad \beta^2 = \frac{\omega^2}{c^2} = \varepsilon_R(\omega) \omega^2
\]

\[
k = \beta + \frac{i}{2} \alpha
\]

\[
\alpha \beta = \varepsilon_I(\omega) \omega^2
\]
\[ \varepsilon_R(\omega) = \varepsilon_0 + \frac{N\varepsilon^2}{m} \sum_{n} \frac{f_n(\omega_n^2 - \omega^2)}{(\omega^2 - \omega_n^2)^2 + \omega^2 \gamma_n^2} \]

\[ \varepsilon_I(\omega) = \frac{N\varepsilon^2}{m} \sum_{n} \frac{f_n \gamma_n \omega}{(\omega^2 - \omega_n^2)^2 + \omega^2 \gamma_n^2} \]

The imaginary part of \( \varepsilon(\omega) \) is sizeable only near a resonance \( \omega \approx \omega_n \). The peak value of \( \varepsilon_I(\omega) \) diminishes as the resonance frequency increases and the peaks get broader because the higher excitations are increasingly more unstable — smaller lifetime \( \tau_n \) therefore larger \( \gamma_n \).
Now suppose $\epsilon_t \ll \epsilon_r$. Then we can find $\alpha$ and $\beta$ approximately as

$$\beta^2 - \frac{\omega^2}{c^2} \approx \beta^2 = \epsilon_r \omega^2 \Rightarrow \beta = \sqrt{\epsilon_r \omega}$$

$$\alpha = \frac{1}{\beta} \epsilon_i \omega \Rightarrow \alpha = \frac{\epsilon_i \omega}{\sqrt{\epsilon_r \omega}} = \frac{\epsilon_i \omega}{\epsilon_r \sqrt{\omega}}$$

If the plane wave is propagating in the z-direction,

$$\mathbf{E}(z,t) = \text{Re} \left[ E_0 e^{i(kz-\omega t)} \right] = \text{Re} \left[ E_0 e^{i(\beta z-\omega t)} \right] e^{-\alpha z}$$

Thus as the wave propagates, the amplitude attenuates. Furthermore, the phase velocity is

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\epsilon_r \omega}} = \frac{c}{n(\omega)}$$

where $n(\omega) = c \sqrt{\epsilon_r \omega}$ is the index of refraction.

Usually $n(\omega)$ is an increasing function of $\omega$.
(Blue light refracts more than red light)

Rainbow: \[ \text{red} \rightarrow \text{blue} \]