It is instructive to look at the dielectric function $\varepsilon(\omega)$ as a mathematical object:

$$\varepsilon(\omega) = \varepsilon_0 + \frac{Ne^2}{\hbar m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2 - i\gamma_n \omega}$$

by considering $\omega$ as a complex variable. Then $\varepsilon(\omega)$ has a very simple analytic structure — it has poles at $\omega^2 - \omega_n^2 + i\gamma_n \omega = 0$

or $\omega = \frac{-i\gamma_n}{Z} \pm \sqrt{\omega_n^2 - \frac{\gamma_n^2}{4}}$

The poles are in the lower half of the complex $\omega$-plane. This behavior can be shown to follow from the principle of causality which, loosely speaking, means that a localized disturbance in space and time has effects in remote regions which are experienced with a time delay because of the finite velocity of light — that is, “cause” precedes “effect”.

A consequence is that we have the famous Kramers-Kronig relation

$$\varepsilon_\text{R}(\omega) = 1 + \frac{2}{\pi} \mathcal{P} \int_0^\infty \frac{\omega' \varepsilon_\text{I}(\omega') \, d\omega'}{\omega'^2 - \omega^2}$$

which connects the real and imaginary parts of $\varepsilon(\omega)$ for real $\omega$. 
Thus dispersive aspects $E_p(u)$ and absorptive aspects $E_I(u)$ are linked together. There is an analogous connection between the real and imaginary parts of the forward scattering amplitudes in particle physics that was a popular subject in the late 1950's and early 1960's. (c.f. Optical theorem.)

Connection between analyticity and causality

The dielectric function $\varepsilon(u)$ has poles in the lower half $u$-plane, so $\varepsilon(u)$ is an analytic function of $u$ in the upper half $u$-plane. We now demonstrate the connection between analyticity and causality.

Suppose we have a "cause" described by some function of time $c(t)$ and its "effect" described by $E(t)$. These are related by a response function $R(t)$ through

$$E(t) = \int_{-\infty}^{+\infty} R(t-t') c(t') \, dt'$$

Remember $X(t) = \int_{-\infty}^{+\infty} G(t, t') F(t') dt'$ from mechanics

\[\text{Response} \quad \text{Green function}\]
Introduce Fourier Transforms $X = E, R, C$.

\[ X(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} \hat{X}(\omega) d\omega \]

\[ \hat{X}(\omega) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega t} X(t) dt \]

Then we find $\hat{E}(\omega) = \hat{R}(\omega) \hat{C}(\omega)$

where we made use of the Dirac delta function

\[ \delta(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega t} dt \]

The statement of causality is that $R(t-t') = 0$ for $t < t'$. How can this property be ensured?

Suppose $\hat{R}(\omega)$ is analytic in the upper half plane and that $\hat{R}(\omega) \to 0$ sufficiently fast as $|\omega| \to \infty$.

Then for $t-t' < 0$ we may evaluate the integral

\[ R(t-t') = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-i\omega(t-t')} \hat{R}(\omega) d\omega = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega|t-t'|} \hat{R}(\omega) d\omega \]
by “closing the contour” in the upper half plane because on an infinite semicircle in the upper half plane

\[ \omega = \omega_R + i \omega_I \]

\[ e^{i \omega |t-t'|} - \omega_I |t-t'| \]

\[ e \rightarrow 0 \text{ as } \omega_I \rightarrow \infty. \]

Thus for \( t < t' \)

\[ R(t-t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i \omega |t-t'|} d\omega = \frac{1}{c} \int_{c}^{c} e^{i \omega |t-t'|} d\omega = 0 \]

because of Cauchy’s Theorem: If \( f(z) \) is analytic in region \( \Omega \), then \( \oint_{c} f(z) dz = 0 \) for any curve \( c \) in \( \Omega \). Thus analyticity in the upper half complex \( \omega \)-plane implies causality.
Wave packets in a dispersive medium

Consider a medium in which absorption is negligible. Suppose the electric field is linearly polarized along \( \hat{e}_1 \) and the wave propagates along the \( z \)-axis. Then

\[
\vec{E}(z,t) = \text{Re} \, \vec{E}(z,t)
\]

where \( \vec{E}(z,t) \) is now taken not as a plane wave but rather as a superposition of plane waves—what is a "wave packet"

\[
\vec{E}(z,t) = \vec{e}_1 \, E(z,t) = \vec{e}_1 \int_{-\infty}^{+\infty} \frac{dk_z}{\sqrt{2\pi}} \, \hat{E}(k_z) \, e^{i[k_z z - \omega(k_z) t]}
\]

The magnetic field will then be given by

\[
\vec{B}(z,t) = \vec{e}_2 \, B(z,t) = \vec{e}_2 \int_{-\infty}^{+\infty} \frac{dk_z}{\sqrt{2\pi}} \, \hat{B}(k_z) \, e^{i[k_z z - \omega(k_z) t]}
\]

where \( \vec{e}_1 \times \vec{e}_2 = \hat{z} \) and \( \hat{B}(k_z) = \frac{k_z c}{\omega(k_z)} \hat{E}(k_z) \)

The frequency of the wave must be an even function of \( k_z \):

\[
\omega(-k_z) = \omega(k_z)
\]

for an isotropic medium.
We next assume that $\tilde{E}(k_2)$ is sharply peaked about some value $k_0$ which we take to be positive. We then make a Taylor expansion of $\omega(k_2)$ about $k_0$:

$$\omega(k_2) = \omega_0 + \omega_0'(k_2 - k_0) + \frac{1}{2} \omega_0''(k_2 - k_0)^2 + \ldots$$

where $\omega_0 = \omega(k_0)$, $\omega_0' = \left[ \frac{d\omega(k_2)}{dk_2} \right]_{k_2 = k_0}$, $\omega_0'' = \left[ \frac{d^2\omega(k_2)}{dk_2^2} \right]_{k_2 = k_0}$

The quantity $\omega_0'$ is called the group velocity of the wave packet at wave number $k_0$.

For a dispersionless medium then

$$\omega = \frac{|k_2|}{n^2}$$

where $n = \frac{\text{constant index of refraction}}{}$

and $\omega' = \frac{\omega}{n} = V = \text{phase velocity of the wave}$.

In such a dispersionless medium we have

$$E(z,t) = \int_{-\infty}^{+\infty} \frac{dk_2}{\sqrt{4\pi}} \tilde{E}(k_2) e^{ik_2(z-vt)}$$

$$= \int_{0}^{+\infty} \frac{dk_2}{\sqrt{4\pi}} \tilde{E}(k_2) e^{ik_2(z-vt)} + \int_{-\infty}^{0} \frac{dk_2}{\sqrt{4\pi}} \tilde{E}(k_2) e^{ik_2(z+vt)}$$

$$= E_+(z-vt) + E_-(z+vt)$$

Right-moving $k_2 > 0 \rightarrow \quad k_2 < 0 \quad$ left-moving $\leftarrow$