It is tempting to say that all the quantities defined above — $A^\mu$, $J^a$, and $F^{a\nu}$ — are 4-tensors, but that is a matter of physics, not merely definition. We now look at the quantities in question and show that they really are tensors under Lorentz transformations.

(i) $J^\mu$ — 4 current density

Consider a small amount of charge at rest $d\mathbf{q}$. We assume a continuous charge distribution. Then the charge density is

$$J_\mu = \frac{d\mathbf{q}}{d\mathbf{V}_0}$$

(the $0$ subscript means “proper”)

where $d\mathbf{V}_0$ is the volume element in which $d\mathbf{q}$ sits.

If we now move the volume element with velocity $\mathbf{u}$, it will contract in the direction of motion with

$$d\mathbf{V} = \sqrt{1 - \frac{u^2}{c^2}} d\mathbf{V}_0$$

and charge density

$$J = \frac{d\mathbf{q}}{d\mathbf{V}} = \gamma J_0$$

Note that we have taken $d\mathbf{q}$ as Lorentz invariant.

Conservation of charge actually requires this. The current density is

$$\mathbf{J} = \gamma \mathbf{u} = \gamma J_0 \mathbf{u}$$
Thus if we define \( J^0 = c \rho \) then
\[
J^\mu = J^0 U^\mu \quad \text{where } U^\mu \text{ is the local velocity field of the charged "fluid".}
\]

If we have more than one species of charged fluid then
\[
J^\mu (x) = \sum_s \phi_s (x) U^\mu (x)
\]

Since \( \phi_s (x) \) is a scalar field—it is the charge density of the \( s \)th species in a frame which is instantaneously at rest with respect to the local charge of species \( s \)—and \( U^\mu (x) = \left[ \gamma (s) c, \gamma (s) \vec{u} (x) \right] \) is a 4-vector velocity field, then \( J^\mu (x) \) is also a 4-vector field.

(ii) \( F^{\mu \nu} \)

Quite aside from Maxwell's equations, the electric and magnetic fields are defined operationally through the electromagnetic force. The force law is
\[
\frac{d \vec{P}}{dt} = q \left( \vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) \quad \text{(c.g.s. units)}
\]

Since the magnetic force is perpendicular to the particle velocity, then only the electric force does work.
Thus \( \frac{dK}{dt} = q \, \vec{u} \cdot \vec{E} \) where \( K \) is the kinetic energy of the particle. Since the rest mass of the particle is a constant, we may write
\[
\frac{dP^o}{dt} = \frac{q}{c} \, \vec{u} \cdot \vec{E} \quad (**)
\]
The equations (*) and (**) can be combined into
\[
\frac{dP^\mu}{dt} = \frac{q}{c} \frac{dx^\mu}{dt} \, F^{\mu\nu}
\]
or multiplying both sides by \( \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \) we have
\[
\frac{dP^\mu}{dt} = \frac{q}{c} \, U_\nu \, F^{\mu\nu}
\]
Now the left-hand side is known to be a 4-vector and \( U_\nu \) is an arbitrary 4-vector. It therefore follows from the following theorem that \( F^{\mu\nu} \) is a second rank 4-tensor.

**Homework:** Prove that if \( A_\mu \) is an arbitrary covariant 4-vector and \( B^\mu \) is a contravariant 4-vector and \( C^{\mu\nu} A_\nu = B^\mu \), then \( C^{\mu\nu} \) is a contravariant 4-tensor.
Thus Maxwell's equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^{\nu}$$

$$\gamma^\nu F^\alpha_\beta (x) + \gamma^\beta F^\alpha_\mu (x) + \gamma^\mu F^{\alpha\beta} (x) = 0$$

hold in all inertial frames. Remember $\gamma_\alpha$ and $\gamma^\mu$ just convert tensors into higher rank tensors and index contraction reduces the tensor rank by 2.

Field invariants

From the space time 4-vector, we constructed the invariant interval between events

$$\Delta S^2 = \Delta x^{\mu} \Delta x_{\mu}$$

$\Delta S^2$ has the same value in all frames. There are also invariants for the electromagnetic field. We must construct scalars from $F^{\mu\nu}$.

(i) Linear invariants

$F^{\mu\nu}$ is a scalar field, but

$$F^{\mu\nu} = F_0 + \sum_{k=1}^{3} F_k = F^0 - \sum_k F_{kk}$$

however $F^{\mu\mu}$ (not summed) = 0 since $F^{\mu\nu}$ is antisymmetric. Therefore $F^{\mu\mu} = 0$.
(ii) Quadratic invariants

\[ F^{\mu\nu} F_{\nu\lambda} \] is a scalar field. We have

\[ F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} \] with

\[
\begin{pmatrix}
  E^1 \\
  -E^1 & E^2 \\
  -E^2 & E^3 \\
  -E^3 & -B^2 & B^1 \\
  B^2 & 0 & -B^1 \\
  -B^3 & B^1 & 0
\end{pmatrix}
\]
as a matrix.

From this matrix and the one given previously for \([F_{\mu\nu}]\) we calculate

\[
-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} F^{\mu\nu} F_{\nu\lambda} = \frac{1}{2} \operatorname{Tr} \left( [F_{\mu\nu}] \cdot [F_{\mu\nu}] \right) = E^2 - B^2 + \vec{B} \cdot \vec{B}^* = E^2 - B^2
\]

Thus \(E^2 - B^2\) is a Lorentz invariant. If \(E > B\) in one inertial frame, then \(E > B\) in all inertial frames.

Now consider the totally antisymmetric quantity

\[
\epsilon_{\mu\nu\alpha\beta} = \begin{cases} 
+1 & \text{if } \mu\nu\alpha\beta \text{ even permutation of } 0123 \\
-1 & \text{if } \mu\nu\alpha\beta \text{ odd permutation of } 0123 \\
0 & \text{if any of } \mu\nu\alpha\beta \text{ are the same}
\end{cases}
\]
We require this symbol to have the same definition in all inertial frames. Thus \( \varepsilon_{\mu\nu\rho} \) must transform according to the rule

\[
\varepsilon_{\mu\nu\rho} = \det[L] \mu_\alpha \nu_\omega \rho_\beta \varepsilon_{\alpha\omega\beta}
\]

where \([L]\) is the matrix of coefficients \( \lambda_\alpha \).

But the permutation symbol can be used to define the determinant of a matrix through the relation

\[
\lambda_\alpha \lambda_\nu \lambda_\omega \lambda_\beta \varepsilon_{\alpha\nu\omega\beta} = \det[L] \varepsilon_{\mu\nu\rho}
\]

Thus we have

\[
\varepsilon_{\mu\nu\rho} = (\det[L])^2 \varepsilon_{\mu\nu\rho}
\]

But \( \lambda_\alpha \lambda_\mu = \delta_\alpha^\mu \)

\[
(g_\mu^\alpha \delta_\alpha^\rho) \lambda_\nu \lambda_\beta = \delta_\nu^\beta
\]

\[
9_{ab} (L^T)^b g^{a\mu} \lambda_\mu \lambda_\beta = \delta_\beta^\beta
\]

adjacent indices summed over
\( \Rightarrow \) matrix multiplication
where \[ [g_{xg}] = [g_{ou}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

and \[ L = \begin{pmatrix} l_0 & l_0 & l_0 & l_0^3 \\ l_0 & l_1 & l_1^2 & l_1^3 \\ l_2 & l_2 & l_2^2 & l_2^3 \\ l_3 & l_3 & l_3^2 & l_3^3 \end{pmatrix} \]

\[ G \overset{L^T}{\rightarrow} G \overset{L}{\rightarrow} \overset{II}{II} \]

\[ \frac{\left[ \text{det}(G) \right]^2}{1} \cdot \text{det}(L^T) \cdot \text{det}(L) = 1 \]

\[ = \text{det}(L) \]

\[ \Rightarrow \left[ \text{det}(L) \right]^2 = 1 \Rightarrow \text{det}(L) = \pm 1 \]

The equation was

\[ \bar{\epsilon}_{\mu
\nu
\alpha
\beta} = \left[ \text{det}(L) \right]^2 \epsilon_{\mu
\nu
\alpha
\beta} \]

so \[ \bar{\epsilon}_{\mu
\nu
\alpha
\beta} = \epsilon_{\mu
\nu
\alpha
\beta} \] frame independent as required.
A quantity which transforms according to

\[ T_{\bar{a}_1 \cdots \bar{a}_m}^{\bar{b}_1 \cdots \bar{b}_n} = \text{det}(\xi) \xi_{\bar{a}_1} \cdots \xi_{\bar{a}_m} \xi^{\bar{b}_1} \cdots \xi^{\bar{b}_n} T_{\bar{a}_1 \cdots \bar{a}_m}^{\bar{b}_1 \cdots \bar{b}_n} \]

is called a pseudo tensor of rank \( m + n \). It differs from a tensor by the factor \( \text{det}(\xi) \). Note if \( \text{det}(\xi) = +1 \) then there is no difference. \( \text{det}(\xi) = -1 \) arises from space inversion or time reversal (but both space inversion and time reversal together give \( \text{det}(\xi) = +1 \). Sometimes the word "orthochronous" is used to indicate no time reversal).

The quantity \( F^{\mu \nu} \) is a tensor. The quantity

\[ g_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \beta} F^{\rho \beta} \]

is a second rank pseudo tensor which is said to be "dual" to \( F^{\mu \nu} \).

Aside: In three space dimensions under \( SO(3) \) rotations

\( \varepsilon_{ijk} \) (Levi-Civita symbol) is a 3rd rank pseudo tensor and \( A_i = \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_{jk} \) is a pseudovector which is dual to the antisymmetric second rank tensor \( A_{jk} \)

\[ A = \begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix} \]
\[
\begin{bmatrix}
T_{\mu\nu}
\end{bmatrix} =
\begin{pmatrix}
0 & -B^1 & -B^2 & -B^3 \\
B^1 & 0 & -E^3 & E^2 \\
B^2 & E^3 & 0 & -E^1 \\
B^3 & -E^2 & E^1 & 0
\end{pmatrix}
\]

and we can construct the contravariant form

\[
y^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}, \quad \begin{bmatrix}
T^{\mu\nu}
\end{bmatrix} =
\begin{pmatrix}
0 & B^1 & B^2 & B^3 \\
-B^1 & 0 & -E^3 & E^2 \\
-B^2 & E^3 & 0 & -E^1 \\
-B^3 & -E^2 & E^1 & 0
\end{pmatrix}
\]

Next we form the invariant with regard to proper (no space inversion or time reversal) Lorentz transformation

\[
\frac{1}{4} \sum_{\mu\nu} F^{\mu\nu} = \vec{E} \cdot \vec{B}
\]

Thus \( \vec{E} \cdot \vec{B} \) is invariant under proper Lorentz transformations. If \( \vec{E} \cdot \vec{B} = 0 \) in one frame, then \( \vec{E} \cdot \vec{B} = 0 \) in all inertial frames. In particular, suppose \( \vec{B} = 0 \) in some frame and \( \vec{E} \neq 0 \). Then in all other frames \( \vec{E}' \cdot \vec{B}' = 0 \), that is \( \vec{E}' \) and \( \vec{B}' \) are perpendicular. And we can't find a frame in which \( \vec{E}' = 0 \) since \( E^2 - B^2 > 0 \) in the original frame and \( E^2 - B^2 \) is a Lorentz invariant, thus \( E'^2 > B'^2 \) in the new frame.
Under improper Lorentz transformations (spatial inversion or time reversal) then $\vec{E} \cdot \vec{B}$ changes sign.

Under inversion, the vector $\vec{E}$ does not change sign (the components $E^i$ do, but so do the basis vectors). However, from $\vec{D} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{d\vec{E}}{dt}$ we see that $\vec{B}$ is a pseudo-vector ("axial" vector)—that is, the components $B^k$ do not change sign under inversion, but the basis vectors do, so $\vec{B}$ changes sign under inversion. Similarly under time reversal, $\vec{J}$ changes sign and hence $\vec{B}$ again changes sign.

Since $F^{\mu\nu}$ and $\gamma^{\mu\nu}$ are determined in terms of $\vec{E}$ and $\vec{B}$, then all Lorentz scalars (or pseudoscalars) must be made out of $\vec{E} \cdot \vec{B}$ or $E^2 - B^2$.

Note that $E^2 - B^2$ is an energy density and is the 00 component of a second rank tensor—the energy–momentum stress tensor $T^{\mu\nu}$, thus

\[ T^{\mu\nu} \gamma_{\mu\nu} \sim E^2 - B^2 \]

\[ F^{\mu\nu} F_{\mu\nu} = 0 \quad \text{etc.} \]

There are no other invariants besides $\vec{E} \times \vec{B}$ and $E^2 - B^2$ (well... and 0).
Behavior of Fields under Lorentz Transformation

Because our experience and intuition about the electromagnetic field is with \( \vec{E} \) and \( \vec{B} \), and not \( F^{\mu\nu} \), we give the transformation law for the fields directly. It is simplest to use a matrix notation. For the special Lorentz transformation (parallel axes with origin of \( \vec{K} \) moving with velocity \( \vec{v} \) along the 1-axis of frame \( \vec{K} \)), we have

\[
L = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\vec{x} = L \vec{x}
\]

Let \( F \) be the matrix \( [F^{\mu\nu}] \) with contravariant components

\[
\bar{F} = L F L^T \iff \bar{F}^{\mu\nu} = \bar{\ell}^\alpha \bar{\ell}^\beta F_{\alpha\beta} = \bar{\ell}^\alpha \bar{\ell}^\beta (L^T)_{\alpha\beta}
\]

\[
\begin{pmatrix}
0 & -E_1^3 & -E_2^3 & E_3^3 \\
E_1^0 & 0 & -\vec{B}^2 & \vec{B}^1 \\
E_2^0 & \vec{B}^3 & 0 & -\vec{B}^1 \\
E_3^0 & -\vec{B}^2 & \vec{B}^1 & 0
\end{pmatrix} = \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & -E_1^3 & -E_2^3 & E_3^3 \\
E_1^0 & 0 & -\vec{B}^2 & \vec{B}^1 \\
E_2^0 & \vec{B}^3 & 0 & -\vec{B}^1 \\
E_3^0 & -\vec{B}^2 & \vec{B}^1 & 0
\end{pmatrix} \begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
\[
\mathbf{F} = \\
\begin{pmatrix}
0 & -E' & -\gamma[E^2 - \beta B^2] & -\gamma[E^2 + \beta B^2] \\
E' & 0 & -\gamma[B^2 - \beta E^2] & \gamma[B^2 + \beta E^2] \\
\gamma[E^2 - \beta B^2] & \gamma[B^2 - \beta E^2] & 0 & -B' \\
\gamma[E^2 + \beta B^2] & -\gamma[B^2 + \beta E^2] & B' & 0 \\
\end{pmatrix}
\]

Thus we have along the direction of motion:
\[E' = E\]
and \[B' = B\]

and perpendicular to the direction of motion:
\[E^2 = \gamma[E^2 - \beta B^2]\]
\[B^2 = \gamma[B^2 + \beta E^2]\]
\[E^3 = \gamma[E^3 + \beta B^2]\]
\[B^3 = \gamma[B^3 - \beta E^2]\]

If you want the inverse transformation, switch barved and unbarved fields and change the sign of \[\beta = \frac{v}{c}\].

If the boost direction is not aligned along the 1-axis:
\[E_\parallel = E_\parallel\]
and \[B_\parallel = B_\parallel\]
\[E_\perp = \gamma(E_\perp + \beta \times B)\]
\[B_\perp = \gamma(B_\perp - \beta \times E)\]

where
\[E_\parallel = \frac{(\hat{E}, \hat{v}) \hat{v}}{v^2}\]
and
\[E_\perp = \hat{E} - E_\parallel = \frac{\hat{v} \times (\hat{E} \times \hat{v})}{v^2}\]
\[ \vec{E} = (1-\gamma) \frac{\vec{v}(\vec{v} \cdot \vec{E})}{v^2} + \gamma (\vec{E} + \frac{\vec{v}}{c} \times \vec{B}) \]

\[ \vec{B} = (1-\gamma) \frac{\vec{v}(\vec{v} \cdot \vec{B})}{v^2} + \gamma (\vec{B} - \frac{\vec{v}}{c} \times \vec{E}) \]

\[ \vec{E} = (1-\gamma) \frac{\vec{v}(\vec{v} \cdot \vec{E})}{v^2} + \gamma (\vec{E} - \frac{\vec{v}}{c} \times \vec{B}) \]

\[ \vec{B} = (1-\gamma) \frac{\vec{v}(\vec{v} \cdot \vec{B})}{v^2} + \gamma (\vec{B} + \frac{\vec{v}}{c} \times \vec{E}) \]

As an example, consider the case of a point charge at rest in frame K. Then for \( v \ll c \), \( \gamma \approx 1 \) and we have

\[ \vec{E} \approx \vec{E} = \frac{kq \vec{r}}{r^3} = kq \frac{\vec{r} - \vec{v}t}{|\vec{r} - \vec{v}t|^3} \quad (k=1 \text{ in c.g.s.}) \]

\[ \vec{B} \approx \frac{\vec{v}}{c} \times \vec{E} = \frac{q}{c} \frac{\vec{v}}{c} \times \frac{(\vec{r} - \vec{v}t)}{|\vec{r} - \vec{v}t|^3} \quad (\text{Biot-Savart Law}) \]

For a general transformation between inertial frames with low relative velocity we have

\[ \vec{E} = \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \quad \quad \vec{E} = \vec{E} - \frac{\vec{v}}{c} \times \vec{B} \]

\[ \vec{B} = \vec{B} - \frac{\vec{v}}{c} \times \vec{E} \quad \quad \vec{B} = \vec{B} + \frac{\vec{v}}{c} \times \vec{E} \]
The Galilean invariance of Newton's Second Law gives the transformation above for $\vec{E}$ but not for $\vec{B}$. The reason that Galilean invariance cannot give the correct low-velocity transformation law for $\vec{B}$ is that it would require an $O(\frac{1}{c^2})$ term in Newton's law and that cannot arise in Newtonian Mechanics. [$O(\frac{1}{c^2})$ terms are always relativistic corrections whereas $O(\frac{1}{c})$ are not.]

As an application of the field transformation law for arbitrary velocity between inertial frames we again consider a point charge at rest in $\vec{R}$. Then

\[ \vec{E} = \frac{Q \vec{r}}{r^3}, \quad \vec{B} = 0 \]

In frame $K$, the particle moves with velocity $\vec{v}$.

Suppose motion is in the $1$-direction: $\vec{v} = v$

\[
\begin{align*}
\vec{X}' &= \gamma (X' - v t) & X' &= \gamma (X' + v t) \\
\vec{X}^2 &= x^2 & x^2 &= \bar{x}^2 \\
\vec{X}^3 &= x^3 & x^3 &= \bar{x}^3 \\
\vec{t} &= \gamma (t - \frac{v}{c^2} X') & t &= \gamma (\bar{t} + \frac{v}{c^2} \bar{x}')
\end{align*}
\]

Suppose you sit in $K$ at $X^1 = 0$, $X^2 = b$, $X^3 = 0$. 
In R, the observer has coordinates

\[ \bar{x}' = -v \bar{t}, \quad \bar{x}^2 = b, \quad \bar{x}^3 = 0, \quad \bar{t} = \gamma t. \]

hence \[ \bar{x}' = -v \gamma t \] so that

\[ \bar{E}' = \frac{\gamma \bar{x}'}{\bar{t}^3}, \quad \bar{E}^2 = \frac{\gamma b}{\bar{t}^3}, \quad \bar{E}^3 = 0, \quad \bar{B} = 0. \]

or

\[ \bar{E}' = -\frac{\gamma \gamma v t}{\bar{t}^3} \quad \text{and} \quad \bar{t} = \sqrt{(\bar{x}')^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2} \]

\[ \bar{t} = \sqrt{\gamma^2 v^2 t^2 + b^2} \]

hence

\[ \bar{E}' = -\frac{\gamma \gamma v t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad \bar{E}^2 = \frac{\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad \bar{E}^3 = 0, \quad \bar{B} = 0 \]

The electric field can then be transformed to the K frame:

\[ E' = E = \frac{\gamma \gamma v t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad E^2 = \gamma E^2 = \frac{\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}}, \quad E^3 = 0. \]

For the magnetic field, we have \[ \bar{B} = 0 \] so

\[ \bar{B} = (1 - \gamma) \frac{v \times (\bar{v} \cdot \bar{E})}{v^2} + \gamma \bar{E} \quad \text{and} \quad \bar{B} = \gamma \frac{\vec{v}}{c} \times \bar{E} \]

\[ \Rightarrow \bar{B} = \frac{\vec{v}}{c} \times \bar{E} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ \frac{\bar{v}}{c} & 0 & 0 \\ E^1 & E^2 & E^3 \end{vmatrix} \Rightarrow B^1 = 0, \quad B^2 = 0, \quad B^3 = \frac{\gamma b \gamma v}{(\gamma^2 v^2 t^2 + b^2)^{3/2}} \]
We plot the field components $E^1$ and $E^2$ as a function of time. It is straightforward to calculate that $|E^1|$ reaches a maximum value $\frac{2 \gamma}{\sqrt{2} \gamma^2 b^2}$ at $t = \frac{b}{\sqrt{2} \gamma}$ while $E^2$ reaches a maximum value of $\frac{\gamma b}{b^2}$ at $t = 0$.

In both cases, the width of the peaks is $\Delta t = \frac{b}{\gamma \nu}$. Because of its inertia, a charged particle detector at $x' = 0$, $x^2 = b$, $x^3 = 0$ may well not detect $E^1$ for $\gamma > 1$ ($\nu \approx c$), but it will certainly see $E^2$.

Next we introduce a vector $\vec{r}_s$ with coordinates $x'_s = -\nu t$, $x^2_s = b$, $x^3_s = 0$. 
This vector drawn from the location of the charge in K at time \( t \) to the observation point \((O, b, 0)\).

\[
x_s^2 = r_s \sin \alpha = b
\]

\[
x_s' = r_s \cos \alpha = vt
\]

From the previous expressions we have

\[
E' = \frac{\varphi Y'}{Y' (\gamma_s^2 x_s'^2 + (x_s')^2)^{\gamma_2}}
\]

\[
E^2 = \frac{\varphi Y x_s^2}{[\gamma_r^2 (x_s^2) + (x_s^2)]^{\gamma_2}}
\]

\[
\Rightarrow \frac{E'}{E^2} = \frac{x_s'}{x_s^2} \Rightarrow \vec{E} \text{ is radial in K!}
\]

\[
[\gamma_r^2 (x_s')^2 + (x_s')^2] = r_s^2 \left[ \gamma_s^2 \cos^2 \alpha + \sin^2 \alpha \right] = r_s^2 \left[ \gamma_s^2 (1 - \sin^2 \alpha) + \sin^2 \alpha \right]
\]

\[
= r_s^2 \gamma_s^2 \left[ 1 + \left( \frac{1}{\gamma_s^2} - 1 \right) \sin^2 \alpha \right] = r_s^2 \gamma_s^2 \left[ 1 - \frac{v^2}{c^2} \sin^2 \phi \right]
\]

\[
\vec{E} = \frac{\varphi r_s^2}{r_s^3 \gamma_s^2 (1 - \frac{v^2}{c^2} \sin^2 \phi)^{3/2}}
\]

Thus the electric field lines in K are radial but anisotropic.

Field line spacing opens up because of motion.

Field spacing at \( v = 0 \).
**Motion in a Uniform Magnetic Field**

In the absence of an electric field we have

\[
\frac{d \vec{p}}{dt} = q \frac{\vec{u}}{c} \times \vec{B}, \quad \vec{E} = 0
\]

from which it follows that \(\vec{u} \cdot \frac{d \vec{p}}{dt} = 0\), but \(\vec{p}\) is of the form \(\vec{p} = f(u) \vec{u}\) where \(f(u) = \gamma = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}}

Thus \(\frac{d \vec{p}}{dt} = f(u) \frac{d \vec{u}}{dt} + f'(u) \frac{du}{dt} \vec{u}\)

and so

\[
\vec{u} \cdot \frac{d \vec{p}}{dt} = \vec{u} \cdot \frac{d \vec{u}}{dt} f(u) + f'(u) \frac{du}{dt} u^2 = 0
\]

Now use \(\frac{1}{2} d(u^2) = \frac{1}{2} d(\vec{u} \cdot \vec{u}) = \vec{u} \cdot d\vec{u}\)

\[
0 = \frac{1}{2} \frac{d(u^2)}{dt} f(u) + f'(u) u \frac{1}{2} \frac{d(u^2)}{dt}
\]

\[
\Rightarrow 0 = \frac{1}{2} \frac{d(u^2)}{dt} [f(u) + uf'(u)]
\]

but \(f(u) + uf'(u) = \gamma + \gamma^2 \frac{u^2}{c^2} \neq 0 \Rightarrow \frac{d(u^2)}{dt} = 0\)

Hence the particle moves with constant speed just as in the non-relativistic case. We may then write

\[
\frac{d}{dt} \vec{p} = \frac{d}{dt} (m \gamma \vec{u}) = m \gamma \frac{d\vec{u}}{dt} = q \frac{\vec{u}}{c} \times \vec{B}
\]
\[ \frac{d\vec{u}}{dt} = \frac{q}{m^* c} \vec{u} \times \vec{B} \quad \text{where} \quad m^* = \frac{m}{\sqrt{1 - \frac{u^2}{c^2}}} = m \gamma \]

is sometimes referred to as the "relativistic mass".

For a uniform magnetic field, the particle moves on a circle (helix more generally) with angular velocity

\[ \vec{\omega} = -\frac{q \vec{B}}{m^* c} \quad \text{(cyclotron frequency)} \]

with radius \[ R = \frac{u}{|\vec{u}|} \quad \text{for velocity} \quad \vec{u} \perp \vec{B} \quad \text{--- i.e., no motion along} \ \vec{B}. \]

Thus

\[ R = \frac{m^* u c}{|\vec{u}| B} = \frac{pc}{|\vec{u}| B} = \frac{\sqrt{E^2 - m^2 c^4}}{|\vec{u}| B} \]

where \( E \) is the energy and \( m \) is the rest mass.

**Homework**: Determine the motion of a charged particle in constant uniform electric and magnetic fields which are perpendicular to one another. Choose a coordinate system with \( \vec{E} = E \hat{x} \) and \( \vec{B} = B \hat{y} \). Take initial conditions

\[ x(0) = 0, \quad y(0) = 0, \quad z(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 0, \quad \dot{z}(0) = v. \]