## Homework Solutions 004

*Steve Sekula*, 22 February 2010 (created 17 February 2010)

### Point Distributions

Points were distributed as follows for each problem:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Total</th>
<th>Point Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CH4-7</strong></td>
<td>5</td>
<td>Used the Uncertainty Principle to answer the question (3 Points) and arrived at a correct answer (2 Points)</td>
</tr>
</tbody>
</table>
| **SS-8** | 10    | Part 1: Correctly identified the real (1 Points) and imaginary (1 Point) parts, computed the magnitude (1 Point), and found another way to combine $z$ and $z^*$ to make a real number (2 Points)  
Part 2: Decomposed the function into its real and imaginary parts (3 Points) and computed the sum to demonstrate its real-only constitution (2 Points) |
<p>| <strong>CH4-36</strong> | 15    | Understood what &quot;well-defined&quot; meant (5 Points), applied that and calculated the wave number and angular frequency, (5 Points), and recognized that the plane wave was the right way to encapsulate this information (5 Points) |
| <strong>CH4-37</strong> | 15    | Used the relationship between wave number and momentum to solve for momentum (5 Points), used the relationship between total and kinetic energy and angular frequency for a non-relativistic particle to solve for kinetic energy (5 Points), and used the relationship between energy and momentum in the non-relativistic realm to solve for mass (5 Points) |
| <strong>CH4-43</strong> | 15    | Applied the uncertainty principle (5 Points), recognized that a maximum uncertainty in one implies a minimum uncertainty in the other (5 Points) |</p>
<table>
<thead>
<tr>
<th></th>
<th>Points), and solved for the minimum speed from the minimum momentum (5 Points)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CH4-44</strong></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>Understood what was being asked when we are told to find the spatial extent (maximum size) or a region in which borderline relativistic speeds (minimum speeds = minimum momentum) can be achieved (5 Points), applied the uncertainty principle to the problem (5 Points), reasoned an answer to the first question (5 Points) and finally to the second question (5 Points)</td>
</tr>
<tr>
<td><strong>CH4-48</strong></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>Applied the uncertainty principle for energy and time (10 Points), recognized that the width means the maximum uncertainty on mass-energy (5 Points), and computed the minimum uncertainty on the lifetime of the meson (5 Points)</td>
</tr>
</tbody>
</table>

Deductions outside of the above:

- 1 point is deducted for failing to box the numerical answer
- 1 point is deducted for incorrectly applying significant figures
- other points are deducted as outlined in the homework policy

HARRIS, **CH4-7** (5 Points)

**SOLUTION**

According to the Uncertainty Principle, $\Delta x \Delta p_x \geq \frac{\hbar}{2}$. That means that if the uncertainty in one quantity, say $x$, is zero ($\Delta x = 0$) then in order for the principle to hold the other uncertainty must be infinite. This is the only way that the product can remain finite, if undefined. Therefore, if the uncertainty in position is zero, the uncertainty in momentum must be infinite. Likewise, if the uncertainty in momentum is zero, the uncertainty in position must be infinite.

**SS-8** (10 Points)
The following exercises are meant to get you thinking about complex numbers and get some exercise using them.

1. Given the complex number \( z = 5 + 10i \), identify the real and imaginary parts and calculate the magnitude, \(|z| = \sqrt{z^*z}\), of the number. The magnitude is a real number. Can you construct another real number from \( z \) and \( z^* \)? If so, what is it?

**SOLUTION**
The real part of \( z \) is 5, and the imaginary part of \( z \) is 10. The magnitude of \( z \) is

\[
|z| = \sqrt{z^*z} = \sqrt{(5 - 10i)(5 + 10i)} = \sqrt{25 + 100} = 11.8.
\]

Another real number can be obtained simply by adding \( z \) and \( z^* \):

\[
z + z^* = 10.
\]

2. Consider the apparently complex function \( e^{i\theta} + e^{-i\theta} \), where \( \theta \) is a real-valued variable. Show that this function is, in fact, only real-valued.

**SOLUTION**
Begin by writing \( e^{i\theta} \) in terms of its real and imaginary parts. This results in

\[
e^{i\theta} = \cos(\theta) + i\sin(\theta).
\]

Since \( e^{-i\theta} \) is the complex conjugate of \( e^{i\theta} \), we may already conclude from the first part of this problem that the sum must be real; however, let's see this. If this is indeed the complex conjugate, then

\[
e^{-i\theta} = \cos(\theta) - i\sin(\theta).
\]

We then see that the sum is

\[
e^{i\theta} + e^{-i\theta} = 2\cos(\theta)
\]

HARRIS, *CH4-36* (15 Points)

**SOLUTION**
First of all, let us consider what "well-defined" means. From the discussion
of the uncertainty principle, we see that this indicates zero uncertainty in the momentum of the particle. That means that we know exactly the value of the wave number, \( k \), and also the angular frequency, \( \omega \). This particle is non-relativistic (\( p = m\nu \rightarrow \nu = 0.002c \)), so \( k = p/\hbar \) and \( E = p^2/2m = \hbar\omega \). We should therefore compute \( k \) and \( \omega \) right away:

\[
k = 2\pi/\lambda = 2\pi p/\hbar = 4.74 \times 10^9 \text{m}^{-1}
\]

and

\[
\omega = E/\hbar = 2\pi E/\hbar = 2\pi p^2/(2m\hbar) = 1.3 \times 10^{15}\text{Hz}.
\]

There are no forces acting on the electron, as far as we know, so we can assume that the wave function describing its motion is just a plane wave:

\[
\Psi(x, t) = Ae^{i(kx-\omega t)}.
\]

Plugging in our wave number and angular frequency:

\[
\Psi(x, t) = Ae^{i((5 \times 10^9)x-(1 \times 10^{15})t)}
\]

HARRIS, CH4-37 (15 Points)

**SOLUTION**

This problem is the reverse of CH4-36. Let's begin with the plane wave:

\[
\Psi(x, t) = Ae^{i((1.58 \times 10^{12})x-(7.91 \times 10^{16})t)}.
\]

What is the particle's momentum? This is most easily arrived at from the wave number:

\[
k = p/\hbar = 1.58 \times 10^{12} \text{m}^{-1}
\]

from which we can determine

\[
p = 1.67 \times 10^{-22} \text{kg} \cdot \text{m/s}.
\]

What is the particle's kinetic energy? We're told in the problem that this particle is non-relativistic, so we can simply use

\[
E = p^2/2m = \hbar\omega = \hbar(7.91 \times 10^{16}\text{Hz}) = 8.35 \times 10^{-18}\text{J}.
\]

Finally, what is the mass of the particle? From the previous formula, we can
solve for $m$:

$$m = p^2/(2E) = (1.67 \times 10^{-22}\text{kg} \cdot \text{m/s})^2/(2 \cdot (8.35 \times 10^{-16}\text{J})) = 1.67 \times 10^{-27}\text{kg.}$$

HARRIS, *CH4-43* (15 Points)

**SOLUTION**

Begin with the Uncertainty Principle: $\Delta x \Delta p_x \geq \hbar/2$. If we know the position "to within" a number, that means that the maximum uncertainty on the position is that number - in our case, $\sim 5 \times 10^{-15}\text{m}$. To answer the question about what speeds we might find it moving, we need to apply this business about "maximum uncertainty on position" and the Uncertainty Principle. If the maximum uncertainty on position is given, then the Uncertainty Principle tells us the minimum uncertainty on the momentum - that is, that the momentum could be as little as a certain number, or much more. Thus the minimum uncertainty on momentum is

$$\Delta p_x \geq \hbar/(2\Delta x) = 1.06 \times 10^{-20}\text{kg} \cdot \text{m/s}.$$ 

Therefore, if $\Delta p_x$ must be larger than or equal to this number, we can figure out what at speeds the particle may be moving:

$$\Delta p_x = m \Delta \nu_x \geq 1.06 \times 10^{-20}\text{kg} \cdot \text{m/s}$$

and thus

$$\nu_x \geq (1.06 \times 10^{-20}\text{kg} \cdot \text{m/s})/(1.67 \times 10^{-27}\text{kg}) = 6.3 \times 10^6\text{m/s} = 6 \times 10^6\text{m/s}.$$ 

In other words, the neutron must be moving at a minimum speed of about 2% of the speed of light, but it could be moving faster than that (up to the speed of light).

HARRIS, *CH4-44* (20 Points)

**SOLUTION**

Again, we have to start from the uncertainty principle, $\Delta x \Delta p_x \geq \hbar/2$. We are told to find the maximum size of a spatial region ("... to how small a region ...") to become reasonably sure that an electron is moving at
borderline relativistic speeds, or at a minimum speed of $0.05c$. If the minimum speed is $0.05c$, we can calculate the minimum momentum:

$$\Delta p = \gamma \nu m = (1.0013) \times (9.1 \times 10^{-31} \text{kg}) \times (1.497 \times 10^7 \text{m/s}) = 1.36 \times 10^{-23} \text{kg} \cdot \text{m/s}.$$  

(We also observe just how borderline such speeds are - the gamma factor for these speeds is very nearly 1 to our level of quantitative interest, so we could have also just as easily used $p = \nu m$). We can now use the uncertainty principle to find the maximum size to which such a particle can be confined to insure this minimum momentum:

$$\Delta x \geq \hbar/(2\Delta p) = 3.87 \times 10^{-12} \text{m}.$$  

How does this compare to the dimensions of the Hydrogen atom? Given that the orbital extent of the electron in the hydrogen atom is up to $10^{-10} \text{m}$, it seems unlikely that an electron confined to such a wider space will be likely to routinely achieve relativistic speeds. What about for the inner shell of the lead atom, where the extent of an orbital electron is more like $10^{-12} \text{m}$? In that case, we're within a factor of 4 of the upper limit we computed, which means that it's much more likely that we'll find relativistic electrons in the innermost shell of lead.

HARRIS, CH4-48 (20 Points)

**SOLUTION**

To answer this question, we must apply the other uncertainty principle for energy and time: $\Delta E \Delta t \geq \hbar/2$. If we imagine the $\rho^0$ meson ("rho zero meson") to be at rest, then all of its energy is mass-energy. The uncertainty on that mass is given by the "width", and is 150 MeV. This represents a maximum uncertainty on the mass-energy of the meson. From this, we can compute the minimum time that the rho meson exists:

$$\Delta t = \hbar/(2\Delta E) = \hbar/(2 \times 150 \times 10^6 \text{eV} \times 1.6 \times 10^{-19} \text{J/eV}) = 2.2 \times 10^{-24} \text{s}.$$  

By comparison, we learned that the muon lives 2.2$\mu$s, so the rho meson lives a lot shorter life - about $10^{12}$ times shorter!