Nuclear Physics: Models of the Nucleus and Radioactivity
(11.3-11.5)
Steve Sekula, 8 April 2010 (created 7 April 2010)

**Review**

- We applied our quantum mechanics and statistical mechanics to study the nucleus
- We were presented with experimental observations of the nucleus:
  - nuclear density is constant as a function of mass number/nucleus radius
  - there is a "hard core" of repulsion in the nucleus, preventing shorter bond distances that about 1fm
  - at a bond distance of 2fm, the strong nuclear force drops to zero strength
  - the strong nuclear force is strongest between two nucleons when their spins are aligned
- From this, we built a model of the nucleus that:
  - predicted (correctly) that binding energy per nucleon increases to a point, then falls off as un-challenged Coulomb repulsion between a large number of widely spaced protons works to destabilize the nucleus
  - predicted (correctly) that stable nuclei at low A have N=Z, and for large A have N>Z
- We also talked about binding energy per nucleus, and learned that binding LOWERS the total internal energy (mass) of a nucleus compared to the sum of the constituent masses. This means it TAKES ENERGY to break up the nucleus, or that energy is given off when the nucleus forms. For the deuteron, the nucleus of deuterium, we learned that it takes 1.11 MeV per nucleon to pull apart the deuteron.

**Binding Energy Per Nucleon**

From our existing model, we can then compute the total binding energy, and from there get to the BE/A. Naively, we would want to do the following:
Note that for a fixed $A = N + Z$, if binding energy INCREASES the total mass of the nucleus DECREASES relative to the sum of the masses of the nucleons. Strongly bound nuclei are less-massive nuclei in comparison to the sum of their parts, while nuclei whose masses are closer to the sum of their parts are less well-bound.

But usually, a table of atomic data returns the ATOMIC MASS, which includes the mass of the electrons. A more useful (practical) formula is:

$$BE = Zm_p + Nm_n - M_{NUCLEUS}$$

where the first term is the total mass of electrons and protons in an atom with $Z$ protons (hydrogen's mass is derived almost entirely from the sum of its constituents), the second term is the total mass of the neutrons, and the third term subtracts the atomic mass of the element, including all its protons, neutrons, electrons, and binding energy.

You can then divide this $BE$ by $A$ to get the binding energy per nucleus.

**Models of the Nucleus: the Liquid Drop Model**

There are two existing useful full-fledged models of the nucleus. These are widely used in making calculations about the behavior and properties of new nuclei.

Each model emphasizes different aspects of the data, and while there are overlaps in their predictive ability each has strengths and weaknesses.

Until and unless the nucleus is completely understood from a theoretical perspective, there will ALWAYS be room for complimentary models.

**QUESTION:** How is the nucleus like a drop of water?

- **GUIDE:**
  - Water molecules have a potential much like that of the nucleus - at too short a distance, water molecules repel one another - there is a "hard core". They are incompressible past a certain point (water's
maximum density occurs at 4-degrees Celsius, not at freezing).
- Water molecules clump and form drops that are spherical. Why?
  Molecules inside the surface (in the volume) are bonded well to one
  another. Those near the surface are more weakly bonded. This
  lowers the binding energy of the drop, making the potential well
  more shallow. Nature wants to achieve the lowest energy state, so a
  state where the surface area is minimized is the best way to do this.
  A sphere achieves this feature, and is why droplets are round.

So a nucleus is like a water drop in several ways:

- hard core (1fm) - incompressible
- interior particles attracted to all neighbors

The liquid drop model attempts to calculate the binding energy by
assembling theoretical pieces of the nucleus (volume attraction, surface
effects, coulomb repulsion, and N/Z asymmetry) with data glueing the
pieces together.

**Volume Term**

If the nucleons safely tucked away from the surface were the only
contributor, more nucleons would mean better binding since all nucleons
would be bonded to all neighbors. One expects the total BE to grow as A.
Thus this piece is:

$$c_1 A$$

**Surface Term**

However, there are inevitably nucleons which are not bonded in all
directions, and have fewer bonds. These REDUCE the depth of the well,
decreasing the BE. The reduction should be proportional to the surface
area, $4\pi r^2 = 4\pi A^{2/3} R_v^2$ so

$$-c_2 A^{2/3}$$
**Coulomb Term**

The protons all repel each other, with a force going as $1/r^2$. The energy associated with this goes as $1/r$. Consider the repulsion between all pairs of protons. How many pairs of protons are there in a nucleus? $Z(Z-1)$ is the answer. We expect this term to look like:

$$-c_2 Z(Z-1)/A^{1/3}$$

**Asymmetry Term**

We know that, ignoring Coulomb Repulsion, $N=Z$ is the favored stable state for $A$ nucleons. An imbalance - an asymmetry - causes a reduction in the BE and a less-stable nucleus. What might that reduction in energy look like?

Consider fixed $A$, with $N=Z$ originally. If we draw a picture of such a state for a nucleus, we can suppose that the levels are filled to equal heights.

**QUESTION:**

- what is the name of the top-most filled energy level?
  - The FERMI ENERGY

Now, let’s convert some of our neutrons into protons. They have to go fill up higher energy proton states. If we convert $j$ neutrons into $j$ protons, how does $Z-N$ change?

$$Z + j - (N - j) = Z - N + 2j$$

If the energy levels are equally spaced with gaps equal to $\delta E$, then the energy level to which each nucleon goes is:

$$E_P + (j/2)\delta E$$
why $j/2$? Each PAIR of nucleons can fit in a state of energy $E_n$, so the energy levels reachable by $j$ nucleons is reduced by $1/2$. If the energy of each particle increases by $(j/2)\delta E$, the total energy of the system increases by $(j^2/2)\delta E$.

It can be shown that $\delta E \propto 1/A$. We can then rewrite $j$ in terms of $N$ and $Z$, and arrive at the effect of the asymmetry:

$$-c_4(N - Z)^2/A$$

**The Semi-Empirical Formula**

We have arrived at a formula for the binding energy of a nucleus:

$$BE = c_1 A - c_2 A^{2/3} - c_3 Z(Z - 1)/A^{1/3} - c_4(N - Z)^2/A$$

By taking measured values of $Z$, $A$, $N$, and binding energy (from the mass) for several nuclei, we can solve for the coefficients:

- $c_1 = 15.8$
- $c_2 = 17.8$
- $c_3 = 0.71$
- $c_4 = 23.7$

(All units in $\text{MeV}$)

This is a semiempirical formula because it has theory pieces stitched together with data. That makes it a model. It has predictive power (for new, unknown nuclei), but it doesn't predict the value of the coefficients.

(See slides for predictions of $BE$ vs. measurements).

The model is very good overall, with some key places where it fails. Helium is badly managed by this model. For something like Helium, $N$ and $Z$ are small, and the average behaviors assumed in the SE formula likely fall apart. Helium is much more tightly bound than predicted. To explain
Helium, the "Shell Model" works better.

**The Shell Model**

An example of the complexity of the nucleus is evident in just how different the two useful models actually are. The liquid drop model is very classical, while the shell model is purely quantum.

It begins by treating nucleons in 3-D, subject to a central force (much like Hydrogen). In the shell model, nucleons are treated as occupying quantum states in an average potential well that largely ignores interparticle forces. Only nucleons in the highest energy states can participate in physical phenomena, just as in a solid of atoms.

Unique predictions of the shell model:

- quantized angular momentum of nucleons (verified)
- tendencies for N and Z to be "magic numbers" or, at least, even numbers.

As in solids, the motion of nucleons in the highest energy states are essentially free from collisions, and most nucleons don't participate in kinetic motion. Only those at the highest energy levels have states to move into. There is again a Fermi energy.

The form of the potential is rather speculative (not a simple Coulomb force). A rounded finite well is a common choice. In your homework, you applied the infinite square well and found MeV energy differences between levels - this is characteristic of nuclear phenomena, and actually works quite well.

Careful inspection of N vs. Z (see slides) reveals that a lot of stable nuclei prefer N or Z or both equal to "magic numbers": 2, 8, 20, 50, 82, 126. This observation can be confirmed in the shell model, taking into account interactions between spin and orbital angular momentum.

The shell model also explains the tendency for N and Z to be even (the pairing effect) - filled shells are the situations of lowest energy and most stability for nuclei, and even numbers achieve that.

**NMR**

See slides.
Let's try to create a non-lethal, non-invasive imaging technique.

- Angular momentum is quantized in the nucleus. What is an inevitable consequence of quantized angular momentum? MAGNETIC MOMENT. Nuclei will respond to magnetic fields
- Put material in a strong magnetic field. What happens to the nuclear spins?
- Now hit the nuclei with photons. This causes the spins to flip (anti-align)
  - DISCUSSION: What kind of photons would you want to use?
- Choose your magnetic field so that radio waves are enough to cause transitions between angular momentum states in the field.
- When the momentum flips back, EM radiation is given off in all directions. You can look for the source of the radiation from the body.
- Scanning the magnetic field over the body in small steps let's you build up an image of the body
  - how? Different materials have different atomic and nuclear content, and respond differently to radio waves in the same field. Fix the field, change the radio waves, and see how different materials respond.

**Radioactivity**

A common language for radioactive decay is to compute "Q" - the kinetic energy available to particles after decay.

\[ Q = (m_i - m_f)c^2 \]

Kinetic energy increases as mass/internal energy decreases.

- Alpha Radiation: helium nuclei pp-nn - very tightly bound, ejected en masse. Reduces Z and N by an even number each, which is preferable to achieve stability. A Parent Nucleus emits a Daughter Nucleus and an alpha particle. Given that most of the kinetic energy in classical physics is taken by the lightest particle, the alpha gets most of the KE.
- Gamma radiation
- Beta Radiation: \( ^{12}_5 B \rightarrow ^{12}_6 C + ^0_{-1} \beta^- + \bar{\nu} \)
- Spontaneous fission: results in nuclei of about 1/2 the mass of the
original, and spare neutrons.

**Radioactive Decay Law**

The number of decays per unit time is proportional to the number of nuclei in your sample - more sample, more decays per unit time. Thus:

\[
\frac{dN}{dt} \propto N
\]

Let us denote the constant of proportionality by \( \lambda \) - this is not a wavelength!!! It is called the "Decay Constant." Let us also put in a minus sign, since we know that the number of nuclei DECREASES with time - \( \frac{dN}{dt} < 0 \).

\[
\frac{dN}{dt} = -\lambda N
\]

We want to then find the relationship between the number of nuclei you start with, \( N_0 \), and the number still left at some later time, \( t \). Thus we can do the following:

\[
\frac{dN}{N} = \lambda dt
\]

\[
\int_{N_0}^{N} \frac{1}{N'} dN' = \lambda \int_0^t dt = \lambda t
\]

\[
\ln \left( \frac{N}{N_0} \right) = \lambda t
\]

\[
N = N_0 e^{-\lambda t}
\]

Only for large numbers of nuclei is the decay rate smooth - so this formula is most accurate when \( N \) is large. By decay rate, we typically mean the number of decays per unit time, and we give this as:

\[
R = \lambda N
\]
In addition, a very useful concept is the "half-life" - the time it takes for half the sample to decay:

\[ \frac{1}{2}N_0 = N_0 e^{-\lambda T_{1/2}} \]

Solving for \( T_{1/2} \):

\[ T_{1/2} = \frac{\ln 2}{\lambda} \]

or, more useful, you can use the half-life to determine the decay constant:

\[ \lambda = \frac{\ln 2}{T_{1/2}} \]

Radioactive dating uses the ratios of daughter nuclei and parent nuclei in a sample to determine how old the sample is. For instance, lead is produced from uranium and thorium. Measuring the ratio of lead to these parents allows you to work backward, knowing \( T_{1/2} \), and figure out when the sample came into being.

A useful example is carbon dating. A radioactive isotope of carbon, carbon-14, is constantly produced in the biosphere by the action of cosmic rays raining down on earth. While it decays, it is replenished, and the ratio of carbon-14 to carbon-12 is roughly constant over time. Living organisms are constantly exchanging carbon with their environment, but when they die they stop doing so. After they die, the ratio of carbon-12 to carbon-14 in their body changes in a well-defined way. The half-life of carbon-14 is about 5000 years, so carbon dating is accurate to within an order or magnitude of the half-life. After that, it's hard to measure accurately the amount of carbon-14 left in the sample in order to date it.

**Application of Radioactive Decay: The Age of the Earth**

How old is the earth? This question was one of the first scientific assaults on Charles Darwin's Theory of Evolution. Not the Biblical Age, which was estimated to be about 6000 years; rather, William Rumford, aka Lord Kelvin, used 19th-century thermodynamics to estimate the age of the earth
based on the temperature of the earth. He settled on a range between 20-400 million years, and argued that this was not long enough for the slow evolution advanced by Darwin and his colleagues. It was a serious challenge to the theory, and hard to explain away. But Kelvin himself opened the door to the solution by saying that unless there was some heretofore unknown source of heat in the earth, his estimates were correct.

He was correct - there were two heretofore unknown sources of heat in the earth: convection of the viscous fluid mantle of the earth (which put the age of the earth to at least 2-3 billion years old), and radioactive decay (which takes you the last billion and a half). The release of energy from decay of Uranium and other unstable elements can supply heat to the earth long after the initial collapse and cooling of the planetary material began, while convection continues to distribute heat from the core.

How do you date the earth? The age of the solar system can be estimated by looking at the ratio of Uranium to Lead in meteorites. Uranium decays gradually down to Lead, which is stable. Thus the ratio in a sample, combined with the half-life and decay chain information, can be used to date the meteorites. Other typical ways to do this are to look for Argon from unstable Potassium decay. Various calcium compounds are also very useful in meteorites, as they allow more accurate dating.

The oldest meteorites found on earth date themselves to 4.54 billion years ago, placing an upper limit on the age of the earth (the solar system must be about that old, and the earth would have formed shortly after the sun came into being). The oldest minerals on earth have been dated to 4.4 billion years. This puts the earth somewhere between 4.4 and 4.54 billion years old. The sun is only about 30 million years older than earth.

Darwin died before radioactivity was understood, and thus died not only before genetics were understood as a means to pass traits, but also before an accurate age of the earth could be determined.

Next Time

- Fission and Fusion
- Fundamental Particles and Interactions