Topics in Spin - Homework 1

March 11, 2014

Problem SS-1: Creating Orthonormal Basis Vectors

Consider the following set of basis vectors,

\[ |I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \ |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}. \]

Show how one goes from this set of basis vectors to the orthonormal basis vectors,

\[ |1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ |2\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ |3\rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}. \]

Problem SS-2: More on the Pauli Spin Matrices

Consider the Pauli Spin Matrices, which play a central role in spin angular momentum for a spin - \( \frac{1}{2} \) system:

\[ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

1. Demonstrate that the Pauli Spin Matrices are unitary (that is, that \( UU^\dagger = I \), the identity matrix).

2. Demonstrate the relationships among the spin matrices,

\[ \sigma_x \sigma_y = i \sigma_z, \ \sigma_y \sigma_z = i \sigma_x, \ \sigma_z \sigma_x = i \sigma_y. \]

(This feature is generally known as cyclic permutation - that is, that as long as you preserve the cyclic order \( i, j, k \) of the indices (e.g. moving the symbols like they are on a cyclic conveyor belt, such that when one “falls off the end” it returns to the beginning: \( i, j, k \rightarrow k, i, j \rightarrow j, k, i \)), then it is true that \( \sigma_i \sigma_j = i \sigma_k \). The way of enforcing this using mathematical shorthand is the Levi-Civita Symbol \( \varepsilon_{ijk} \), which allows us to conveniently write \( \sigma_i \sigma_j = i \varepsilon_{ijk} \). The feature of this symbol is that \( \varepsilon_{ijk} = \varepsilon_{kji} = 1 \), while \( \varepsilon_{ikj} = \varepsilon_{jki} = -1 \) (that is, under the swap of any single pair of indices, the function yields -1).
3. Demonstrate the following commutation relations:
\[ [\sigma_i, \sigma_j] \equiv \sigma_i \sigma_j - \sigma_j \sigma_i = 2i\sigma_k. \]

4. Demonstrate that the spin matrices anti-commute, that is
\[ \{\sigma_i, \sigma_j\} \equiv \sigma_i \sigma_j + \sigma_j \sigma_i = 0. \]

Remember these properties - they are immensely useful!

**Problem SS-3: Exercise in Eigenvalues and Eigenvectors**

Find the eigenvalues and eigenvectors of the following matrix:
\[
M = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}
\]

(HINT: review how to compute the determinant of an \(n \times n\) matrix - this is the key to solving eigenvalue/eigenvector problems)