1. Consider the finite difference equation

\[ a_{n+2} = a_{n+1} + (n + 1) a_n \]

with \( a_0 = a_1 = 1 \). Find a generating function \( F(s) \) such that

\[ a_n = \frac{\partial^n F(s)}{s^n} \bigg|_{s=0} . \]

(The difference equation above could come from a variety of different problems: from the relation between coefficients in a power series solution to a differential equation, or a recursion relation among special functions, or the discrete approximation to a differential equation, with \( a_n \approx a(x) \) and \( x = n \Delta x \).)

2. Use the generating function for Hermite polynomials to generate the first four polynomials. Use them to state the first four harmonic oscillator wavefunctions. (Include the normalization we computed in lecture.)

3. Use the condition that the harmonic oscillator ground state is annihilated by the lowering operator to compute its position-space wavefunction, and normalize it. Then use the raising operator to produce the first three excited harmonic oscillator wavefunctions.

4. Use position space wavefunctions and the generating function for the Hermite polynomials to give a general expression for the harmonic oscillator matrix elements

\[ \langle n| x|m \rangle . \]

(Use the method we used to get the general normalization.)

5. Repeat the previous problem using raising and lowering operator methods.