1. Extract the leading behavior at large (positive) $x$ for the one-dimensional Schrödinger equation with a potential $V(x) = k|x|$. (This is what the Coulomb potential looks like in one dimension. It’s also a pretty good model for the confining potential for quarks.)

We know that the behavior should be some type of exponential, so start by replacing $\psi(x) \equiv \exp(S(x))$ and find the dominant behavior for $S(x)$. (This is sometimes called the controlling behavior.) Now include the first correction to $S$. These together give the leading behavior. (There’s an excellent discussion in Bender and Orszag beginning in section 3.4 if you’d like to learn more about this.)

2. Consider a spin-1 particle (with no orbital degrees of freedom). Let

$$H = AS_z^2 + B(S_x^2 - S_y^2),$$

where $S_i$ are $3 \times 3$ spin matrices, and $A \gg B$. Treating the $B$ term as a perturbation, find the eigenstates of $H^0 = AS_z^2$ that are stable under the perturbation. Calculate the energy shifts to first order in $B$. How are these related to the exact answers?

3. Use the generating function for Legendre polynomials to express $1/|r_1 - r_2|$ as a series in $r_1/r_2$ and $P_\ell(\cos \theta_{12})$. (Assume $r_2 > r_1$. Note that when $r_1 > r_2$, the roles of $r_1$ and $r_2$ switch. This problem should only take a couple lines. The result is very useful for doing integrals in spherical coordinates which involve this potential.)

4. In this exercise you will derive a sum rule for spherical harmonics from some of the simple properties of rotation operators. We will make use of this relation in lecture.

As you know, the angular momentum operators generate rotations. The unitary operator

$$U(\alpha, \beta, \gamma) = \exp(-i\alpha J_3) \exp(-i\beta J_2) \exp(-i\gamma J_3)$$

rotates a state through the Euler angles $\alpha, \beta, \gamma$. In particular, $U(\phi, \theta, 0)$ rotates an eigenstate at a position along the $z$ axis to the angles $(\theta, \phi)$. For an eigenstate of $J^2$ and $J_3$, $|\ell m\rangle$, rotations only mix the $m$’s. The effect is represented by the rotation matrix $D$, with

$$U(\alpha, \beta, \gamma) |\ell m\rangle = |\ell m'\rangle D^\ell (\alpha, \beta, \gamma)_m^{m'}.$$
(The $m'$ is summed over.) The spherical harmonics have a very physical interpretation as the overlap between angular momentum eigenstates and eigenstates of angular position,

$$\langle \Omega | \ell m \rangle = Y_{\ell m}(\Omega)$$

with $|\Omega\rangle \equiv |\theta, \phi\rangle$ normalized such that

$$\langle \theta', \phi'| \theta, \phi \rangle = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

and

$$\int d\Omega \langle \Omega | \Omega \rangle = 1.$$ 

Note that

$$Y_{\ell m}(0, \phi) = \left[ \frac{2\ell + 1}{4\pi} \right]^{1/2} \delta_{m,0}$$

$$P_{\ell}(\cos \theta) = \left[ \frac{4\pi}{2\ell + 1} \right]^{1/2} Y_{\ell 0}(\theta, \phi).$$

Use these relations to show that

$$Y^*_{\ell m}(\theta, \phi) = \left[ \frac{2\ell + 1}{4\pi} \right]^{1/2} D_{\ell}^{\star}(\phi, \theta | 0)^{m}_{0}$$

and that

$$P_{\ell}(\cos \theta_{12}) = \left[ \frac{4\pi}{2\ell + 1} \right] \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega_2) Y^*_{\ell m}(\Omega_1).$$

(To prove the second equation, you may wish to consider a coordinate system where $\Omega_1$ is along the $z$ axis, and rotate to a general coordinate system.)

Finally, combine this result with that of the previous problem to express $1/|r_1 - r_2|$ as a series in $Y_{\ell m}(\Omega_2) Y^*_{\ell m}(\Omega_1)$. You will want to use this result in the following problem.

5. Unlike the electron, the proton is not a point particle but composite, and has a finite size. Suppose its charge distribution can be described by a spherically symmetric distribution $\rho(r)$. Treat the difference between the Coulomb potential due to $\rho(r)$ and that of a point source as a perturbation, and compute the first-order energy shift of the $S$ states in the hydrogen atom. You may assume that the atomic wavefunctions vary little over the proton’s radius, and give the shift in terms of the proton’s mean-squared radius

$$\langle r^2 \rangle_{\text{proton}} = (4\text{GeV}^{-1})^2.$$ 

Why is there no shift for states other than $S$?