**Part I: Drell-Yan Process**

**History:**
- Discovery of J/ψ, Upsilon, W/Z, and “New Physics” ??

**Calculation of** $q q \rightarrow \mu^+\mu^-$ **in the Parton Model**
- Scaling form of the cross section
- Rapidity, longitudinal momentum, and $x_F$

**Comparison with data:**
- NLO QCD corrections essential (the K-factor)
- $\sigma(pd)/\sigma(pp)$ important for d-bar/ubar
- W Rapidity Asymmetry important for slope of $d/u$ at large x

**Where are we going?**
- $P_T$ Distribution
- W-mass measurement
- Resummation of soft gluons

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**Historical Background**

Our story begins in the late 1960's at CERN
Brookhaven National Lab
Alternating Gradient Synchrotron

An Early Experiment:

The Goal: \( p + N \rightarrow W + X \)
They found: \( p + N \rightarrow \mu^+ \mu^- + X \)

What is the explanation???

In DIS, we have two choices for an interpretation:
- lepton current \( L_{\mu\nu} \)
- hadron current \( W_{\mu\nu} \)

Conserved Current Interactions

What about Drell-Yan???

The Parton Model

Discovery of the J/Psi Particle

The Process: \( p + Be \rightarrow e^+ e^- X \)

very narrow width \( \Rightarrow \) long lifetime

Experimental Observation of a Heavy Particle \( J/\psi \)

at BNL AGS
The November Revolution

related by crossing ...

Drell-Yan
Brookhaven AGS

e⁺e⁻ Production
SLAC SPEAR
Frascati ADONE

More Discoveries with Drell-Yan

1974: The J/Psi (charm) discovery
\( p+N \rightarrow J/\psi \)
... 1976 Nobel Prize

1977: The Upsilon (bottom) discovery
\( p+N \rightarrow \Upsilon \)

1983: The W and Z discovery
\( p + \bar{p} \rightarrow W/Z \)
... 1984 Nobel Prize

The Future of Drell-Yan

Where do we find

New Physics??

- New Higgs Bosons
- New W' or Z'
- SUSY
- ... unknown...
Let's calculate the partonic $\hat{\sigma}$ in the partonic CMS

Let's compute the Born process: $q + \bar{q} \rightarrow e^+ + e^-$

Gathering factors and contracting $g^{\mu\nu}$, we obtain:

$$-iM = iQ_i \frac{e^2}{q^2} [\bar{u}(p_2) Y^\mu u(p_1)] [\bar{u}(p_3) Y_\mu v(p_4)]$$

Squaring, and averaging over spin and color, ....

$$|M|^2 = \left( \frac{1}{2} \right)^2 \left( \frac{1}{3} \right)^2 Q_i^2 \frac{e^4}{q^2} Tr [p_2 Y^\mu p_1 Y^\nu] Tr [p_3 Y_\mu p_4 Y_\nu]$$
Let's work out some parton level kinematics

\[ p_1 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1,0,0, +1 \end{pmatrix} \]
\[ p_2 = \frac{\sqrt{s}}{2} \begin{pmatrix} 1,0,0, -1 \end{pmatrix} \]
\[ p_3 = \frac{\sqrt{s}}{2} \left( 1, \sin(\theta), 0, +\cos(\theta) \right) \]
\[ p_4 = \frac{\sqrt{s}}{2} \left( 1, -\sin(\theta), 0, -\cos(\theta) \right) \]

Defining the Mandelstam variables ...

\[ s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \]
\[ t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \]
\[ u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \]

We'll now compute the matrix element \( M \)

Manipulating the traces, we find ...

\[
\begin{align*}
\text{Tr} \left[ p_2 Y^a p_1 Y^a \right] &= 4 \left[ p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} (p_1 \cdot p_2) \right] \\
&= 4 \left[ (p_2 \cdot p_1) (p_2 \cdot p_3) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right] \\
&= 2 \left[ s^2 + u^2 \right] \\
\text{Tr} \left[ p_2 Y^a p_1 Y^a \right] &= 4 \left[ p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} (p_1 \cdot p_2) \right] \\
&= 2 \left[ s^2 + u^2 \right] \\
\end{align*}
\]

Where we have used:

\[ s = 2 (p_1 \cdot p_2) = 2 (p_3 \cdot p_4) \]
\[ t = 2 (p_1 \cdot p_3) = 2 (p_2 \cdot p_4) \]
\[ u = 2 (p_1 \cdot p_4) = 2 (p_2 \cdot p_3) \]

Putting all the pieces together, we have:

\[ |M|^2 = Q_i^2 \alpha^2 \frac{25 \pi^2}{3} \left( \frac{s^2 + u^2}{s^2} \right) \]

with \[ q^2 = (p_1 + p_2)^2 = s \]

\[ \alpha = \frac{\epsilon^2}{4\pi} \]

... and put it together to find the cross section

\[
d\hat{\sigma} \approx \frac{1}{2s} |M|^2 \ d\Gamma
\]

In the partonic CMS system

\[
d\Gamma = \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \ (2\pi)^4 \delta (p_1 + p_2 - p_3 - p_4) = \frac{d\cos(\theta)}{16\pi}
\]

Recall,

\[ i = -\frac{\sqrt{s}}{2} (1-\cos(\theta)) \quad \text{and} \quad u = -\frac{\sqrt{s}}{2} (1+\cos(\theta)) \]

so, the differential cross section is ...

\[
\frac{d\hat{\sigma}}{d\cos(\theta)} = Q_i^2 \alpha^2 \frac{\pi}{6} \frac{1}{s} (1+\cos^2(\theta))
\]

and the total cross section is ...

\[
\hat{\sigma} = Q_i^2 \alpha^2 \frac{\pi}{6} \frac{1}{s} \int_{-1}^{1} d\cos(\theta) (1+\cos^2(\theta)) = \frac{4\pi \alpha^2 Q_i^2}{9s} \equiv \hat{\sigma}_0
\]

Some Homework:

#1) Show:

\[
\frac{d^3 p}{(2\pi)^3 2E} = \frac{d^3 p}{(2\pi)^3} \ (2\pi)^4 \delta^+(p^2 - m^2)
\]

This relation is often useful as the RHS is manifestly Lorentz invariant

#2) Show that the 2-body phase space can be expressed as:

\[
d\Gamma = \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \ (2\pi)^4 \delta (p_1 + p_2 - p_3 - p_4) = \frac{d\cos(\theta)}{16\pi}
\]

Note, we are working with massless partons, and \( \theta \) is in the partonic CMS frame
Some More Homework:

#3) Let's work out the general 2→2 kinematics for general masses.

<p>| | | |</p>
<table>
<thead>
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</thead>
<tbody>
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</tr>
</tbody>
</table>

a) Start with the incoming particles.

Show that these can be written in the general form:

\[
\begin{align*}
p_1 &= (E_1, 0, 0, +p) & p_1^2 &= m_1^2 \\
p_2 &= (E_2, 0, 0, -p) & p_2^2 &= m_2^2 \\
\end{align*}
\]

... with the following definitions:

\[
\begin{align*}
E_{1,2} &= \frac{\hat{s} \pm m_1^2 \mp m_2^2}{2 \sqrt{\hat{s}}} \\
p &= \frac{\Delta \{\hat{s}, m_1^2, m_2^2\}}{2 \sqrt{\hat{s}}} \\
\Delta \{a, b, c\} &= \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)}
\end{align*}
\]

Note that \(\Delta(a, b, c)\) is symmetric with respect to its arguments, and involves the only invariants of the initial state: \(s, m_1^2, m_2^2\).

b) Next, compute the general form for the final state particles, \(p_3\) and \(p_4\). Do this by first aligning \(p_3\) and \(p_4\) along the z-axis (as \(p_1\) and \(p_2\) are), and then rotate about the y-axis by angle \(\theta\).

---

Next, we'll compute the hadronic CMS

---

What does the angular dependence tell us?

Observe, the angular dependence:

\[
q + \bar{q} \rightarrow e^+ + e^-
\]

\[
\frac{d \hat{\sigma}}{d \cos(\theta)} = Q_e^2 \alpha^2 \frac{\pi}{6} \frac{1}{\hat{s}} (1 + \cos^2(\theta))
\]

Characteristic of scattering of spin ½ constituents by a spin 1 vector

\(q\) \hspace{1cm} \(\bar{q}\)

\(e^+\) \hspace{1cm} \(e^-\)

\(\gamma\)

Note, for the photon, the mirror image of the above is also valid; hence the symmetric distribution.

The \(W\) has V-A couplings, so we'll find: \((1+\cos\theta)^2\)

---

Kinematics in the Hadronic Frame

\[
\begin{align*}
P_1 &= \frac{\sqrt{\hat{s}}}{2} (1, 0, 0, +1) & P_1^2 &= 0 \\
P_2 &= \frac{\sqrt{\hat{s}}}{2} (1, 0, 0, -1) & P_2^2 &= 0 \\
s &= (P_1 + P_2)^2 = \frac{\hat{s}}{x_1 x_2} = \frac{\hat{s}}{\tau} & \text{Therefore} & \tau = x_1 x_2 = \frac{\hat{s}}{s} \equiv \frac{Q^2}{s} \\
\end{align*}
\]

Fractional energy\(^2\) between partonic and hadronic system

\[
\frac{d \sigma}{d Q^2} = \sum_{q, \bar{q}} \int dx_1 \int dx_2 [q(x_1) \bar{q}(x_2) + q(x_1) q(x_2)] G_0 \delta (Q^2 - \hat{s})
\]

Hadronic cross section

Parton distribution functions

Partonic cross section
Scaling form of the Drell-Yan Cross Section

Using: \( \hat{\sigma}_0 = \frac{4\pi \alpha^2}{9 \hat{s}} Q_i^2 \) and \( \delta (Q^2 - \hat{s}) = \frac{1}{s x_1} \delta (x_2 - \frac{\tau}{x_1}) \)

we can write the cross section in the scaling form:

\[
Q^4 \frac{d\sigma}{dQ^2} = \frac{4\pi \alpha^2}{9} \sum_{q,q'} Q_i^2 \int_0^1 \frac{dx_1}{x_1} \tau \left[ q(x_1)\bar{q}(x_1) + \bar{q}(x_1)q(x_1) \right]
\]

Notice the RHS is a function of only \( \tau \), not \( Q \).

This quantity should lie on a universal scaling curve.

Cf., DIS case, & scattering of point-like constituents

So, we're ready to compare with data

(or so we think...)

Longitudinal Momentum Distributions

Partonic CMS has longitudinal momentum w.r.t. the hadron frame

\[
\begin{align*}
p_1 &= x_1 P_1 \\
p_2 &= x_2 P_2 \\
p_{12} &= (p_1 + p_2) = (E_{12}, 0, 0, p_L) \\
E_{12} &= \frac{\sqrt{s}}{2} (x_1 + x_2) \\
p_L &= \frac{\sqrt{s}}{2} (x_1 - x_2) \equiv \frac{\sqrt{s}}{2} x_F
\end{align*}
\]

\( x_F \) is a measure of the longitudinal momentum

The rapidity is defined as:

\[
y = \frac{1}{2} \ln \left( \frac{E_{12} + p_L}{E_{12} - p_L} \right) = \frac{1}{2} \ln \left( \frac{x_1}{x_2} \right)
\]

\[
dx_1 dx_2 = d\tau dy
\]

\[
dQ^2 dx_F = dy d\tau \sqrt{x_F^2 + 4 \tau}
\]

\[
\frac{d\sigma}{dQ^2 dx_F} = \frac{4\pi \alpha^2}{9 Q^4} \frac{1}{\sqrt{x_F^2 + 4 \tau}} \tau \sum_{q,q'} Q_i^2 \left[ q(x_1)\bar{q}(x_1) + \bar{q}(x_1)q(x_1) \right]
\]

Let's compare data and theory

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Energy</th>
<th>From</th>
<th>( K = m_{c\ell}/m_{c\ell} )</th>
</tr>
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<tbody>
<tr>
<td>E288</td>
<td>52 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>NA10</td>
<td>50 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>NA14</td>
<td>40 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>E269</td>
<td>50 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>NA17</td>
<td>40 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>E261</td>
<td>50 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
<tr>
<td>E265</td>
<td>50 GeV</td>
<td></td>
<td>0.7 \pm 0.2</td>
</tr>
</tbody>
</table>

Oooops, we need the QCD corrections

\[ K = 1 + \frac{2 \pi \alpha}{3} s \ (\ldots) + \ldots = e^{\frac{2 \pi \alpha s}{3}} \]

Drell-Yan can give us unique and detailed information about PDF's.

We'll now examine two examples:

1) Ratio of pp/pd cross section

2) W Rapidity Asymmetry

Excellent agreement between data and theory

pp & pN processes sensitive to anti-quark distributions

A measurement of \( \bar{d}(x)/\bar{u}(x) \)
Antiquark asymmetry in the Nucleon Sea
FNAL E866/NuSea

800 GeV \( p + p \) and \( p + d \rightarrow \mu^+ \mu^- X \)

Fermilab E866 - Drell-Yan

\( 0.032 \) Systematic error not shown

Cross section ratio of $pp$ vs. $pd$

Obtain the neutron PDF via isospin symmetry:

$$
\begin{align*}
\frac{\sigma^{pd}}{2\sigma^{pp}} & \approx \frac{1}{2} \left( 1 + \frac{d_2}{u_2} \right) \\
& \approx \frac{1}{2} \left( 1 + \frac{\bar{d}_2}{\bar{u}_2} \right)
\end{align*}
$$

In the limit $x_1 \gg x_2$:

$$
\sigma^{pp} \propto \frac{4}{9} u(x_1) \bar{u}(x_2) + \frac{1}{9} d(x_1) \bar{d}(x_2)
$$

$$
\sigma^{pd} \propto \frac{4}{9} u(x_1) \bar{d}(x_2) + \frac{1}{9} d(x_1) \bar{u}(x_2)
$$

For the ratio we have:

$$
\frac{\sigma^{pd}}{2\sigma^{pp}} \approx \frac{1}{2} \left( 1 + \frac{\bar{d}_2}{\bar{u}_2} \right)
$$

As promised, this provides information about the sea-quark distributions.

EXERCISE: Verify the above.

E866 required significant changes in the hi-x sea distributions

With increased flexibility in the parameterization of the sea-quark distributions, good fits are obtained

Does the theory match the data???

$$
\frac{\sigma^{pd}}{2\sigma^{pp}} \approx \frac{1}{2} \left( 1 + \frac{\bar{d}_2}{\bar{u}_2} \right)
$$

Fermilab E866 - Drell-Yan

Next ...

2) W Rapidity Asymmetry
Where do the W’s and Z’s come from???

\[
\frac{d\sigma}{dy}(W^\pm) = \frac{2\pi G_F}{\sqrt{2}} \sum_q |V_{q\tau}|^2 \left[ q(x_a) \bar{q}(x_b) + q(x_b) \bar{q}(x_a) \right]
\]

Flavour decomposition of W cross sections

For anti-proton:

\[
\begin{align*}
  & \text{u}(x_a) \Rightarrow \bar{u}(x) \\
  & \text{d}(x) \Rightarrow \bar{d}(x)
\end{align*}
\]

Therefore

\[
\begin{align*}
  & \frac{d\sigma}{dy}(W^+) \approx \frac{2\pi G_F}{\sqrt{2}} \left[ u(x_a) d(x_b) \right] \\
  & \frac{d\sigma}{dy}(W^-) \approx \frac{2\pi G_F}{\sqrt{2}} \left[ d(x_a) u(x_b) \right]
\end{align*}
\]

Unfortunately, we don’t measure the W directly since W→eν.

Still the lepton contains important information

\[
A(y) = \frac{\frac{d\sigma}{dy}(I^+)}{\frac{d\sigma}{dy}(I^+)} - \frac{\frac{d\sigma}{dy}(I^-)}{\frac{d\sigma}{dy}(I^-)}
\]

A bit of calculation

With the previous approximation,

\[
A \approx \frac{u(x_a)d(x_b) - d(x_a)u(x_b)}{u(x_a)d(x_b) + d(x_a)u(x_b)} = \frac{R_{du}(x_b) - R_{ud}(x_a)}{R_{du}(x_b) + R_{ud}(x_a)}
\]

where

\[
R_{du}(x) = \frac{d(x)}{u(x)}
\]

We can make Taylor expansions:

\[
x_{1,2} = x_0 e^{\pm y} \approx x_0 (1 \pm y)
\]

\[
R_{du}(x) \approx R_{du}(x_0) \pm y x_0 R'_{du}(x_0) \sqrt{y}
\]

Thus, the asymmetry is:

\[
A(y) = -y x_0 \frac{R'_{du}(x_0)}{R_{du}(x_0)}
\]

The form of the d/u ratio at large x as a function of

1) Parameterization

2) Nuclear Corrections

Dotted – With Nuc. Cor. and d/u Forced to 0.2
Dashed – With Nuc. Cor. and CTEQ5 Parameterization
Solid – CTEQ5M (No Nuc. Cor.)

**End of Part I: Where have we been???

History:
- Discovery of J/ψ, Upsilon, W/Z, and “New Physics” ???

**Calculation of \[ q q \rightarrow \mu^{+}\mu^{-} \text{ in the Parton Model} \]
- Scaling form of the cross section
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**Comparison with data:**
- NLO QCD corrections essential (the K-factor)
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**Where are we going?**
- \( P_T \) Distribution
- W-mass measurement
- Resummation of soft gluons

---

**Part II: W Boson Production as an example**

**Finding the W Boson Mass:**
- The Jacobian Peak, and the W Boson \( P_T \)
- Multiple Soft Gluon Emissions
- Single Hard Gluon Emission

**Road map of Resummation**
- Summing 2 logs per loop: multi-scale problem (\( Q,q_T \))
- Correlated Gluon Emission
- Non-Perturbative physics at small \( q_T \)

**Transverse Mass Distribution:**
- Improvement over \( P_T \) distribution

**What can we expect in future?**
- Tevatron Run II
- LHC

---

**Drell-Yan Process: Part II**

**Fred Olness**

**SMU**

**Side Note: From \( pp\rightarrow \gamma/Z/W \), we can obtain \( pp\rightarrow \gamma/Z/W \rightarrow l^{+}l^{-} \)**

Schematically:

\[
\frac{d\sigma}{dQ^2\,dt}(q\bar{q} \rightarrow l^{+}l^{-}g) = \frac{d\sigma}{d\alpha}(q\bar{q} \rightarrow \gamma^{*}g) \times \frac{d\sigma}{d\alpha}(\gamma^{*} \rightarrow l^{+}l^{-})
\]

For example:

\[
\frac{d\sigma}{dQ^2\,dt}(q\bar{q} \rightarrow l^{+}l^{-}g) = \frac{d\sigma}{d\alpha}(q\bar{q} \rightarrow \gamma^{*}g) \times \frac{\alpha}{3\pi Q^2}
\]
Part II: W Boson Production as an example

How do we measure the W-boson mass?

\[ u + \bar{d} \rightarrow W^+ \rightarrow e^+ \nu \]

- Can’t measure W directly
- Can’t measure ν directly
- Can’t measure longitudinal momentum

We can measure the \( p_T \) of the lepton

How can we use this to extract the W-Mass???

Now that we’ve got the picture, here’s the math... (in the W CMS frame)

\[ p_T^2 = \frac{\hat{s}}{4} \sin^2 \theta \]

\[ \cos \theta = \sqrt{1 - \frac{4 p_T^2}{\hat{s}}} \]

\[ \frac{d \cos \theta}{dp_T^2} = \frac{2}{\hat{s}} \frac{1}{\cos \theta} \]

So we discover the \( p_T \) distribution has a singularity at \( \cos \theta = 0 \), or \( \theta = \pi/2 \)

The Jacobian Peak

Suppose lepton distribution is uniform in \( \theta \)

The dependence is actually \((1+\cos \theta)\), but we’ll take care of that later

What is the distribution in \( p_T \)?

We find a peak at \( p_T^{\text{max}} \approx M_W/2 \)

1) The W-mass is important fundamental quantity of the Standard Model

2) \( p_T \) Distribution is important for measuring the W-mass

BUT !!!

Measuring the Jacobian peak is complicated if the W boson has finite \( p_T \)
The W-Mass is an important fundamental quantity

\[ \chi^2/N_{\text{exp}} = 0.4/4 \]

- 80.360 +/- 0.370 - UA2 (W \to ev)
- 80.410 +/- 0.180 - CDF(Run 1A, W \to ev, \mu

What gives the W

P_T ???

What about the intrinsic k_t of the partons?

Assume a Gaussian form:

\[ \frac{d^2\sigma}{d^2p_T} \approx \sigma_0 e^{-\frac{p_T^2}{2k_T^2}} \]

\[ \langle k_T \rangle = 760 \text{ MeV} \]

good agreement

proslems
For high $P_T$, we need a hard parton emission

\[ q \xrightarrow{\gamma f/Z/W} g \]

Compton

\[ \frac{1}{P_T^2} \]

Perturbative

Gaussian

\[ e^{-\frac{2}{P_T^2}} \]

Combination of Gaussian & QCD corrections

Road map for Resummation

BEFORE

\[ \Rightarrow \]

AFTER

The complete $P_T$ spectrum for the W boson

The full $P_T$ spectrum for the W-boson showing the different theoretical regions

NLO $P_T$ distribution for the W boson

In the limit $P_T \to 0$

\[ \frac{d\sigma}{d\tau dy dp_T^2} \approx \left( \frac{d\sigma}{d\tau dy} \right)_{\text{Born}} \times \frac{4\alpha_s}{3\pi} \frac{\ln s}{P_T^2} \]

\[ \int_0^s \frac{d\sigma}{d\tau dy dp_T^2} dp_T^2 = \left( \frac{d\sigma}{d\tau dy} \right)_{\text{Born}} + \mathcal{O}(\alpha_s) \]

finite

singular

effect of gluon emission

assume this exponentiates

Pani & Petronzio, NP B154, 427 (1979)
Resummation of soft gluons: Step #1

Differentiating the previous expression for \( d^2\sigma/d\tau dy \)

\[
\frac{d\sigma}{d\tau dy dp_T^2} \approx \left( \frac{d\sigma}{d\tau dy} \right)_{\text{Born}} \times \frac{4\alpha_s}{3\pi} \ln \frac{s}{p_T^2} \times \exp \left( -\frac{2\alpha_s}{3\pi} \ln \frac{s}{p_T^2} \right)
\]

We just resummed (exponentiated) an infinite series of soft gluon emissions

\[ e^{-\frac{\alpha_s}{s} L^2} \approx 1 - \alpha_s L^2 + \frac{(\alpha_s L^2)^2}{2!} - \frac{(\alpha_s L^2)^3}{3!} + \ldots \]

Soft gluon emissions treated as uncorrelated

\[ L = \ln \frac{s}{p_T^2} \]

I've skipped over some details..

Parisi & Pettorino, NP B154, 427 (1979)
Curci, Greco, Srivastava, PRL 43, 834 (1979); NP B159, 451 (1979)
Jeff Owens, 2000 CTEQ Summer School Lectures

1) We summed only the leading logarithmic singularity

\[
\frac{d\sigma}{dq_T^2} \sim \frac{\alpha_s L}{q_T} \left\{ 1 + \alpha_s L^2 + \alpha_s^2 L^4 + \ldots \right\}
\]

The terms we are missing are suppressed by \( \alpha_s L \), not \( \alpha_s^! \)

If (somehow) we could sum the sub-leading log ...

\[
\frac{d\sigma}{dq_T^2} \sim \frac{\alpha_s L}{q_T} e^{\alpha_s (L^2/2)}
\]

Now, the terms we are missing are suppressed only by \( \alpha_s^! \)

2) We assumed exponentiation; proof is non-trivial

Review where the logs come from

Review one-scale problem \((Q)\) resummation via RGE

Review two-scale problem \((Q,q_T)\) resummation via RGE+ Gauge Invariance
Where do the Logs come from?

Total Cross Section: $\sigma(e^+e^-)$ at 3 Loops

$$\sigma(Q^2) = \sigma_0 + \left(\frac{\alpha_s(Q^2)}{4\pi}\right)^2 - \frac{\alpha_s(Q^2)}{2} c_{\beta}^2 \frac{123}{2} - 144\frac{\alpha_s^2}{3} - c_{\beta}^2 T_{\alpha}(1 - 22 + 10\beta(0))$$

$$+ \left(\frac{\alpha_s(Q^2)}{4\pi}\right)^3 - \frac{\alpha_s(Q^2)}{2} c_{\beta}^2 \frac{123}{2} - 144\frac{\alpha_s^2}{3} - c_{\beta}^2 T_{\alpha}(1 - 22 + 10\beta(0))$$

Drell-Yan at 2 Loops:

$$R_{\alpha}(\beta,\mu^2) = \left[\frac{\alpha_s}{4\pi}\right] R(1 - \alpha) C_{\gamma}^2 \frac{1}{4\pi} - 24\left(\frac{\alpha_s^2}{3}\right) + \frac{24}{5} \frac{\alpha_s^2}{3} + \frac{12}{25} \frac{\alpha_s^2}{3}$$

Renormalization Group Equation

More Differential Quantities $\Rightarrow$ More Mass Scales $\Rightarrow$ More Logs!!!

$$\frac{d\sigma}{dQ^2} \sim \ln \left(\frac{Q^2}{\mu^2}\right)$$

How do we resum logs? Use the Renormalization Group Equation

For a physical observable $R$:

Using the chain rule:

$$\mu \frac{dR}{d\mu} = 0$$

$$\beta(\alpha_s(\mu)) \Rightarrow \int \frac{dx}{\alpha_s(x)} = \frac{\alpha_s(Q^2)}{\beta(x)}$$
Renormalization Group Equation: OVER SIMPLIFIED!

\[
\left\{ \mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s(\mu)) \frac{\partial}{\partial \alpha_s(\mu^2)} \right\} R(\mu^2, \alpha_s(\mu^2)) = 0
\]

If we expand \( R \) in powers of \( \alpha_s \), and we know \( \beta \), we then know \( \mu \) dependence of \( R \).

\[
R(\mu, Q, \alpha_s(\mu^2)) = R_0 + \alpha_s(\mu^2) R_1 \left[ \ln \left( \frac{Q^2}{\mu^2} \right) + c_1 \right] + \alpha_s^2(\mu^2) R_2 \left[ \ln^2 \left( \frac{Q^2}{\mu^2} \right) + \ln \left( \frac{Q^2}{\mu^2} \right) + c_2 \right] + O(\alpha_s^3(\mu^2))
\]

Since \( \mu \) is arbitrary, choose \( \mu = Q \).

\[
R(Q, Q, \alpha_s(Q^2)) = R_0 + \alpha_s(Q^2) R_1 \left[ 0 + c_1 \right] + \alpha_s^2(Q^2) R_2 \left[ 0 + 0 + c_2 \right] + ...
\]

We just summed the logs

Two-Scale Problems

For \( R(\mu, Q, \alpha_s) \), we could resum \( \ln(Q^2/\mu^2) \) by taking \( Q = \mu \).

What about \( R(\mu, Q, q_T, \alpha_s) \); how do we resum \( \ln(Q^2/\mu^2) \) and \( \ln(q_T^2/\mu^2) \).

Are we stuck? Can’t have \( \mu^2 = Q^2 \) and \( \mu^2 = q_T^2 \) at the same time!

Solution: Use Gauge Invariance; cast in similar form to RGE

Use axial-gauge with axial vector \( \xi \).
This enters the cross section in the form: \( (\xi \cdot p) \).

\[
\frac{d \sigma}{d \mu^2} = 0 \quad \text{RGE allows us to vary} \ \mu \ \text{to resum logs}
\]

\[
\frac{d \sigma}{d (\mu^2 \cdot \xi^2)} = 0 \quad \text{Gauge invariance allows us to vary} \ (\xi \cdot p) \ \text{to resum logs}
\]

The details will fill multiple lectures:

3) We assumed gluon emission was uncorrelated

\[
\frac{d \sigma}{d \tau dy dp_T^2} \approx \ln \left( \frac{p_T^2}{\mu^2} \right) \times \exp \left( -\frac{2 \alpha_s}{3\pi} \ln \frac{s}{p_T^2} \right)
\]

This leads to too strong a suppression at \( P_T = 0 \).
Need to impose momentum conservation for \( P_T \).

A particle can receive finite \( k_T \) kicks, yet still have \( P_T = 0 \).

A convenient way to impose transverse momentum conservation is in impact parameter space (b-space) via the following relation:

\[
\delta^{(2)} \left( \sum_{i=1}^{n} k_{i,T} - p_T \right) = \frac{1}{(2\pi)^2} \int d^2 b \ e^{-i\hat{b} \cdot \hat{p}_T} \prod_{i=1}^{n} e^{-i\hat{k}_{i,T} \cdot \hat{b}}
\]

4) We encounter Non-Perturbative Physics

\[
S(b; Q) = \int_{0.5}^{Q^2} \frac{d \mu^2}{\mu^2} \left( A(\alpha_s(\mu^2)) \ln \left( \frac{Q^2}{\mu^2} \right) + B(\alpha_s(\mu^2)) \right)
\]

as \( \mu \to \infty, \ \alpha_s(-\mu) \to \infty \). PROBLEM!!!

Solution: Use a Non-Perturbative Sudakov form factor \( S_{NP} \) in the region of large \( b \) (small \( q_T \))

\[
\bar{S}(b) \sim e^{S(b)} \to e^{S(b,c)} \cdot e^{S_{NP}(b)}
\]

with \( b^* = \frac{b}{\sqrt{1 + b b_{max}^2}} \)

Note, as \( b \to \infty, b^* \to b_{max}^* \).
A Brief (but incomplete) History of Non-Perturbative Corrections

Original CSS: \[ S_{NP}^{\text{CSS}}(b) = b \left( g_1 + g_1 \ln(b_{\text{max}} Q^2) \right) \]


Davies, Webber, and Stirling (DWS): \[ S_{NP}^{\text{DWS}}(b) = b \left[ g_1 + g_1 \ln(b_{\text{max}} Q^2) \right] \]


Ladinsky and Yuan (LY): \[ S_{NP}^{\text{LY}}(b) = b \left[ g_1 + g_1 \ln(1000 \cdot 4 s^2 \cdot b) \right] + g_2 b \ln(b_{\text{max}} Q^2) \]


"BLNY": \[ S_{NP}^{\text{BLNY}}(b) = b \left[ g_1 + g_1 \ln(1000 \cdot 4 s^2 \cdot b) \right] + g_2 b \ln(b_{\text{max}} Q^2) \]


"qT resummation": \[ \tau^\text{NP}(q_T) = 1 - e^{-q_T^2} \quad \text{(not in b-space)} \]


Functional Extrapolation:


Analytical Continuation:


Recap: Where have we been???

1) We now summed the two leading logarithmic singularities, \( \alpha_s(L^2+L) \).

2) We still assumed exponentiation; but sketched ingredients of proof.

The existence of two scales \( (Q,p_T) \equiv (Q,q_T) \) yields 2 logs per loop

Use Renormalization Group + Gauge Invariance

Transformation to b-space

3) Gluon emission was assumed to be uncorrelated.

Impose momentum conservation for \( P_T \) (In b-space)

4) Introduced Non-Perturbative function for small \( q_T \) (large b) region.

Putting it all together: \[ \sigma_{\text{TOT}} = \sigma_{\text{RESUM}} + \sigma_{\text{PERT}} + \sigma_{\text{ASYM}} \]

Let's expand out the resummed expression:

\[ \frac{d\sigma}{d q_T} \sim \frac{\alpha_s L}{q_T^2} e^{\frac{\alpha_s}{2} (L^2+L)} \sim \frac{1}{q_T^2} \left\{ \alpha_s L + \alpha_s^2 (L^3+L^2+\ldots) \right\} \]

Compare the above with the perturbative and asymptotic results:

\[ d\sigma_{\text{resum}} \sim \{ \alpha_s L + \alpha_s^2 (L^3+L^2+0+0) + \alpha_s^3 (L^5+L^4)+\ldots \} \]
\[ d\sigma_{\text{pert}} \sim \{ \alpha_s L + \alpha_s^2 (L^3+L^2+1) + \alpha_s^3 (0+0) \} \]
\[ d\sigma_{\text{asy}} \sim \{ \alpha_s L + \alpha_s^2 (L^3+L^2+0+0) + \alpha_s^3 (0+0) \} \]

Note that \( \sigma_{\text{ASYM}} \) removes overlap between \( \sigma_{\text{RESUM}} \) and \( \sigma_{\text{PERT}} \).

We expect:

\( \sigma_{\text{RESUM}} \) is a good representation for \( q_T \sim 0 \)

\( \sigma_{\text{PERT}} \) is a good representation for \( q_T \sim m_W \)

I've left out A LOT of material
Putting it all together: \( \sigma_{\text{TOT}} = \sigma_{\text{RESUM}} + \sigma_{\text{PERT}} - \sigma_{\text{ASYM}} \)

\[ \sigma_{\text{RESUM}} \text{ for } q_T \sim 0 \]
\[ \sigma_{\text{PERT}} \text{ for } q_T \sim M_W \]

cross section \( d\sigma/dq_T \)

transverse momentum \( q_T \)

Let's compare with some real results

We'll look at Z data where we can measure both leptons for \( Z \rightarrow e^+e^- \)

different \( S_{NP}(b,Q) \) functions yield difference at small \( q_T \).

Let's return to the measurement of \( M_W \)
Transverse Mass Distribution

We can measure $d\sigma/dp_T$ and look for the Jacobian peak. However, there is another variable that is relatively insensitive to $p_T(W)$.

Transverse Mass

$$M_T^2(e,\nu) = (|p_{e\nu}| + |p_\nu|)^2 - (p_{e\nu} + p_\nu)^2$$

Invariant Mass

$$M^2(e,\nu) = (|p_{e\nu}| + |p_\nu|)^2 - (p_{e\nu} + p_\nu)^2$$

In the limit of vanishing longitudinal momentum, $M_T \sim M$. $M_T$ is invariant under longitudinal boosts.

$M_T$ can also be expressed as:

$$M_T^2(e,\nu) = 2|p_{e\nu}||p_\nu|(1 - \cos \Delta \phi_{e\nu})$$

For small values of $p_T(W)$, $M_T$ is invariant to leading order.

Exercise:

a) Verify the above definitions of $M_T$ are set equal.

b) For $p_{e\nu} = +p^* + p_T(W)/2$ and $p_\nu = -p^* + p_T(W)/2$; verify $M_T$ is invariant to leading order in $p_T(W)$.

Transverse Mass Distribution and $M_W$ Measurement

The Future:

- Tevatron Run II ... happening now
- LHC ... happening soon

M$_T$ distribution is much less sensitive to $P_T$ of $W$

Still, we need $P_T$ distribution of $W$ to extract mass and width with precision

PDF and $p_T(W)$ uncertainties will need to be controlled:

- Currently uncertainty: $\sim$10-15 MeV/$c^2$

Statistical precision in Run II will be miniscule... placing an enormous burden on control of modeling uncertainties.

Combined World Measurements of $M_W$

<table>
<thead>
<tr>
<th>LEP II (ee $\to$ WW)</th>
<th>World Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>80.360 +/- 0.310</td>
<td>80.436 +/- 0.037</td>
</tr>
<tr>
<td>80.410 +/- 0.180</td>
<td>80.410 +/- 0.180</td>
</tr>
<tr>
<td>80.470 +/- 0.089</td>
<td>80.470 +/- 0.089</td>
</tr>
<tr>
<td>80.433 +/- 0.079</td>
<td>80.433 +/- 0.079</td>
</tr>
<tr>
<td>80.452 +/- 0.062</td>
<td>80.452 +/- 0.062</td>
</tr>
<tr>
<td>80.427 +/- 0.046</td>
<td>80.427 +/- 0.046</td>
</tr>
<tr>
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<td>80.436 +/- 0.037</td>
</tr>
</tbody>
</table>

T. Affolder, et al. [CDF Collaboration], PRD64, 052001 (2001)

Measurement of the W boson mass with the Collider Detector at Fermilab,
Part II: Drell-Yan Process: Where have we been???

Finding the W Boson Mass:
- The Jacobian Peak, and the W Boson $p_T$
- Multiple Soft Gluon Emissions
- Single Hard Gluon Emission

Road map of Resummation
- Summing 2 logs per loop: multi-scale problem ($Q,q_T$)
- Correlated Gluon Emission
- Non-Perturbative physics at small $q_T$

Transverse Mass Distribution:
- Improvement over $p_T$ distribution

What can we expect in future?
- Tevatron Run II
- LHC

The W-Mass is an important fundamental quantity

Thanks to...
- Jeff Owens
- Chip Brock
- C.P. Yuan
- Pavel Nadolsky
- Randy Scalise
- Wu-Ki Tung
- Steve Kuhlmann
- Dave Soper
- and my other CTEQ colleagues

and the many web pages where I borrowed my figures
Attention:

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References:

Ellis, Webber, Stirling

Barger & Phillips, 2nd Edition

C.T.E.Q. Handbook

C.T.E.Q. Pedagogical Page:

C.P. Yuan, 2002

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Calculate

2003 CTEQ Summer School

Drell-Yan Process

Lecture #3

Fred Olness

C. P. Yuan

Michigan State University

June 2002

Outline

1. Parton Model
2. Factorization Theorem
3. Feynman Rules and Feynman Diagrams
4. Immediate Problems (Singularities)
5. Virtual Corrections
6. Real Emission Contribution
7. Dimensional Regularization
8. Perturbative Parton Distribution Functions

Summary of NLO [O(a_s^3)] Corrections

The sample calculations which are on the web at cteq.org under the Miscellaneous heading, and in addition lecture notes, and Mathematica notebooks of Feynarts and Feyncalc can be found at the CTEQ web site. In Lecture 1 we will examine a NLO DY calculation. We will be using the notes from C.P. Yuan (beautifully typeset by Qing-Hong Cao).
\[ \sum_{ij} \int_{1}^{2} \int_{0}^{1} \phi_{i}/h \left( x, Q_{2} \right) H_{ij} \left( Q_{2} x_{1} x_{2} S \right) \phi_{j}/h \left( x_{2}, Q_{2} \right) \]

Nonperturbative, but universal, hence, measurable

\[ \int_{Q_{0}}^{Q_{2}} \frac{dQ_{2}}{2Q_{2}} \left( \frac{\partial^{2}}{\partial Q_{2}^{2}} + i \frac{x_{2}^{(0)}}{x_{2}^{(2)}} \right) \]

Procedure:

1. Complete in QCD
2. Extract in QCD
3. Complete in QCD with all parameters
4. Use in the above equation with \( \int_{Q_{0}}^{Q_{2}} \frac{dQ_{2}}{2Q_{2}} \left( \frac{\partial^{2}}{\partial Q_{2}^{2}} + i \frac{x_{2}^{(0)}}{x_{2}^{(2)}} \right) \)
In "Cut-diagram" notation

\[
(\varepsilon - 1) \frac{g}{2} \cdot (\varepsilon - 1) \cdot \frac{m^2 \gamma^2}{2} \cdot \frac{1}{2} = \frac{p_0^2}{(0)^2}
\]

In dimensions

\[
(\varepsilon - 1) = (1 - (d-4)\varepsilon) \cdot \frac{\varepsilon}{2} \cdot (\varepsilon - 1) (\varepsilon - 1) = (1 - (d-4)\varepsilon) \cdot \frac{\varepsilon}{2} \cdot (\varepsilon - 1) (\varepsilon - 1)
\]

\[
= (1 - (d-4)\varepsilon) \cdot \frac{\varepsilon}{2} = \varepsilon - d - 4
\]

Using the Nambu-Goto prescription:

\[
\begin{array}{c}
\int \left[ \frac{\varepsilon}{2} \cdot \frac{\varepsilon}{2} \cdot \frac{\varepsilon}{2} \cdot \frac{\varepsilon}{2} \cdot \frac{\varepsilon}{2} \right]
\end{array}
\]

\[
\int \left[ (\varepsilon - 1) \varphi^2 + 2 \varphi + \frac{m^2 \gamma^2}{2} \right] \frac{1}{2} = \frac{p_0^2}{(0)^2}
\]

Reverse the form cross section in dimensions

Immediate problems (Singularities)
The result for each subprocess

In the following, I shall summarize

Subprocesses can be found in Appendix E

The detailed calculation for each

Tools needed for an NLO calculation

Calculations

Strong Coupling 2 in $n$ dimensions

$$J_{\text{hadron}} = \left(0^0\right)$$

$\mathcal{O}^\mu$: color octet

Sort of color-magnetic

$\mathcal{O}^\mu$: sort of color and magnetic

$$\left(\mathcal{O}^\mu\right)^{\mu\nu\rho} = \left(0^0\right)^{\mu\nu\rho}$$

Note: $\mu$ is the neutrino state.

$$\left(\mathcal{O}^\mu\right)^{\mu\nu\rho} = \left(0^0\right)^{\mu\nu\rho}$$

Pole remainder is free of ultraviolet singularity.

In Feynman gauge

Virtual Corrections ($\mathcal{O}^\mu$)

Real Emission Contribution ($\mathcal{O}^\mu$)
Factorization Theorem

\[
\left\{ \frac{\alpha s x}{\alpha + (\alpha s x) x + y} \bigg| \frac{\alpha s x}{\alpha + (\alpha s x) x + y} \bigg| \frac{\alpha s x}{\alpha + (\alpha s x) x + y} \right\} \frac{\delta_{\alpha x}}{\delta x} + \frac{\frac{\alpha s x}{\alpha + (\alpha s x) x + y}}{\frac{\alpha s x}{\alpha + (\alpha s x) x + y}} \frac{\delta_{\alpha x}}{\delta x} + \frac{\frac{\alpha s x}{\alpha + (\alpha s x) x + y}}{\frac{\alpha s x}{\alpha + (\alpha s x) x + y}} \frac{\delta_{\alpha x}}{\delta x} = \frac{\delta_{\alpha x}}{\delta x}
\]

\[
\frac{\partial}{\partial x} \int_{\alpha x}^{\alpha x} f(x) \, dx = f(x)
\]

where

\[
\delta_{\alpha x} = \frac{\partial}{\partial x} \int_{\alpha x}^{\alpha x} f(x) \, dx
\]

\[
\left\{ \left[ x - 1 \right] \left( y - \frac{x}{x} \right) + \left[ x - 1 \right] \left( y - \frac{1}{x} \right) \right\} \frac{\partial}{\partial x} \frac{\delta_{\alpha x}}{\delta x} + \left( \frac{\delta_{\alpha x}}{\delta x} \frac{x}{x} \right) \left( \frac{\delta_{\alpha x}}{\delta x} \frac{x}{x} \right) = \frac{\delta_{\alpha x}}{\delta x}
\]

\[
\left\{ \left[ x - 1 \right] \left( y - \frac{x}{x} \right) + \left( \frac{\alpha s x}{\alpha s x} \right) \left( x - 1 \right) \left( y - \frac{x}{x} \right) \right\} \frac{\partial}{\partial x} \frac{\delta_{\alpha x}}{\delta x} + \left( \frac{\delta_{\alpha x}}{\delta x} \frac{x}{x} \right) \left( \frac{\delta_{\alpha x}}{\delta x} \frac{x}{x} \right) = \frac{\delta_{\alpha x}}{\delta x}
\]

\[
\text{Take off the factor SIVITY in the scheme.}
\]

\[
\text{Perturbative PDF}
\]
ultraviolet signatures
strong couplings etc. to eliminate the
it is also necessary to renormalize the
strong interaction (eq. 9-1)
but when the Born level process involves

...</p>
After averaging over colors and spins
\[ \langle d \cdot t \rangle = \frac{1}{3}, \quad \langle d \cdot t \rangle = \frac{1}{3}, \quad \langle d \cdot t \rangle = \frac{1}{3}. \]

Define the Mandelstam variables
\[ (s, t, u) = \frac{1}{2} (p^2, q^2, p^2 - q^2). \]

Consider the real emission process.
There is a rigorous factorization proof...

A formal proof was constructed by numerous groups.

This proof was explicitly extended to the case of massive quarks (Collins, 1998)

THOUGH EXPERIMENT
To keep things simple, let's consider scattering off a parton target.

Application of Factorization Formula at Leading Order (LO)

Basic Factorization Formula

\[ \sigma = f \otimes \omega \otimes d + O(\Lambda^2/Q^2) \]

At Zeroth Order:

\[ \sigma^0 = f^0 \otimes \omega^0 \otimes d^0 + O(\Lambda^2/Q^2) \]

Use: \( f^0 = \delta \) and \( d^0 = \delta \) for a parton target.

Therefore:

\[ \sigma^0 = f^0 \otimes \omega^0 \otimes d^0 = \delta \otimes \omega^0 \otimes \delta = \omega^0 \]

\[ \sigma^0 = \omega^0 \]

Warning: This trivial result leads to many misconceptions at higher orders

Application of Factorization Formula at Next to Leading Order (NLO)

Combined Result:

\[ \omega^0 + \omega^1 = \sigma^0 + \sigma^1 - \{ f^1 \otimes \sigma^0 + \sigma^0 \otimes d^1 \} \]

TOT \quad HE \quad HC

SUB

\( f^1 \otimes \sigma^0 \)

\( \sigma^0 \otimes d^1 \)

Heavy Excitation

-heavy Excitation

\[ \text{TOT} = \text{HE} + \text{HC} - \text{SUB} \]
Splitting Kernel to $\alpha_s$ order

\[ \phi_{\nu, j}(x, \epsilon) = \delta(1-x) \delta_{ij} + \frac{\alpha_s}{2\pi} (1 - \frac{1}{\epsilon}) \left( \frac{\mu^2}{M^2} \right)^\epsilon \rho_{\nu, j}^{(1)}(x) \]

**HOMEWORK PROBLEM: WILSON COEFFICIENTS**

Use the Basic Factorization Formula

\[ \sigma = f \otimes \omega \otimes d + O(\Lambda^2 / Q^2) \]

**At Second Order:**

\[ \sigma^2 = f^2 \otimes \omega^0 \otimes d^0 + \ldots \]

\[ f^1 \otimes \omega^1 \otimes d^0 + \ldots \]

Therefore:

\[ \omega^2 = ???? \]

- Compute $\omega^2$ at second order.
- Make a diagrammatic representation of each term.

---

**HOMEWORK PROBLEM: CONVOLUTIONS**

Part 1) Show these 3 definitions are equivalent; work out the limits of integration.

\[ f \otimes g = \int f(x) g(y) \delta(z - x \cdot y) \, dx \, dy \]

\[ f \otimes g = \int f(x) g \left( \frac{z}{x} \right) \frac{dx}{x} \]

\[ f \otimes g = \int f \left( \frac{z}{y} \right) g(y) \frac{dy}{y} \]

Part 2) Show convolutions are the "natural" way to multiply probabilities.

If $f$ represents the heads/tails probability distribution for a single coin flip, show that the distribution of 2 coins is $f \otimes f$ and 3 coins is $f \otimes f \otimes f$.

\[ f \otimes g = \int f(x) g(y) \delta(z - (x + y)) \, dx \, dy \]

\[ f(x) = \frac{1}{2} (\delta(1-x) + \delta(1+x)) \]

**BONUS: How many processes can you think of that don't factorize?**
KLN Theorem: cancellations of soft singularities

HE = $\int f(P \rightarrow a) \otimes \sigma (a \rightarrow c)$

DGLAP Equation

$\frac{df_i}{d\log \mu^2} = \frac{\alpha_s}{2\pi} P_{j \rightarrow i} \otimes f_j + ...$

Splitting Function

$1_P g ightarrow q = \frac{1}{2} [x^2 + (1-x)^2] + \left( \frac{M_H^2}{\mu^2} \right) x(1-x)$

Mass-Independent Evolution.

Why is it valid?

In Summary:

Near threshold ($M_H \sim Q$), mass effects cancel between HE and SUB

Above threshold ($M_H \ll Q$), mass effects can be ignored

Effect of Heavy Quark Mass in the Calculation

valid near threshold ($M_H \sim Q$)

$1P$ splittings must match
Effect of Heavy Quark Mass in the Calculation is Trivial

\[ HE = \int f(P \rightarrow a) \otimes \sigma (a \rightarrow c) \]

\[ HC = \int f(P \rightarrow g) \otimes \sigma (g \rightarrow c) \]

\[ SUB = \int f(P \rightarrow g) \otimes P(g \rightarrow a) \otimes \sigma (a \rightarrow c) \]