Dimensional Regularization meets Freshman E&M

References:
Dimensional Analysis: The Pythagorean Theorem

Goal: Demonstrate $a^2 + b^2 = c^2$

Method: Dimensional Analysis:

Total Area:

$$A_c = c^2 f(\theta, \phi)$$

Total Area

$$A_a + A_b = a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

Total Area = Total Area

$$c^2 f(\theta, \phi) = a^2 f(\theta, \phi) + b^2 f(\theta, \phi)$$

$$c^2 = a^2 + b^2$$
Potential due to Infinite Line of Charge

\[
V(r) = \frac{\lambda}{4 \pi \varepsilon_0} \int \frac{1}{\sqrt{x^2 + r^2}} \, dx
\]

with \( \lambda = Q/x \) and \( dQ = \lambda \, dx \)

Scale Invariance:

\[
V(r) \text{ Function of only } \lambda \text{ and } r
\]

\[
V(kr) = \frac{\lambda}{4 \pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2 + (kr)^2}} = \frac{\lambda}{4 \pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{y^2 + r^2}} = V[r]
\]

Implies: \( V(r_1) = V(r_2) \) after rescaling: \( x = ky \)

Choices:

\begin{itemize}
  \item \( V(r) = c \) \quad \text{a finite constant}
  \item \( V(r) = \infty \) \quad \text{infinite}
  \item \( V(r) = V(r/\Lambda) \) \quad \exists \text{another length scale}
      \quad \text{(Scale invariance is broken)}
\end{itemize}
Cutoff Regularization Method

Use L as cutoff

\[ V[\mathbf{r}] = \frac{\lambda}{4 \pi \varepsilon_0} \int_{-L}^{L} \frac{1}{\sqrt{x^2 + r^2}} \, dx = \frac{\lambda}{4 \pi \varepsilon_0} \log \left[ \frac{L + \sqrt{L^2 + r^2}}{-L + \sqrt{L^2 + r^2}} \right] \]

Problem:
- \( V[\mathbf{r}] \) depends on artificial regulator L
- We cannot remove the regulator L:
  \[ \lim_{L \to \infty} V[\mathbf{r}] \]

All physical quantities are independent of the regulator!!!

Electric Field:

\[ E = -\frac{\partial V[\mathbf{r}]}{\partial \mathbf{r}} = \frac{\lambda}{2 \pi \varepsilon_0 r} \frac{L}{\sqrt{L^2 + r^2}} \]

Energy \( \sim \delta V \):

\[ \delta V = V[\mathbf{r}_1] - V[\mathbf{r}_2] = \frac{\lambda}{4 \pi \varepsilon_0} \log \left[ \frac{r_2^2}{r_1^2} \right] \]

Problem solved at the expense of an extra scale, L
BUT, we have a broken symmetry--translational invariance!!!
Dimensional Regularization Method

Idea: Compute $V[r]$ in $n$-dimensions

Use: $d^n x = d\Omega_n x^{n-1} \, dx$

$\begin{array}{|c|c|} 
\hline
n & d\Omega(n) \\
\hline
1 & 2 \\
2 & 2 \pi \\
3 & 4 \pi \\
4 & 2 \pi^2 \\
\hline
\end{array}$

$\frac{\mu^{n-1}}{\Gamma(n/2)}$

Must introduce $\mu^{n-1}$ to ensure $V[r]$ has correct dimension

$$V[r] = \frac{\lambda}{4 \pi \varepsilon_0} \int_0^\infty d\Omega[n] \frac{x^{n-1}}{\mu^{n-1}} \frac{d\mathbf{x}}{\sqrt{x^2 + r^2}}$$

Result: with $n=1-2\varepsilon$

$$V[r] = \frac{\lambda}{4 \pi \varepsilon_0} \frac{\Gamma\left[\frac{1-n}{2}\right]}{\left(\frac{r}{\mu} \sqrt{\frac{\pi}{\varepsilon}}\right)^{1-n}} = \frac{\lambda}{4 \pi \varepsilon_0} \left(\frac{\mu^{2\varepsilon}}{\pi^\varepsilon r^{2\varepsilon}}\right) \Gamma[\varepsilon]$$

Problem:

- $V[r]$ depends on artificial regulator $\mu$  
- We cannot remove the regulator $\mu$:

$$V[r] \xrightarrow[\varepsilon \to 0]{}$$

All physical quantities are independent of the regulator!!!
Dimensional Regularization Method

**All physical quantities \( R \) are independent of the regulator!!!**

\[ \Rightarrow \text{Renormalization Group Equation: } \frac{d\sigma}{d\mu} = 0 \]

**Electric Field:**

\[ E = -\frac{\partial V[r]}{\partial r} = \frac{\lambda}{4\pi\varepsilon_0} \left( \frac{2\varepsilon\mu^2}{\pi\varepsilon} \frac{\Gamma[\varepsilon]}{r^{1+2\varepsilon}} \right) \rightarrow \frac{\lambda}{2\pi\varepsilon_0 r} \]

**Energy \( \sim \delta V \):**

\[ \delta V = V[r_1] - V[r_2] = \frac{\lambda}{4\pi\varepsilon_0} \log \left[ \frac{r_2^2}{r_1^2} \right] \]

**Same as before:**

Problem solved at the expense of an extra scale, \( \mu \)

**Different from before:**

Translational invariance symmetry preserved

**Dimensional Regularization Respects Symmetries**
Expand \( V[r] \) in powers of \( \varepsilon \):

\[
V[r] = \frac{\lambda}{4 \pi \varepsilon_0} \left( \frac{1}{\varepsilon} + \ln \left[ \frac{e^{-\gamma E}}{\pi} \right] + \ln \left[ \frac{\mu^2}{r^2} \right] + O[\varepsilon] \right)
\]

Let's invent a Minimal Subtraction (MS) prescription:

\[
V_{\text{MS}}[r] = \frac{\lambda}{4 \pi \varepsilon_0} \left( \ln \left[ \frac{e^{-\gamma E}}{\pi} \right] + \ln \left[ \frac{\mu^2}{r^2} \right] + O[\varepsilon] \right)
\]

or even a Modified Minimal Subtraction (MS-bar) prescription:

\[
V_{\text{MS}}[r] = \frac{\lambda}{4 \pi \varepsilon_0} \left( + \ln \left[ \frac{\mu^2}{r^2} \right] + O[\varepsilon] \right)
\]

After renormalization, we can remove regulator \((\varepsilon \to 0)\), but we will still have \(\mu\)-dependence and scheme dependence in \( V[r] \).

Again, physical observables are independent of \( \mu \) and scheme,

\[
(V_{\text{MS}}[r_1] - V_{\text{MS}}[r_2]) = (V_{\text{MS}}[r_1] - V_{\text{MS}}[r_2]) = \delta V
\]

but only if you use a single scheme consistently.

\[
(V_{\text{MS}}[r_1] - V_{\text{MS}}[r_2]) = (V_{\text{MS}}[r_1] - V_{\text{MS}}[r_2]) \neq \delta V
\]

Mixed results introduce scheme dependence in physical observables.
Consider the basic parton factorization formula:

\[
\sigma = f \cdot \omega
\]

The renormalization group equation reads:

\[
\frac{d\sigma}{d\mu} = 0 = \frac{df}{d\mu} \omega + f \frac{d\omega}{d\mu}
\]

f and \(\omega\) depend on \(\mu\), but \(\sigma\) does not!!!

After rearrangement (and a Mellin transform):

\[
\frac{1}{\tilde{f}} \frac{d\tilde{f}}{d\ln[\mu]} = -\gamma = - \frac{1}{\tilde{\omega}} \frac{d\tilde{\omega}}{d\ln[\mu]}
\]

The anomalous dimension \(\gamma\) is the separation constant.

(If \(f\) obeyed exact scaling, \(\gamma\) would be zero--hence the term anomalous.)

The DGLAP evolution equation in moment (Mellin) space is:

\[
\frac{d\tilde{f}}{d\ln[\mu]} = -\gamma \tilde{f}
\]

or transformed back to x-space, is:

\[
\frac{df}{d\ln[\mu]} = P \cdot f
\]

The solution of the DGLAP evolution equation in moment (Mellin) space is:

\[
\tilde{f} \sim \mu^{-\gamma}
\]

(Hence, the term anomalous dimension.)
Recap

Regulator provides unique definition of $V, f, \omega$

- Cut off regulator, $L$:
  
  simple, but doesn't respect symmetries

- Dimensional regulator, $\varepsilon$:
  
  respects symmetries: translation, Lorentz, Gauge
  introduces new scale: $\mu$

All physical quantities ($E, \delta V, \sigma$) are independent of regulator

- Renormalization group equation: $d\sigma/d\mu = 0$

We can define renormalized quantities ($V, f, \omega$)

- Renormalized ($V, f, \omega$) are scheme dependent & arbitrary
- Physical quantities ($E, \delta V, \sigma$) are unique and scheme independent if we apply scheme consistently