Noncommuting Gauge Fields
as a Lagrange Fluid

R. Jackiw
MIT
Noncommuting coordinates

\[ [x^i, x^j] = i\theta^{ij} \]  

\[ (*) \]

1930  
(Heisenberg \xrightarrow{} Peierls \xrightarrow{} Pauli \xrightarrow{} Oppenheimer \xrightarrow{} Snyder)

\[ \text{ Suppress} \quad \text{Lowest Landau Level} \quad \text{First paper} \uparrow \quad \text{short-distance singularities} \]

Which coordinate transformations preserve \((*)\)?

\[ \delta x = -f(x) \quad \Rightarrow \quad f^i(x) = \theta^{ij}\partial_j f(x) \quad \Rightarrow \quad \nabla \cdot f = 0 \]

(Subgroup of) volume-preserving diffeomorphisms \(\equiv (\text{volume})'\) preserving

NB: In 2-d, (volume)' preserving \(\approx\) area preserving

**Fluid Mechanics** (Lagrange formulation, Euler formulation)

Lagrange formulation invariant against

volume-preserving diffeomorphisms

**Suggestion**  Coincidence of symmetries

leads to other similarities providing useful information

\[ \Rightarrow \text{noncommuting U}(1) \text{ gauge theory} \]

\[ \sim \text{fluid mechanics in Lagrange formulation} \]
Peierls Substitution

Consider a charged \( e \), massive \( m \) particle moving on the \( (x, y) \) plane in a constant magnetic field along the \( z \) axis

\[
B = \nabla \times A \begin{cases}
A_x = 0 \\
A_y = Bx
\end{cases}
\]

\[
L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{e}{c} \dot{x} A_x + \frac{e}{c} \dot{y} A_y - V(x, y)
\]

Landau levels \( \text{impurity} \)

separation between Landau levels \( O(B/m) \)

strong field \( (B \to \infty) \) only lowest Landau level survives

strong field \( (B \to \infty) \iff (m \to 0) \)

\[
L_0 = \frac{e}{c} B x \dot{y} - V(x, y) \sim p \dot{q} - H(p, q)
\]

\[
[x, y] = \frac{\hbar c}{eB}
\]

[G. Dunne, C. Trugenberger, & R.J., PRD 42, 661 (90); NPB 33(C) (93)]

Alternate viewpoint

\[
\langle LLL | x, y \rangle | L'L'L' \rangle = \langle LLL | xy | L'L'L' \rangle - \langle LLL | yx | L'L'L' \rangle
\]

\[
= \sum_n \left( \langle LLL | x | n \rangle \langle n | y | L'L'L' \rangle - \text{(transpose)}^* \right)
\]

\[
\text{all states} \Rightarrow [x, y] = 0
\]

\[
\text{LLL states} \Rightarrow [x, y] = -i \frac{\hbar c}{eB}
\]

\[
\text{first } N \text{ LL states} \Rightarrow [x, y] = -i \frac{\hbar c}{eB}
\]

[G. Magrò, quant-ph/0302001]
Realizing non-Commutativity

A.) Define Hilbert space on which \( x^i \) act
   (like \( p \) & \( q \) in quantum mechanics)
   Define inner product of states.
   Define trace

or

B.) Weyl-Moyal method (equivalent)

   Ignore non commutativity but replace ordinary product
   (multiplication) of functions of \( \mathbf{x} \) by "star product" *

\[
(f \star g)(\mathbf{x}) \equiv \exp \left( \frac{i}{2} \left( \epsilon^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) \right) f(\mathbf{x}) \, g(\mathbf{y}) \bigg|_{\mathbf{x}=\mathbf{y}}
\]

* product associative, but not commutative

\[
[x^i, x^j]_\star = x^i \star x^j - x^j \star x^i = i\epsilon^{ij}
\]

\[
\text{trace} \leftrightarrow \int dx
\]
Field Theory Defined on non-Commutative Space

e.g. $\varphi^4$

action:

$$ I = \int dx \left[ \frac{1}{2} \partial_\mu \varphi \star \partial^\mu \varphi - \frac{m^2}{2} \varphi \star \varphi - \frac{\lambda}{4} \varphi \star \varphi \star \varphi \star \varphi \right] $$

equation of motion:

$$ \Box \varphi + m^2 \varphi + \lambda \varphi \star \varphi \star \varphi = 0 $$

e.g. $U(1)$ gauge theory, $A_\mu$

gauge transformation:

$$ A_\mu \rightarrow A_\mu + \partial_\mu \theta - i[A_\mu, \theta] \equiv A_\mu + D^\mu \theta $$

field strength:

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] $$

gauge variant:

$$ F_{\mu\nu} \rightarrow F_{\mu\nu} - i[F_{\mu\nu}, \theta] $$

action:

$$ I = -\frac{1}{4} \int dx F^{\mu\nu} \star F_{\mu\nu} $$

equation of motion:

$$ \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] \equiv D^\mu F^{\mu\nu} = \text{sources} $$

like non-Abelian, commuting gauge theory

NO LOCAL GAUGE-INARIANT QUANTITIES

2b
Fluid mechanics in Lagrange formulation

Fluid

\[ X(t, x) \]

\[ \uparrow \text{coordinates} \]

\[ \uparrow \text{labels for fluid coordinates}, \]

\[ X(0, x) = x \text{ (comoving coordinates)} \]

Possibility of relabeling \( x \) leads to invariance against volume-preserving transformations on \( x \) : \( \delta x = -f(x) \), \( \nabla \cdot f = 0 \), provided \( X \) transforms as a scalar \( \delta_f X = f^i \partial_i X = \theta^{ij} \partial_j f \partial_i X \)

Introduce Poisson bracket with help of \( \theta^{ij} \)

(assumed to be nonsingular: \( \theta^{ij} \omega_{jk} = \delta^i_k \))

\[ \{ \mathcal{O}_1, \mathcal{O}_2 \} = \theta^{ij} \frac{\partial}{\partial x^i} \mathcal{O}_1 \frac{\partial}{\partial x^j} \mathcal{O}_2 \]

\[ \{ x^i, x^j \} = \theta^{ij} \]

\[ \delta_f X = \{ X, f \} \]

Introduce vector potential \( \hat{A} \) for evolution of \( X \)

\[ X^i(t, x) = x^i + \theta^{ij} \hat{A}_j(t, x) \]

\[ \delta_f X \Rightarrow \delta_f \hat{A} = \nabla f + \{ \hat{A}, f \} \]

(like a gauge transformation in noncommuting gauge theory)

\[ \{ X^i(t, x), X^j(t, x) \} = \theta^{ij} + \theta^{ik} \theta^{j\ell} \hat{F}_{k\ell} \]

\[ \hat{F}_{k\ell}(x) = \partial_k \hat{A}_\ell - \partial_\ell \hat{A}_k + \{ \hat{A}_k, \hat{A}_\ell \} \]

(like a field strength in noncommuting gauge theory)
(b) Seiberg-Witten map

Can replace noncommuting gauge fields by
(nonlocal) function of commuting gauge fields

\[ \hat{A} = \hat{A}(A) \]

\[ \hat{A}(A^g) = \hat{A}^{G(A,g)}(A) \Rightarrow \text{determines functional relationship between } \hat{A} \text{ and } A \]

[JHEP 9009, 032 (99)]

N.B. \( G(A, g) \) is a noncommuting 1-cocycle

[S.-Y. Pi & R.J., PLB 534, 181 (02)]

Interesting and useful for extracting physical, gauge-invariant information

e.g., plane wave solutions to noncommuting Maxwell theory

[Z. Guralnik, S.-Y. Pi, A. Polychronakos, & R.J., PLB 515, 450 (01)]

\[ \Rightarrow \hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + O(\theta^2) \]

\[ \hat{L}_{\text{EM}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8} \theta^{\alpha\beta} F_{\alpha\beta} F^{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu} + O(\theta^2) \]

equations of motion support plane waves

with Lorentz-noninvariant dispersion law
CS modification of Electromagnetism

CS: 3-d (Euclidean) \(\Rightarrow\) embed in 4-d space-time physics 
(violates Lorentz boosts, CTP, ...)
[Carroll, Field, RJ, PRD 41, 123 (90)]

Abelian gauge theory CS

\[
CS(A) = \frac{1}{4} \varepsilon^{ijk} F_{ij} A_k = \frac{1}{2} A \cdot B
\]

\[
I = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\mu}{2} A \cdot B \right)
\]

\[
= \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} v_\mu \ast F_{\mu\nu} A_\nu \right)
\]

\[
CS \text{ current} : \partial_\mu ( \ast F_{\mu\nu} A_\nu ) = \frac{1}{2} \ast F_{\mu\nu} F_{\mu\nu}
\]

\[
= \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{4} \theta \ast F_{\mu\nu} F_{\mu\nu} \right)
\]

\[
v_\mu = \partial_\mu \theta \quad \theta = \mu t
\]

Only Ampère’s law is modified

\[
-\frac{\partial E}{\partial t} + \nabla \times B = J + \mu B
\]

\(\Rightarrow\) gauge invariant, 2 polarizations, each travels with velocity 
\(\neq c\) (Lorentz boost invariance lost)
\(\neq\) each other (parity lost)

\(\Rightarrow\) Faraday rotation, light from distant galaxies shows no such effect in Nature.
(replace bracket with $-i \times \text{commutator}$)

(a) Coordinate transformations

Noncovariance problem with coordinate transformations

(in commuting gauge theory)

$$\delta f x^\mu = -f^\mu(x), \delta_f A_\mu = L_f A_\mu = f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha$$

$$\delta_f F_{\mu\nu} = L_f F_{\mu\nu} = f^\alpha \partial_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha}$$

Not covariant!

Cure \[R.J., \text{PRL} \ 41, \ 1635 \ (78)\]

$$L_f A_\mu = f^\alpha \left( \partial_\alpha A_\mu - \partial_\mu A_\alpha - i[A_\alpha, A_\mu] \right)$$

$$+ f^\alpha \partial_\mu A_\alpha + i f^\alpha [A_\alpha, A_\mu] + \partial_\mu f^\alpha A_\alpha$$

$$= f^\alpha F_{\alpha\mu} + D_\mu (f^\alpha A_\alpha)$$

$$\delta_f A_\mu = f^\alpha F_{\alpha\mu} \uparrow \text{drop gauge transformation}$$

$$\delta_f F_{\mu\nu} = f^\alpha D_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} \text{ covariant!}$$

(in noncommuting gauge theory noncovariance more severe)

using fluids as guide, find

$$\delta_f \hat{A}_\mu = \frac{1}{2} \{ f^\alpha(X), \hat{F}_{\alpha\mu} \} + \text{plus reordering terms}$$

Note: $f^\alpha$ is restricted to be either linear or volume preserving

$f^\alpha$ enters anticommutator evaluated at $X^i = x^i + \theta^{ij} \hat{A}_j$

[S.-Y. Pi and R.J., \text{PRL} \ 88, \ 111603 \ (02);]

[S.-Y. Pi, A.P. Polychronakos, and R.J., \text{Ann. Phys.} \ 301, \ 157 \ (02)]
Seiberg-Witten map from fluid mechanics
[S.-Y. Pi, A. Polychronakos, & R.J., Ann. Phys. 301, 174 (02)]

**Fluid mechanics in Euler formulation**

Euler variables:
density $\rho(t, r)$ and current $j(t, r)$ or velocity $v(t, r)$ [$j = \rho v$]
[R.J., Lectures on Fluid Dynamics (Springer, 2002)]

Relation between Euler variables and Lagrange variables

$$\rho(t, r) = \int dx \delta(X(t, x) - r)$$

$$j(t, r) = \int dx \dot{X}(t, x) \delta(X(t, x) - r)$$

(make $X$ an independent variable, called $r$;

$$\frac{1}{\rho} = \det \left. \frac{\partial X^i}{\partial x^j} \right|_{x \leftrightarrow r}$$

$$\partial_t \rho + \nabla \cdot j = 0$$

Continuity equation follows from above definitions
Derivation of Seiberg-Witten map in (2+1)-d

\[ j^\mu = \int d^2x \left( \frac{1}{\lambda} \right) \delta(X - r) \]

\[ \partial_\mu j^\mu = 0 \]

\( \varepsilon_{\mu\nu\alpha} j^\alpha \) is 2-form that satisfies Bianchi identity

\[ \propto \partial_\mu a_\nu - \partial_\nu a_\mu + \text{constant} \]

\[ \varepsilon_{ij\rho} \propto \partial_i a_j - \partial_j a_i + \text{constant} \]

Passage to noncommuting gauge theory:

\( X^i \) is operator = \( x^i + \theta^{ij} \hat{A}_j = x^i + \theta e^{ij} \hat{A}_j \)

\[ \int d^2x \text{ is } \theta \times \text{trace} \]

Need ordering – prescribe Weyl ordering by Fourier transform

\[ \int d^2r e^{ik \cdot r} (\partial_i a_j - \partial_j a_i) = -\varepsilon^{ij} \left[ \int dx e^{ik \cdot X} - (2\pi)^2 \delta(k) \right] \]

\[ = -\varepsilon^{ij} \left[ \int dx e^{ik \cdot x + \theta x e^{ij} \hat{A}_j} - (2\pi)^2 \delta(k) \right] \]

\[ \equiv \text{(inverse) Seiberg-Witten map} \]

can be extended to higher dimensions;
reproduces [Y. Okawa and H. Ooguri, PRD 64, 046009 (2001)]