HEAVY-QUARK PRODUCTION

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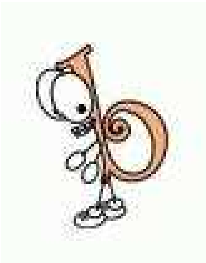
- heavy quarks: c, b and t
- e^+e^- annihilation
  - single-differential massless cross section
  - single-differential massive cross section
- the factorization theorem
- non-perturbative effects
- conclusions

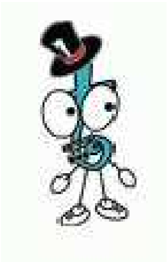
Heavy quarks

$m \gg \Lambda_{\rm QCD} \sim 250$ MeV

charm	$m_c \sim 1.5$ GeV	$m_c/\Lambda_{\rm QCD} \sim 6$	$\alpha_s(m_c) = 0.34$



bottom	$m_b \sim 5$ GeV	$m_b/\Lambda_{\rm QCD} \sim 20$	$\alpha_s(m_b) = 0.21$



top	$m_t \sim 175$ GeV	$m_t/\Lambda_{\rm QCD} \sim 700$	$\alpha_s(m_t) = 0.11$

The smaller the ratio $m/\Lambda_{\rm QCD}$, the bigger the effects of non-perturbative QCD (such as hadronization).

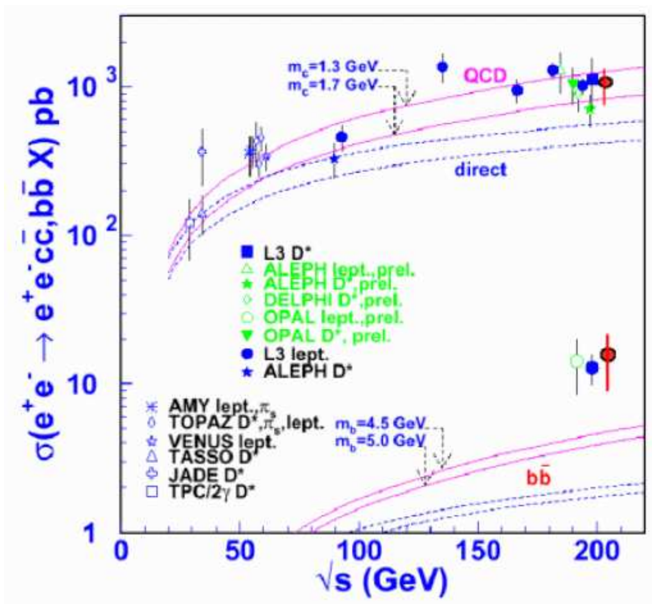
Heavy quarks

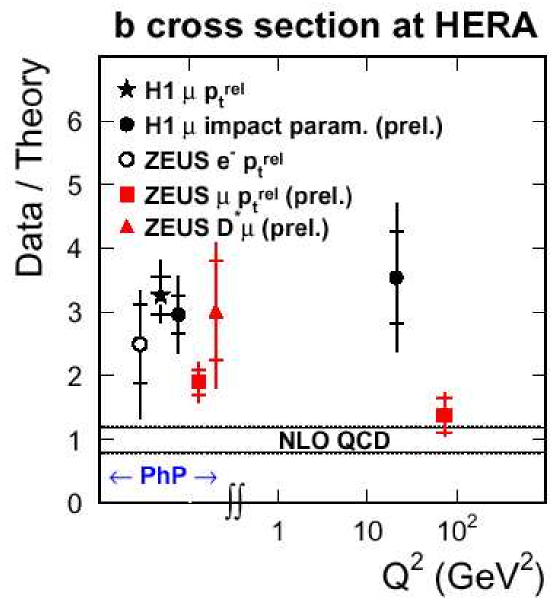
- Extensively studied at e^+e^-, hadron-hadron and photon-hadron colliders.
- Many topics are connected with heavy quarks, both in the production and in the decay mechanism.
  - CKM matrix
  - oscillation (B-$\bar{B}$ mixing) and CP violation
  - measurement of the spin (the top quark decays before hadronize and the products retain all the spin correlations)
  - intrinsic heavy-quark component in the parton distribution functions
  - study of c and b mesons and of cc and bb bound states
  - Higgs boson discovery
  - backgrounds for new physics

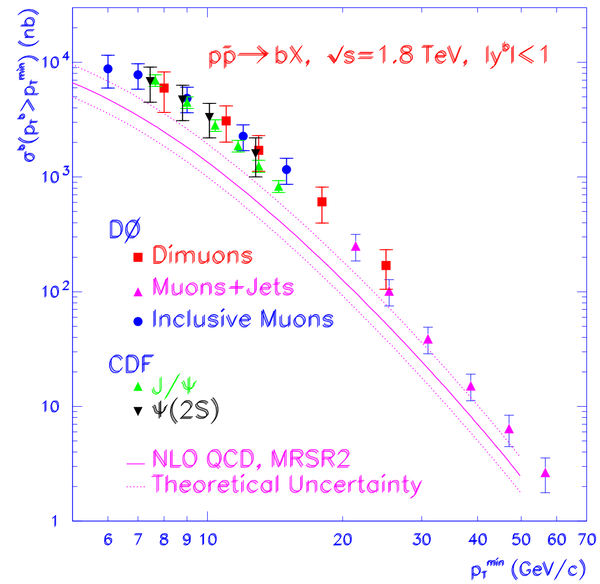
  Many of these topics are going to be discussed in this school.

In this lecture, I will address only heavy-quark production in e^+e^- annihilation, but giving you all the tools to understand the main topics connected with dealing with a massive quark. After all, this is a school.

The b-quark curse





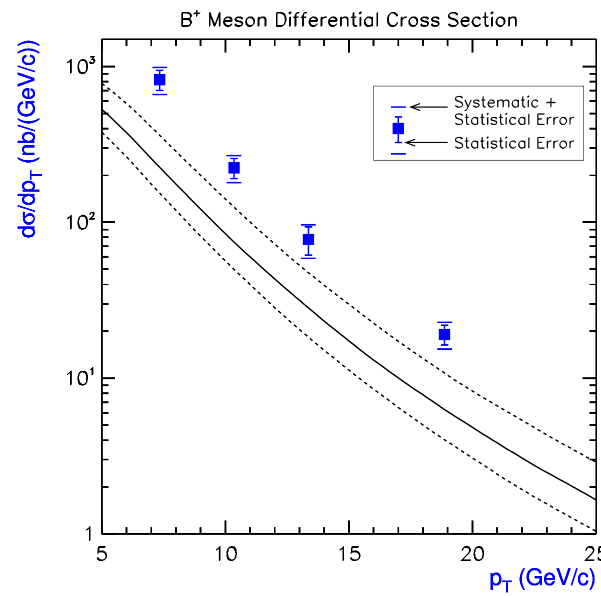


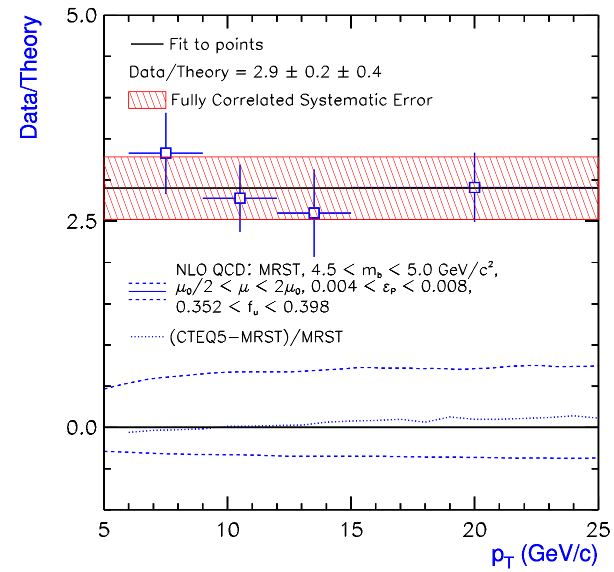
$e^+e^- \to e^+e^- + c\bar{c}\,(b\bar{b}) + X$ $\qquad$ $\gamma p \to b + X$ $\qquad$ $p\bar{p} \to b + X$

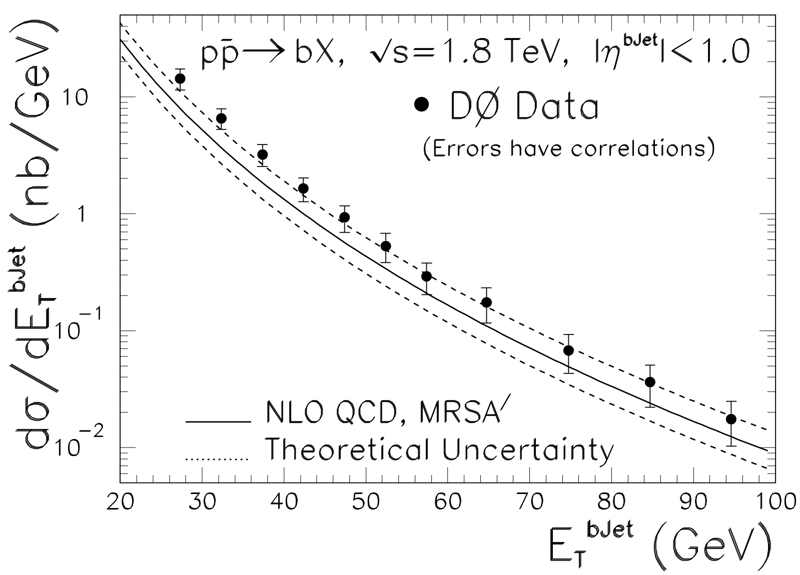
The b-quark puzzle

B^+ CDF Run I data

For about 15 years the single-inclusive p_T spectrum of the b quark, from B mesons spectrum, has been a factor $\sim$ 3 higher than the next-to-leading order prediction

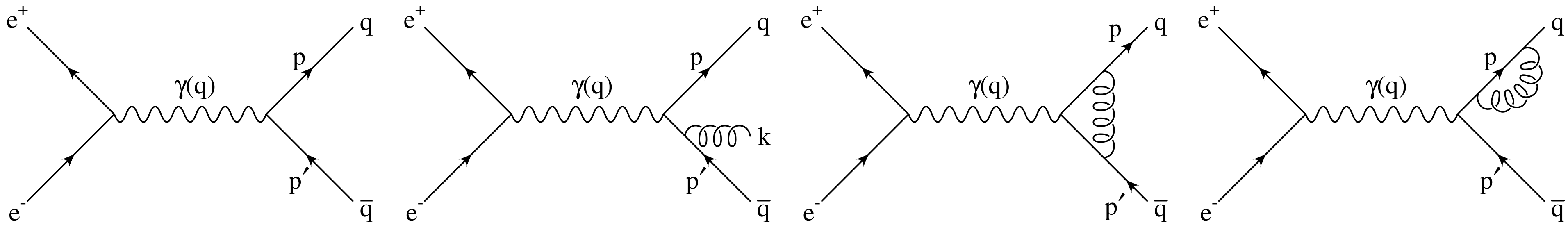






Instead, good agreement between data and theory for the p_T spectrum of b jets, that is jets that contains b quarks (any hadron species)

Differential cross-section at order α_s



$$e^+ \left(p_e'\right) + e^- \left(p_e\right) \to \gamma^*(q) \to q(p) + \bar{q}\left(p'\right) + X \qquad p_e + p_e' = q = p + p' + \ldots$$

$$M = \underbrace{\bar{v}\left(p_e'\right)\left(-i\,e\,\gamma^\mu\right)u\left(p_e\right)}_{L^\mu} \frac{-i}{q^2} H_\mu \qquad |M|^2 = M^\dagger M = \frac{1}{\left(q^2\right)^2} L_{\mu\nu} H^{\mu\nu}$$

$$L_{\mu\nu} = L_\mu^\dagger L_\nu \qquad H_{\mu\nu} = H^{\dagger\mu} H^\nu$$

Due to gauge invariance, $L^\mu q_\mu = H^\mu q_\mu = 0$ and for un-oriented quantities (only depend on q^μ)

$$L_{\mu\nu} = L\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \qquad H_{\mu\nu} = H\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$|M|^2 = \frac{1}{\left(q^2\right)^2} \frac{1}{d-1} \left(L_{\mu\nu} g^{\mu\nu}\right)\left(H_{\alpha\beta} g^{\alpha\beta}\right) \qquad d = 4 - 2\epsilon = \text{space-time dimensions}$$

Exercise: Show this.

Phase-space volume in d dimensions

$$d\sigma = \frac{1}{2E_{e^-} 2E_{e^+} |v_{e^-} - v_{e^+}|} |M|^2 d\phi_n$$

$$\begin{aligned}
d\phi_2 &= \frac{d^{d-1}p}{2\,p_0(2\pi)^{d-1}} \frac{d^{d-1}p'}{2\,p'_0(2\pi)^{d-1}} (2\pi)^d\, \delta^d(q-p-p') \\
&= \frac{1}{8\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(q^2\right)^{-\epsilon} \int_0^1 dv\,[v(1-v)]^{-\epsilon} \frac{1}{N_\phi} \int_0^\pi d\phi\,(\sin\phi)^{-2\epsilon} \qquad N_\phi = 4^\epsilon \pi \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} \\
d\phi_3 &= \frac{d^{d-1}p}{2\,p_0(2\pi)^{d-1}} \frac{d^{d-1}p'}{2\,p'_0(2\pi)^{d-1}} \frac{d^{d-1}k}{2\,k_0(2\pi)^{d-1}} (2\pi)^d\, \delta^d(q-p-p'-k) \\
&= \frac{1}{\Gamma(2-2\epsilon)} \frac{(8\pi)^{2\epsilon}}{2\,(4\pi)^3} (q^2)^{1-2\epsilon} \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \left(1-\cos^2\beta\right)^{-\epsilon} (x_1)^{-2\epsilon} (x_2)^{-2\epsilon}
\end{aligned}$$

where $v = (1-\cos\theta)/2$ (θ and ϕ are the polar angles of p with respect to a reference axis) and

$$\cos\beta = \frac{x_3^2 - x_1^2 - x_2^2}{2\,x_1\,x_2} \qquad x_1 = \frac{2p\cdot q}{q^2} \qquad x_2 = \frac{2p'\cdot q}{q^2} \qquad x_3 = \frac{2k\cdot q}{q^2} = 2 - x_1 - x_2$$

$x_1 \to 1$ when $\vec{k} \parallel \vec{p}'$, $x_2 \to 1$ when $\vec{k} \parallel \vec{p}$ and $x_1, x_2 \to 1$ when the gluon is soft.

Exercise: Check the expressions for the phase-space elements and the integration limits of the variables.

Double-differential cross section: massless case

$$\frac{d\sigma_{q\bar{q}}}{dx_1 dx_2} = 3\sigma_0 \left\{1 + C_F \frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{q^2}\right)^{2\epsilon} H(\epsilon)\left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right] + \mathcal{O}\left(\alpha_s^2\right)\right\}\delta(1-x_1)\,\delta(1-x_2)$$

$$\frac{d\sigma_{q\bar{q}g}}{dx_1 dx_2} = |M|^2_{q\bar{q}g}\, d\phi_3 \div \alpha_s \left(\frac{\mu^2}{q^2}\right)^{2\epsilon}\left\{\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} - 2\epsilon\left[\frac{x_1^2 + x_2^2 + x_3 - 1}{(1-x_1)(1-x_2)} + 1\right] + \mathcal{O}\left(\epsilon^2\right)\right\} d\phi_3$$

$$\begin{aligned}
\sigma_{q\bar{q}g} &\div \int_0^1 dx_1 \int_{1-x_1}^1 dx_2\, (1-x_1)^{-1-\epsilon}(1-x_2)^{-1-\epsilon}(x_1+x_2-1)^{-\epsilon}\left(x_1^2+x_2^2\right) \\
&= 2\int_0^1 dx_1\, x_1^{2-2\epsilon}(1-x_1)^{-1-\epsilon}\int_0^1 dt\, t^{-\epsilon}(1-t)^{-1-\epsilon} \qquad x_2 = 1 - x_1(1-t) \\
&= 2\frac{\Gamma(3-2\epsilon)\Gamma(-\epsilon)}{\Gamma(3-3\epsilon)}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{2}{\epsilon^2} + \ldots
\end{aligned}$$

where

$$\sigma_0 = \frac{4\pi\alpha^2}{3q^2} Q_f^2 \qquad\qquad H(\epsilon) = \frac{(1-\epsilon)(4\pi)^{2\epsilon}}{(3-2\epsilon)\Gamma(2-2\epsilon)} = 1 + \mathcal{O}(\epsilon)$$

- The poles in $\sigma_{q\bar{q}}$ come from the integration over the loop momentum of the virtual diagrams
- The poles in $\sigma_{q\bar{q}g}$, instead, come from the integration over the soft and collinear regions in the phase space.
- None of these poles has an ultraviolet origin. They have nothing to do with renormalization.

Total and single-differential cross section: massless case

$$\sigma_{q\bar{q}} = 3\sigma_0 \left\{ 1 + \frac{2}{3}\frac{\alpha_s}{\pi} H(\epsilon) \left(\frac{\mu^2}{q^2}\right)^{2\epsilon} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\alpha_s^2\right) \right\}$$

$$\sigma_{q\bar{q}g} = 2\sigma_0 \frac{\alpha_s}{\pi} H(\epsilon) \left(\frac{\mu^2}{q^2}\right)^{2\epsilon} \left[+\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right] + \mathcal{O}\left(\alpha_s^2\right)$$

$$\sigma_{\text{tot}} = 3\sigma_0 \left[1 + \frac{\alpha_s}{\pi} + \mathcal{O}\left(\alpha_s^2\right) \right] \Longleftarrow \text{FINITE}$$

What about the single-differential cross section $d\sigma/dx$?
Just do NOT perform the integration over x_1 and you very easily can see that

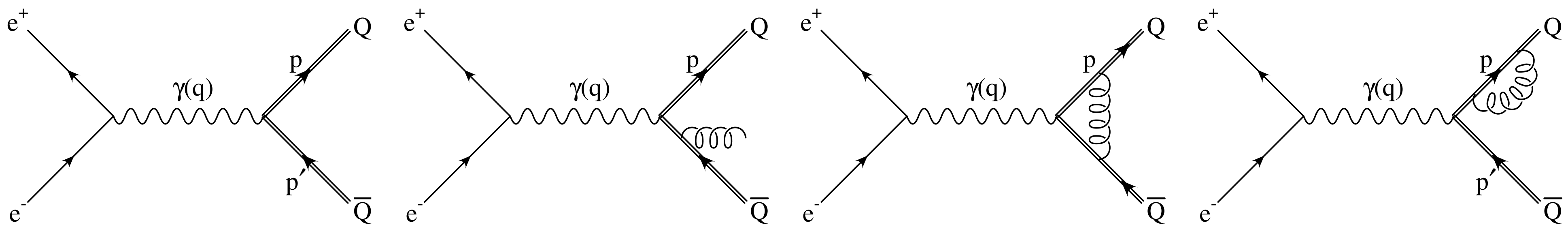
$$\boldsymbol{\frac{d\sigma}{dx}} \textbf{ is NOT FINITE}$$

The reason is that, for the massless case, the single-inclusive cross section is a quantity that is NOT infrared safe, i.e. it is sensitive to soft and collinear phenomena.

In QCD, only quantities that are infrared safe can be computed.

Exercise: Compute the total cross section at next-to-leading order in QCD for $e^+e^- \to \gamma^* \to q\bar{q}$, ignoring all the mass effects.

Single-differential cross section: massive case



$$x = \frac{2\,p\cdot q}{q^2} = \frac{E_Q}{(E_Q)_{\max}}$$

$$\frac{1}{3\sigma_0}\frac{d\sigma}{dx} = \delta(1-x) + \frac{\alpha_s(q^2)}{2\pi} C_F \left\{ 1 + \left(\ln\frac{q^2}{m^2}\right)\left(\frac{1+x^2}{1-x}\right)_+ - \left(\frac{\ln(1-x)}{1-x}\right)_+ (1+x^2) \right.$$
$$\left. +2\frac{1+x^2}{1-x}\log x + \frac{1}{2}\left(\frac{1}{1-x}\right)_+ \left(x^2-6x-2\right) + \left(\frac{2}{3}\pi^2 - \frac{5}{2}\right)\delta(1-x)\right\} + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

$$\int_0^1 \left(\frac{1}{1-x}\right)_+ f(x) \equiv \int_0^1 \left(\frac{f(x)-f(1)}{1-x}\right) \qquad \Longrightarrow \qquad \sigma = 3\sigma_0\left(1+\frac{\alpha_s}{\pi}\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

Exercise: Derive this result. Throw away, when possible, all the power effects of m^2/q^2.

Two main issues

Due to the presence of a **massive quark**, the **single-differential** cross section is now **finite**: the mass m acts as a cut-off for collinear singularities.

If one sends $m \to 0$, the single-differential cross section is no longer finite.

But being finite is not enough! Two main issues need to be addressed:

I) Is this a "well-behaved" perturbation expansion? When $m^2 \ll q^2$, then $\log(m^2/q^2)$ gets bigger and bigger

II) We do NOT "see" free quarks, even if they are heavy, but bound states (mesons and baryons). How can we incorporate these non-perturbative (hadronization) effects?

I) Well-behaved perturbative expansion

Suppose that you compute the first few terms of a perturbative expansion of a physical quantity G

$$G = 1 - 10^3\alpha_s + 5\,10^5\,\alpha_s^2 + \ldots \approx 1 - 100 + 5000 \; + \ldots$$

where we have used $\alpha_s = 1/10$.

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90%	**10%**	**0%**

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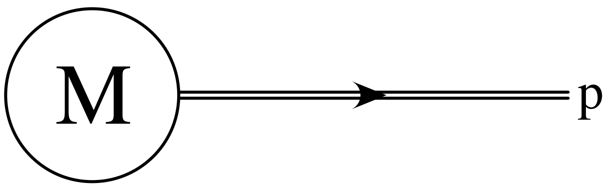
And if I tell you that the exact dependence of G from α_s is

$$G = \exp\left(-1000\,\alpha_s\right) \sim 3.7 \times 10^{-44}$$

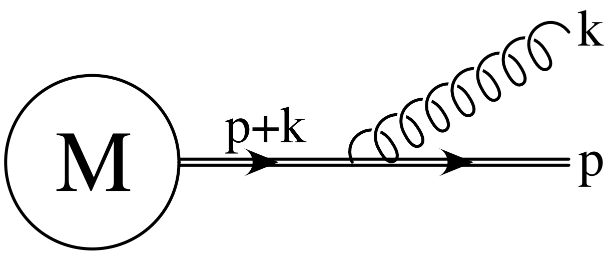
We need to **resum the entire series** to get a meaningful result!

Anatomy of a collinear singularity

Let's consider an arbitrary process M with a massive final quark, with and without gluon radiation



$$\bar{u}(p)\, M(p) \equiv \mathcal{M}_0(p)$$



$$(-i\, g_s\, t^a)\, \bar{u}(p)\, \not{\epsilon} \frac{i}{\not{p}+\not{k}-m} M(p+k) =$$

$$= g_s\, t^a \bar{u}(p)\, \not{\epsilon} \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2} M(p+k) \equiv \mathcal{M}(p,k)$$

Notice that the propagator is protected from the collinear singularity $\theta_{Qg} \to 0$ by the mass m

$$(p+k)^2 - m^2 = 2\, p \cdot k = 2\, k_0 \left(\sqrt{|\vec{p}|^2 + m^2} - |\vec{p}| \cos\theta_{Qg} \right) = 2\, k_0 |\vec{p}| \left(\sqrt{1 + \frac{m^2}{|\vec{p}|^2}} - \cos\theta_{Qg} \right)$$

The Sudakov decomposition

$$A_0(p) = \mathcal{M}_0^\dagger(p)\,\mathcal{M}_0(p) = M^\dagger(p)\,(\not p + m)\,M(p)$$

$$A(p,k) = \mathcal{M}^\dagger(p,k)\mathcal{M}(p,k) = g_s^2\,C_F\,\frac{1}{(2\,p\cdot k)^2}\,M_0^\dagger(p+k)\;N\;M_0(p+k)$$

$$N = \sum_{\text{pol}} \epsilon^{*\mu}\epsilon^\nu\,[\not p + \not k + m]\,\gamma_\mu\,(\not p + m)\,\gamma_\nu\,[\not p + \not k + m]$$

Sudakov decomposition

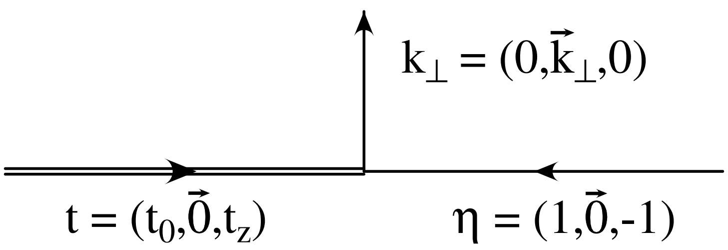
$$p^\nu = z\,t^\nu + \xi'\eta^\nu - k_\perp^\nu \qquad p^2 = t^2 = m^2 \qquad k^2 = 0 \qquad \eta^2 = 0$$

$$k^\nu = (1-z)\,t^\nu + \xi''\eta^\nu + k_\perp^\nu \qquad t\cdot k_\perp = 0 \qquad \eta\cdot k_\perp = 0 \qquad k_\perp^\nu k_{\perp\nu} = -\mathbf{k}_\perp^2$$

z is the fraction of longitudinal momentum carried by the final-state quark, with respect to a vector t that defines the collinear direction.

Sum over physical degrees of freedom (physical gauge)

$$\sum_{\text{pol}} \epsilon^{*\mu}\epsilon^\nu = -g_{\mu\nu} + \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta\cdot k}$$



The quasi-collinear limit

From $p^2 = t^2 = m^2$ and $k^2 = 0$, and defining $p + k = t + \xi\, \eta$

$$\xi' = \frac{(1-z^2)\, m^2 - k_\perp^2}{2\, z\, t\cdot\eta} \qquad\qquad \xi = \xi' + \xi'' = \frac{1}{z(1-z)} \frac{(1-z)^2\, m^2 - k_\perp^2}{2\, t\cdot\eta}$$

$$\xi'' = \frac{-(1-z)^2\, m^2 - k_\perp^2}{2\,(1-z)\, t\cdot\eta} \qquad\qquad 2\, p\cdot k = \frac{1}{z(1-z)}\left[(1-z)^2\, m^2 - k_\perp^2\right]$$

and the quasi-collinear region is defined by $k_\perp \to 0$, $m \to 0$ at fixed ratio $m^2/k_\perp^2$.
After some trivial Diracology

$$\begin{aligned} N &= 4\, p\cdot k \left\{ \left[\frac{1+z^2}{1-z} - \frac{m^2}{p\cdot k}\right] (\not{t} + m) - (1-z)m + \left[\frac{z}{1-z}\xi' - \frac{m^2}{p\cdot k}\xi\right] \not{\eta} - \frac{z}{1-z} \not{k}_\perp \right\} \\ &= 4\, p\cdot k \left[\frac{1+z^2}{1-z} - \frac{m^2}{p\cdot k}\right] (\not{t} + m) + \text{less singular terms} \end{aligned}$$

Notice that in the soft limit ($z \to 1$) we recover the exact factorization formula.

 Exercise: Check the expression for N.

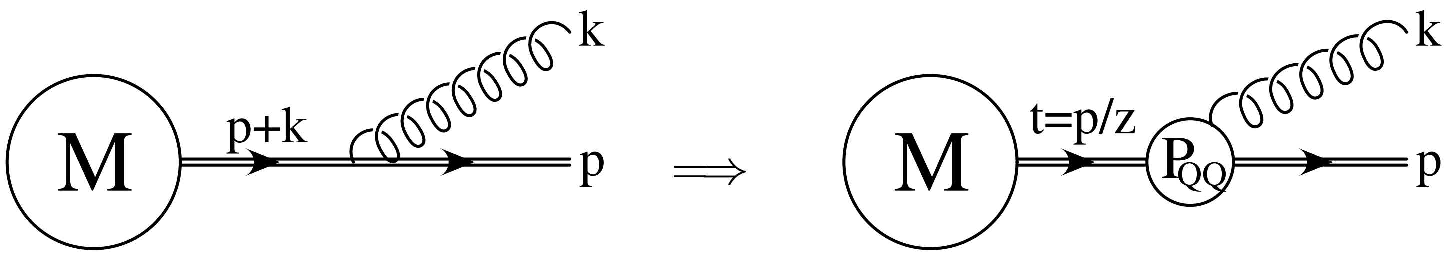
The factorized squared amplitude

$$A(p,k) = \frac{g_s^2 C_F}{(2\, p\cdot k)^2} M_0^\dagger(p+k)\; N\; M_0(p+k)$$

$$\simeq \frac{g_s^2 C_F}{p\cdot k}\left[\frac{1+z^2}{1-z} - \frac{m^2}{p\cdot k}\right] \underbrace{M_0^\dagger(t)\,(\not{t}+m)\,M_0(t)}_{A_0(t)} \simeq 8\pi\alpha_s \frac{z(1-z)}{(1-z)^2 m^2 - k_\perp^2} P_{QQ}\, A_0\left(\frac{p}{z}\right)$$

$$P_{QQ} = C_F\left[\frac{1+z^2}{1-z} - \frac{2z(1-z)m^2}{(1-z)^2 m^2 - k_\perp^2}\right]$$

Altarelli-Parisi splitting function



Notice that, due to the helicity-conserving vertex $q \to qg$, the singular behavior is proportional to $1/(p\cdot k)$ and not $1/(p\cdot k)^2$.

In the physical gauge we have used, only the square of the single Feynman diagram that exhibits the collinear singularity has to be considered. The interference of this diagrams with all the others is finite.

Exercise: Explain this.

Phase-space factorization

$$d\Phi_{n+1} = d\Phi_{n-1} \frac{d^3p}{(2\pi)^3 2p_0} \frac{d^3k}{(2\pi)^3 2k_0} \delta^4(\ldots + p + k)$$

$$p = z\,t + \xi'\eta - k_\perp \qquad k = (1-z)\,t + \xi''\eta + k_\perp \qquad \begin{cases} t = (t_0, 0, 0, t_z) \\ \eta = (1, 0, 0, -1) \\ k_\perp = (0, k_{\perp x}, k_{\perp y}, 0) \end{cases}$$

$$k_0 = (1-z)t_0 + \xi'' \qquad k_z = (1-z)t_z - \xi'' \qquad J = \left|\frac{\partial(k_0, k_z)}{\partial(z, \xi'')}\right| = \eta \cdot t$$

$$\begin{aligned} \frac{d^3k}{(2\pi)^3 2k_0} &= \frac{d^4k}{(2\pi)^3}\delta(k^2) = \frac{1}{(2\pi)^3} d^2k_\perp\, d\xi''\, dz\, J\, \delta\left[(1-z)^2 m^2 + k_\perp^2 + 2\xi''(1-z)\eta \cdot t\right] \\ &= \frac{1}{(2\pi)^3} \frac{1}{2(1-z)}\, d^2k_\perp\, dz \qquad \text{where} \qquad d^2k_\perp = dk_{\perp x}\, dk_{\perp y} \end{aligned}$$

At k fixed, we can make the change of variables $t = p + k - \xi\eta$ so that $d^3p = d^3t$

$$d\Phi_{n+1} \approx d\Phi_{n-1} \frac{d^3t}{(2\pi)^3 2p_0} \frac{d^3k}{(2\pi)^3 2k_0} \delta^4(\ldots + t) = \underbrace{d\Phi_{n-1} \frac{d^3t}{(2\pi)^3 2t_0} \delta^4(\ldots + t)}_{d\Phi_n} \frac{1}{16\pi^2} \frac{dz}{z(1-z)} d\mathbf{k}_\perp^2$$

where $d^2k_\perp = dk_{\perp x}\, dk_{\perp y}$. If there is no dependence from $\vec{k}_\perp$ direction, then we can further write $d^2k_\perp = 2\pi |k_\perp|\, d|k_\perp| = \pi\, d\mathbf{k}_\perp^2$, with $\mathbf{k}_\perp^2 = k_{\perp x}^2 + k_{\perp y}^2 = -k_\perp^2$.

Finally...

In the quasi-collinear limit, we managed to factorize both the squared amplitude and the phase space

$$d\sigma_{n+1} = \int |\mathcal{M}(p,k;\ldots)|^2 d\Phi_{n+1} = \int |\mathcal{M}_0(t;\ldots)|^2 d\Phi_n \, \frac{\alpha_s}{2\pi} \int_0^1 dz \int_0^{\mu_f^2} d\mathbf{k}_\perp^2 \, \frac{1}{\mathbf{k}_\perp^2 + (1-z)^2 m^2} \, P_{QQ}$$

$$= d\sigma_n \, \frac{\alpha_s}{2\pi} C_F \int_0^1 dz \left\{ \frac{1+z^2}{1-z} \log\left[1 + \frac{\mu_f^2}{m^2} \frac{1}{(1-z)^2}\right] + \ldots \right\}$$

where μ_f is some upper limit for the $\mathbf{k}_\perp^2$ integral. This formula can be then iterated to take into account multiple quasi-collinear emissions. The structure of the n-th term of the series, in this approximation, is then known. We can hope to resum all the series.

Notice that

- ✓ if $m = 0$ then

$$\int_0 d\mathbf{k}_\perp^2 \, \frac{1}{\mathbf{k}_\perp^2} \to \infty$$

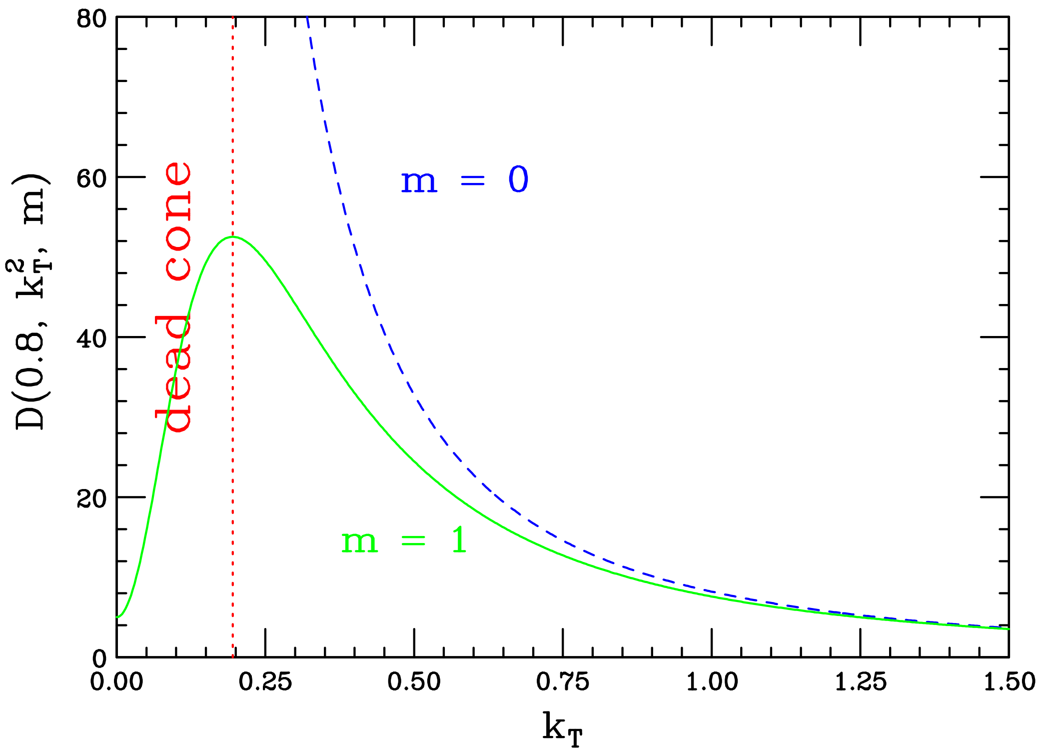
  the presence of the mass m makes the integral finite.

- ✗ the price to pay is the presence of a potentially large logarithm of (μ_f^2/m^2)

The dead cone

The "emission factor" is given by

$$D\left(z, \mathbf{k}_\perp^2, m\right) = C_F \frac{\alpha_s}{2\pi} \frac{1}{(1-z)^2 m^2 + \mathbf{k}_\perp^2} \left[\frac{1+z^2}{1-z} - \frac{2z(1-z)m^2}{(1-z)^2 m^2 + \mathbf{k}_\perp^2}\right]$$



The quark mass m suppresses the emission radiation at small $\mathbf{k}_\perp$: the dead cone

Collinear and soft logarithms

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx} = \sum_{n=0}^{\infty} a^{(n)}(x,q^2,m^2,\mu^2)\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^n$$

$$= \delta(1-x) + \frac{\alpha_s(q^2)}{2\pi}C_F\left\{1 + \left(\ln\frac{q^2}{m^2}\right)\left(\frac{1+x^2}{1-x}\right)_+ - \left(\frac{\ln(1-x)}{1-x}\right)_+(1+x^2)\right.$$

$$\left. +2\frac{1+x^2}{1-x}\log x + \frac{1}{2}\left(\frac{1}{1-x}\right)_+\left(x^2-6x-2\right) + \left(\frac{2}{3}\pi^2 - \frac{5}{2}\right)\delta(1-x)\right\} + \mathcal{O}\left(\frac{m^2}{q^2}\right)$$

- In the limit $q^2 \gg m^2$, the n-th coefficient of the series behaves as

$$a^{(n)} \sim \left(\log\frac{q^2}{m^2}\right)^n$$

  and if $\left(\alpha_s \log\frac{q^2}{m^2}\right) \approx 1$ we cannot truncate the series at some fixed order, because each term in the series is of the same order as the first one $\Longrightarrow$ we have to resum these large contributions and we know how to do this.

- Same for soft logarithms, that arise when $x = E_Q/(E_Q)_{\max} \to 1$, i.e. the gluon becomes soft.

Factorization theorem

In e^+e^- annihilation, in the limit $m^2 \ll q^2$, we can neglect terms proportional to powers of m^2/q^2, and the single inclusive heavy-quark cross section can be written as (factorization theorem)

$$\frac{d\sigma}{dx}(x, q^2, m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z, q^2, \mu_F^2\right) \hat{D}_i\left(\frac{x}{z}, \mu_F^2, m^2\right) \equiv \sum_i \frac{d\hat{\sigma}_i}{dx} \otimes \hat{D}_i$$

$\frac{d\hat{\sigma}_i}{dz}$ $\overline{\text{MS}}$-subtracted partonic cross section, for the production of the parton i (process dependent). This is the massless single-differential cross section, where the poles in ϵ (and eventually some finite part) have been removed, according to the $\overline{\text{MS}}$ prescription

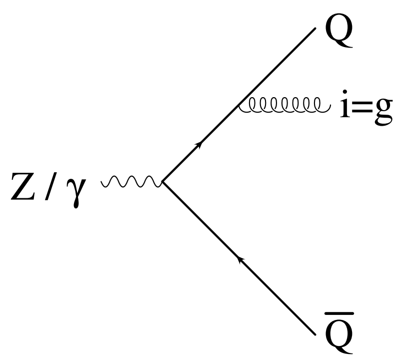
$\hat{D}_i$ $\overline{\text{MS}}$ fragmentation functions for the parton i to "fragment" into the heavy quark Q (process independent)

μ_F factorization scale

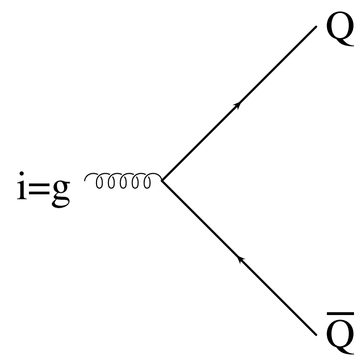
Example

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z,q^2,\mu_F^2\right) \hat{D}_i\left(\frac{x}{z},\mu_F^2,m^2\right)$$

if $i = g$ then



$$\equiv \frac{d\hat{\sigma}_i}{dz} \otimes \hat{D}_i \equiv$$



$m = 0$!

process dependent

$m \neq 0$

process independent

Resummation of collinear logs

$$\frac{d\sigma}{dx}(x,q^2,m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz} \left(z,q^2,\mu_F^2\right) \hat{D}_i \left(\frac{x}{z},\mu_F^2,m^2\right)$$

The logs of (m^2/q^2) in $d\sigma/dx$ are then split into logs of (q^2/μ_F^2) in $d\hat{\sigma}_i/dz$ and into logs of (m^2/μ_F^2) in $\hat{D}_i$.

Important question: **how do we choose μ_F?**

If $\mu_F^2 \approx q^2$ then no large logarithms of q^2/μ_F^2 appear in $d\hat{\sigma}_i/dz$ and its perturbative expansion is reliable.
The large logarithms are moved into the fragmentation functions $\hat{D}_i$.
It seems that the problems caused by the presence of large logs are just shifted, and not solved!

But this is not true. In fact…

DGLAP evolution equations

…the fragmentation functions $\hat{D}_i$ obey the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equations

$$\frac{d\hat{D}_i}{d\log\mu^2}(x,\mu^2,m^2) = \sum_j \int_x^1 \frac{dz}{z} P_{ji}\left(\frac{x}{z},\bar{\alpha}_s(\mu^2)\right) \hat{D}_j(z,\mu^2,m^2) = \sum_j P_{ji} \otimes \hat{D}_j$$

where the splitting functions P_{ji} have the expansion ($\bar{\alpha}_s = \alpha_s/(2\pi)$)

$$P_{ji}\Big(x,\bar{\alpha}_s(\mu)\Big) = \bar{\alpha}_s(\mu)P_{ji}^{(0)}(x) + \bar{\alpha}_s^2(\mu)P_{ji}^{(1)}(x) + \bar{\alpha}_s^3(\mu)P_{ji}^{(2)}(x) + \mathcal{O}\left(\alpha_s^4\right)$$

$P_{ji}^{(0)}$ is the zeroth order Altarelli-Parisi splitting function (previously we have computed $P_{qq} = P_{QQ}(m=0)$).

CLAIM: the DGLAP equations resum correctly all the large logarithms of (m^2/μ^2)

You can convince yourself that this is true by iterating the previous equation.

Proof by iteration

I will drop all the indexes, for ease of notation, and remind you that P starts at order α_s

$$d\hat{D}(\mu^2) = P \otimes \hat{D}(\mu^2)\, d\log\mu^2$$

Upon integrating between μ_0 (small scale) and μ (large scale)

$$\hat{D}(\mu^2) = \hat{D}(\mu_0^2) + \int_{\mu_0^2}^{\mu^2} P \otimes \hat{D}(\mu_1^2)\, d\log\mu_1^2$$

$$= \hat{D}(\mu_0^2) + \int_{\mu_0^2}^{\mu^2} P \otimes \left[\hat{D}(\mu_0^2) + \int_{\mu_0^2}^{\mu_1^2} P \otimes \hat{D}(\mu_2^2)\, d\log\mu_2^2\right] d\log\mu_1^2$$

$$= \hat{D}(\mu_0^2) + P \otimes \hat{D}(\mu_0^2) \log\frac{\mu^2}{\mu_0^2} + \int_{\mu_0^2}^{\mu^2} P \otimes \int_{\mu_0^2}^{\mu_1^2} P \otimes \hat{D}(\mu_2^2)\, d\log\mu_2^2\, d\log\mu_1^2$$

$$= \hat{D}(\mu_0^2) + P \otimes \hat{D}(\mu_0^2) \log\frac{\mu^2}{\mu_0^2} + \int_{\mu_0^2}^{\mu^2} P \otimes \int_{\mu_0^2}^{\mu_1^2} P \otimes \left[\hat{D}(\mu_0^2) + \int_{\mu_0^2}^{\mu_2^2} P \otimes \hat{D}(\mu_3^2)\, d\log\mu_3^2\right] d\log\mu_2^2\, d\log\mu_1^2$$

$$= \hat{D}(\mu_0^2) + P \otimes \hat{D}(\mu_0^2) \log\frac{\mu^2}{\mu_0^2} + P \otimes P \otimes \hat{D}(\mu_0^2)\, \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2}$$

$$+ \int_{\mu_0^2}^{\mu^2} P \otimes \int_{\mu_0^2}^{\mu_1^2} P \otimes \int_{\mu_0^2}^{\mu_2^2} P \otimes \hat{D}(\mu_3^2)\, d\log\mu_3^2\, d\log\mu_2^2\, d\log\mu_1^2$$

$$= \hat{D}(\mu_0^2) + P \otimes \hat{D}(\mu_0^2) \log\frac{\mu^2}{\mu_0^2} + P \otimes P \otimes \hat{D}(\mu_0^2)\, \frac{1}{2}\log^2\frac{\mu^2}{\mu_0^2} + P \otimes P \otimes P \otimes \hat{D}(\mu_0^2)\, \frac{1}{3!}\log^3\frac{\mu^2}{\mu_0^2} + \ldots$$

Leading Log, Next-to-Leading Log...

Introducing the shorthand notation $L = \log\left(m^2/q^2\right)$

✓ with an expansion for P_{ji} up to order α_s, we resum all terms of the form

$$\underbrace{\left.\frac{d\sigma}{dx}(x,q^2,m^2)\right|_{\text{LL}} = \sum_{n=0}^{\infty} \beta^{(n)}(x)\ (\bar{\alpha}_s L)^n}_{(\bar{\alpha}_s L)^n \implies \text{Leading Log}}$$

✓ with an expansion for P_{ji} up to order α_s^2, we resum all terms of the form

$$\underbrace{\left.\frac{d\sigma}{dx}(x,q^2,m^2)\right|_{\text{NLL}} = \sum_{n=0}^{\infty} \beta^{(n)}(x)\ (\bar{\alpha}_s L)^n + \sum_{n=0}^{\infty} \gamma^{(n)}(x)\, \bar{\alpha}_s\, (\bar{\alpha}_s L)^n}_{\bar{\alpha}_s\, (\bar{\alpha}_s L)^n \implies \text{Next-to-Leading Log}}$$

✓ ...so on, so forth.

The time-like splitting functions are known up to the third order [Mitov, Moch and Vogt, hep-ph/0604053]

Initial condition

One piece of information is still missing: $\hat{D}_i$ obey an integral-differential equation. What is the initial condition for $\hat{D}_i(x, \mu^2, m^2)$?

$$\hat{D}_i(x, \mu^2, m^2) = d_i^{(0)} \delta(1-x) + \bar{\alpha}_s(\mu^2)\, d_i^{(1)}(x, \mu^2, m^2) + \mathcal{O}\left(\alpha_s^2\right) \ .$$

Use the factorization theorem

$$\frac{d\sigma}{dx}(x, q^2, m^2) = \sum_i \int_x^1 \frac{dz}{z} \frac{d\hat{\sigma}_i}{dz}\left(z, q^2, \mu^2\right) \hat{D}_i\left(\frac{x}{z}, \mu^2, m^2\right)$$

and match the expansion up to order α_s of the left- and right-hand side to obtain $d_i^{(0)}$ and $d_i^{(1)}$ [Mele and Nason, *Nucl. Phys.* **B361** (91) 626]

$$\begin{aligned}
\frac{d\sigma}{dx}(x, q^2, m^2) &= a^{(0)}(x, q^2, m^2) + a^{(1)}(x, q^2, m^2, \mu^2)\, \bar{\alpha}_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right) \\
\frac{d\hat{\sigma}_i}{dx}(x, q^2, \mu^2) &= \hat{a}_i^{(0)}(x) + \hat{a}_i^{(1)}(x, q^2, \mu^2)\, \bar{\alpha}_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right)
\end{aligned}$$

The $d_i^{(2)}$ terms are known too, and have been computed by [Melnikov and Mitov, `hep-ph/0404143`; Mitov, `hep-ph/0410205`], following a different strategy.

Final recipe for collinear-log resummation

✓ start with $\hat{D}_i(x, \mu_0^2, m^2)$, with $\mu_0^2 \approx m^2$, so that no large logarithms of the ratio μ_0^2/m^2 appear in the initial conditions

$$\hat{D}_i(x, \mu_0^2, m^2) = d_i^{(0)} \delta(1-x) + \bar{\alpha}_s(\mu_0^2)\, d_i^{(1)}(x, \mu_0^2, m^2) + \mathcal{O}\left(\alpha_s^2\right)$$

✓ evolve $\hat{D}_i(x, \mu_0^2, m^2)$ from the low to the high energy scale μ with the DGLAP equation to obtain $\hat{D}_i(x, \mu^2, m^2)$

$$\frac{d\hat{D}_i}{d \log \mu^2} = \sum_j P_{ji} \otimes \hat{D}_j$$

✓ use the factorization theorem to compute the resummed cross section.

$$\frac{d\sigma}{dx} = \sum_i \frac{d\hat{\sigma}_i}{dx} \otimes \hat{D}_i$$

Soft logarithms

In the region of the phase space of multiple soft-gluon emission ($x \to 1$), the differential cross section contains enhanced terms proportional to

$$a^{(n)} \approx c^{(n)}_{\rm LL} \left(\frac{\log^{2n-1}(1-x)}{1-x} \right)_+ + c^{(n)}_{\rm NLL} \left(\frac{\log^{2n-2}(1-x)}{1-x} \right)_+ + \ldots$$

These terms can be organized in towers of $\log N$, where we introduce the Mellin transform

$$f(N) = \int_0^1 dx\, x^{N-1} f(x) \qquad \Longrightarrow \qquad \int_0^1 dx\, x^{N-1} \left(\frac{\log^k(1-x)}{1-x} \right)_+ \approx \log^{k+1} N$$

The large-N contributions come from the regions where $x \to 1$, associated to the bremsstrahlung spectrum of soft and collinear emission.

Up to now, it is known how to resum all the Leading Log and Next-to-Leading Log [Dokshitzer, Khoze and Troyan, `hep-ph/9506425`; Cacciari and Catani, `hep-ph/0107138`]

$$\sum_{n=0}^{\infty} c^{(n)}_{\rm LL}\, \alpha_s^n \log^{n+1} N = \log N\, g_{\rm LL}(\alpha_s \log N) \qquad \sum_{n=0}^{\infty} c^{(n)}_{\rm NLL}\, \alpha_s^n \log^{n} N = g_{\rm NLL}(\alpha_s \log N)$$

II) Non-perturbative effects

✗ The weak point of the factorization theorem comes from the initial condition for the evolution of the fragmentation function, which is computed as a power expansion in terms of $\alpha_s(m)$: irreducible, non-perturbative uncertainties of order $\Lambda_{\rm QCD}/m$ are present.

✗ The soft-gluon resummation functions g_{LL} and g_{NLL} contain singularities at large N which signal the eventual failure of perturbation theory and hence the onset of non-perturbative phenomena.

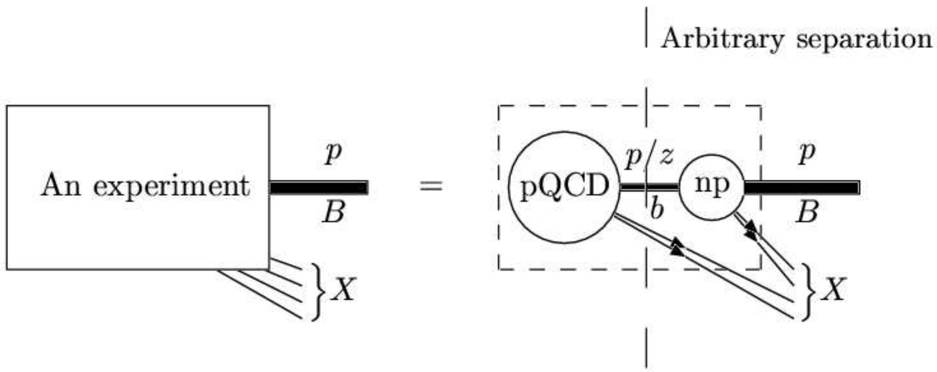
- in the initial condition, the region $(1-x)\,m \approx \Lambda$ ($m/N \approx \Lambda$ in moment space) is sensitive to the decay of excited states of the heavy-flavoured hadrons, where Λ is a typical hadronic scale of a few hundreds MeV.
- in the coefficient functions, when $(1-x)\,q^2 \approx \Lambda^2$, the mass of the recoil system approaches typical hadronic scales.

The matching of perturbative results with non-perturbative physics is a delicate problem, which rests, first of all, on a proper definition of the perturbative series.

Non-perturbative fragmentation function

We assume that all these effects are described by a non-perturbative fragmentation function $D_{\rm NP}^{H}$, that takes into account all low-energy effects, including the process of the heavy quark turning into a heavy-flavoured hadron. The full resummed cross section, including non-perturbative corrections, is then written as

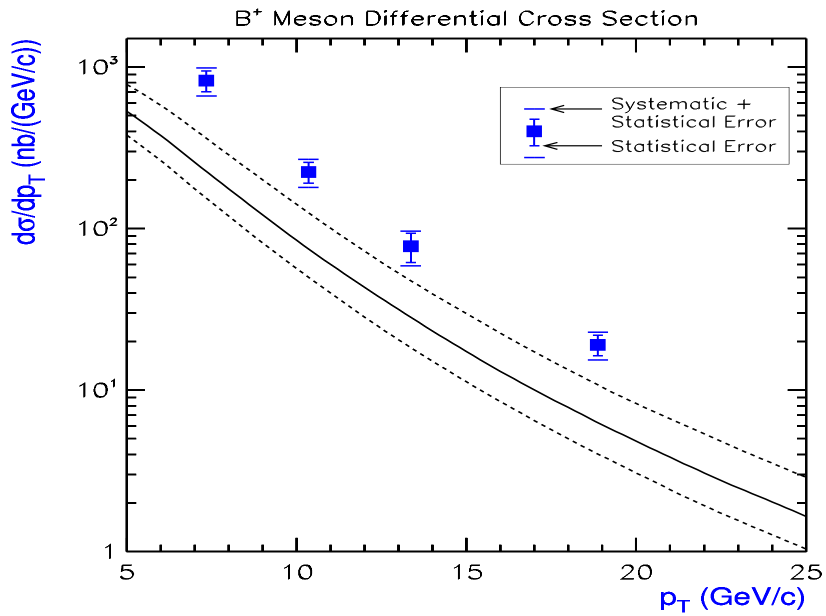
$$\frac{d\sigma^H}{dx}(x,q^2) \quad = \quad \sum_i \frac{d\hat{\sigma}_i}{dx}\left(x,q^2,\mu_F^2\right) \otimes \hat{D}_i\left(x,\mu_F^2,m^2\right) \otimes D_{\rm NP}^{H}(x)$$

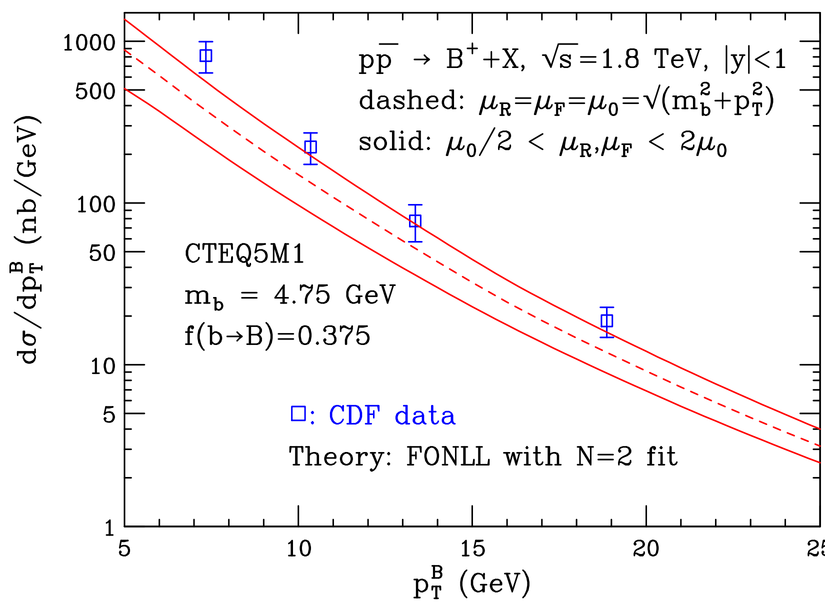


The non-perturbative part $D_{\rm NP}^{H}$ is what is missing to go from the partonic cross section to the hadronic one $\Longrightarrow$ very sensitive to the perturbative part

It is expected to be universal and independent from the production mechanism (short-distance) $\Longrightarrow$ extract it from e^+e^- data (clean environment) and use it in hadronic (messy environment) heavy-quark production.

The b-quark solution





Solution found by [Cacciari and Nason, hep-ph/0204025]. Most of the old data were given in terms of b quark. Information about the deconvolution $B \to b$ lost in the analysis.

- ✓ use the appropriate non-perturbative fragmentation function (45%)
- ✓ use a resummed formalism, matched with a NLO calculation (20%)
- ✓ data/theory from $2.9 \pm 0.2 \pm 0.4$ to $1.7 \pm 0.5 \pm 0.5$

Conclusions

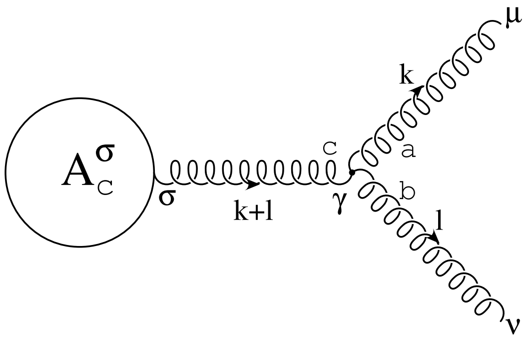
✓ While introducing you to the production of heavy quarks

- I've shown you what happens when an additional scale (the quark, mass in this case) enters a perturbative expansion: it may undermine the convergence of the expansion itself.
- The solution is to resum classes of large contributions, but you need to know the structure of these terms at all orders.
- Using the fact that collinear singularities factorize, we managed to write a factorization theorem and to resum logs of (m^2/q^2)
- And I've shown you how a long-standing puzzle (b-quark production) has been successfully solved.

✓ I've left you several problems to work out, and more in the next slides. Please go through them by yourself. This is the only way to really understand the physics behind. You have plenty of time during this school and you can work out these problems in small groups and cross check between each others.

✓ If you have questions, come and ask me. Do not be shy. I bark but don't byte! I'll be around till the end of the school.

$g \to gg$ and $g \to q\bar{q}$



$$k^\mu = z\, t^\mu + \xi' \eta^\mu + k_\perp^\mu$$
$$l^\mu = (1-z)\, t^\mu + \xi'' \eta^\mu - k_\perp^\mu$$
$$k^2 = l^2 = t^2 = \eta^2 = \eta \cdot \epsilon(k) = \eta \cdot \epsilon(l) = 0$$

$$\mathcal{M}^{ab} = \left\{ A_c^\sigma(l+k)\, \frac{i P_{\sigma\gamma}(k+l)}{(k+l)^2} \,(-g_s)\, f^{abc}\, \Gamma^{\mu\nu\gamma}(-k,-l,k+l) + \mathcal{R}_{ab}^{\mu\nu} \right\} \epsilon_\mu(k)\, \epsilon_\nu(l)$$

$$\Gamma^{\mu\nu\gamma}(-k,-l,k+l) = (-k+l)^\gamma g^{\mu\nu} + (-2l-k)^\mu g^{\nu\gamma} + (2k+l)^\nu g^{\mu\gamma}$$
$$P^{\sigma\gamma}(t) = -g^{\sigma\gamma} + \frac{\eta^\sigma t^\gamma + \eta^\gamma t^\sigma}{\eta \cdot t} \equiv -g_\perp^{\sigma\gamma}$$

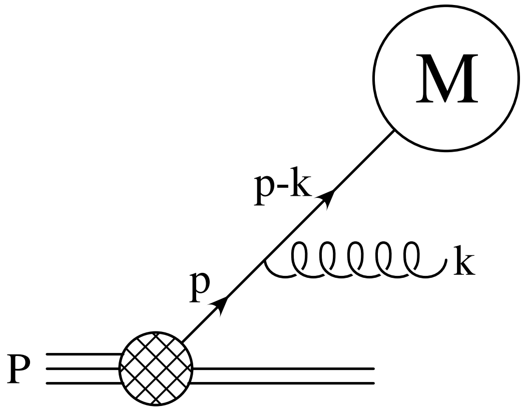
Squaring and summing over the colors and spins of the final gluons ($d = 4 - 2\epsilon$)

$$\sum_{\text{col,spin}} \mathcal{M}^{ab} \mathcal{M}_{ab}^\dagger \simeq \frac{g_s^2}{2\, l \cdot k}\, 4\, C_A \left\{ -\left[-2 + \frac{1}{z} + \frac{1}{1-z} + z\,(1-z) \right] g_{\sigma\sigma'} \right.$$
$$\left. - 2\, z\,(1-z)(1-\epsilon) \left[\frac{k_{\perp\sigma} k_{\perp\sigma'}}{k_\perp^2} - \frac{g_{\perp\sigma\sigma'}}{2 - 2\,\epsilon} \right] \right\} A_c^\sigma(t) A_c^{\dagger\sigma'}(t)$$

Exercise: Check this expression of the Altarelli-Parisi splitting function P_{gg} for $g \to gg$ and derive P_{gq}, that describes $g \to q\bar{q}$ (massless quark)

Initial-state radiation

All masses set to zero. But you can keep finite quark mass.



$$\mathcal{M}_0(p) = M(p)u(p)$$

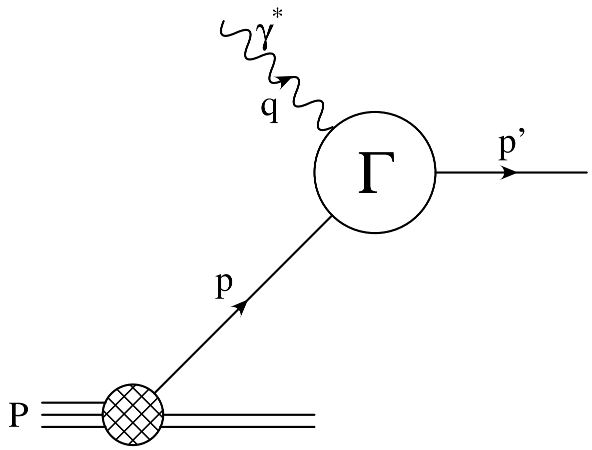
$$\mathcal{M}(p,k) = (-ig_s)\, t^a\, M(p-k)\frac{i}{\not{p}-\not{k}}\not{\epsilon}\, u(p)$$

$$\sigma_0(p) = \frac{N}{2p_0} M^\dagger(p)\not{p}M(p) \qquad N = \text{normalization factor}$$

$$\sigma_g = \frac{\alpha_s}{2\pi}\int dz\, \sigma_0(zp)\, \underbrace{C_F\frac{1+z^2}{1-z}}_{P_{qq}} \int_0 \frac{d\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2}$$

Exercise: Derive the factorization formulae for initial-state collinear singularities

Parton distribution functions I



with $p = x\,P$, P the momentum of the incoming hadron and $f_i(x)dx$ is the probability to find a quark of flavor i, in the incoming hadron, with momentum in the range between xP and $(x+dx)P$

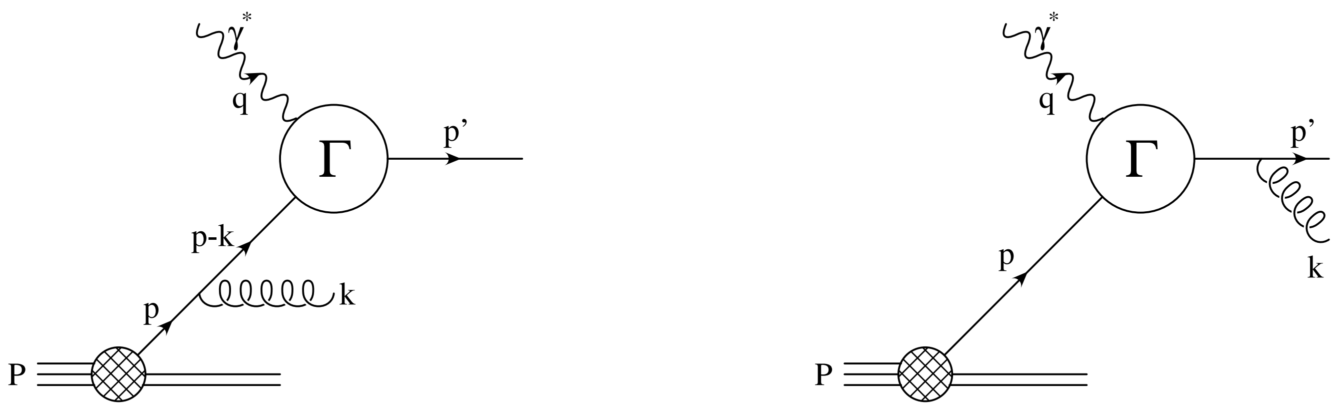
$$\hat{\sigma}_0\left(\gamma^* q_i(p) \to q_i(p')\right) = \frac{1}{\text{flux}} \overline{\sum} \left|M_0\right|^2 \frac{d^3 p'}{(2\pi)^3 2p'_0} (2\pi)^4 \delta^4\left(p' - q - p\right)$$

$$= \frac{2\pi}{\text{flux}} \frac{\overline{\sum}\left|M_0\right|^2}{Q^2} x\,\delta(x - x_{\text{Bj}})\,, \qquad x_{\text{Bj}} \equiv \frac{Q^2}{2P\cdot q}\,, \qquad Q^2 \equiv -q^2$$

$$\sigma_0 = \int dx \sum_i f_i(x)\,\hat{\sigma}_0 = \frac{2\pi}{\text{flux}} \frac{\overline{\sum}\left|M_0\right|^2}{Q^2} \sum_i x_{\text{Bj}}\, f_i(x_{\text{Bj}})$$

Exercise: Derive the expressions for the partonic ($\hat{\sigma}_0$) and hadronic (σ_0) cross sections

Parton distribution functions II



$$\hat{\sigma}_g\left(\gamma^* q_i(p) \to q_i(p')g(k)\right) = \frac{1}{\text{flux}} \overline{\sum} \left|M_g\right|^2 \frac{d^3p'}{(2\pi)^3 2p'_0} \frac{d^3k}{(2\pi)^3 2k_0} (2\pi)^4 \delta^4\left(p' + k - q - p\right)$$

$$\approx \frac{2\pi}{\text{flux}} \int dz \int \frac{d\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2} \frac{\alpha_s}{2\pi} P_{qq}(z) \overline{\sum} \left|M_0\right|^2 2\pi \underbrace{\frac{x_{\text{Bj}}}{zQ^2} \delta\left(x - \frac{x_{\text{Bj}}}{z}\right)}_{\delta\left(p'^2\right)}$$

$$\sigma_g = \int dx \sum_i f_i(x)\, \hat{\sigma}_g \approx \frac{2\pi}{\text{flux}} \frac{\overline{\sum}\left|M_0\right|^2}{Q^2} \sum_i x_{\text{Bj}} \frac{\alpha_s}{2\pi} \int \frac{d\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2} \int_{x_{\text{Bj}}}^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x_{\text{Bj}}}{z}\right)$$

Exercise: Show (again) that in a physical gauge (where only physical degrees of polarization propagate) only the square of the first diagram is collinear divergent, and derive the previous expressions.

DGLAP I

At order α_s, the contribution to the total cross section where only the enhanced collinear terms have been included, is

$$\begin{aligned}\sigma &\approx \sigma_0 + \sigma_g \\ &= \frac{2\pi}{\text{flux}} \frac{\overline{\sum}\left|M_0\right|^2}{Q^2} \sum_i x_{\text{Bj}} \left\{ f_i(x_{\text{Bj}}) + \frac{\alpha_s}{2\pi} \int \frac{d\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2} \int_{x_{\text{Bj}}}^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x_{\text{Bj}}}{z}\right) \right\}\end{aligned}$$

Let's define the "renormalized" parton-distribution function (pdf)

$$f_i\left(x, \mu_F^2\right) \equiv f_i(x) + \frac{\alpha_s}{2\pi} \int_{\mu_0^2}^{\mu_F^2} \frac{d\mathbf{k}_\perp^2}{\mathbf{k}_\perp^2} \int_x^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x}{z}\right)$$

that must be finite, since we actually measure total cross sections!

μ_0 is a lower cut-off scale (some hadronic scale).

The "price" to pay is that the pdf is now scale dependent. This is also known as scaling violation.

DGLAP II

But mostly important

$$\frac{\partial f_i\left(x,\mu_F^2\right)}{\partial \log \mu_F^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\, P_{qq}(z)\, f_i\left(\frac{x}{z}\right)$$

that at order α_s is equivalent to

$$\frac{\partial f_i\left(x,\mu_F^2\right)}{\partial \log \mu_F^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\, P_{qq}(z)\, f_i\left(\frac{x}{z},\mu_F^2\right) = \frac{\alpha_s}{2\pi} P_{qq}\otimes f_i$$

Now you can add by yourself all the other splitting processes and get

$$\begin{aligned}
\frac{\partial q_i}{\partial \log \mu_F^2} &= \frac{\alpha_s}{2\pi}\left(P_{qq}\otimes q_i + P_{qg}\otimes g\right) \\
\frac{\partial g}{\partial \log \mu_F^2} &= \frac{\alpha_s}{2\pi}\left(P_{gg}\otimes g + \sum_i P_{gq}\otimes q_i\right)
\end{aligned}$$

where q_i and g are the pdf of the quark of flavor i and of the gluon, respectively.

Exercise: Derive all these expressions.