The physics of parton showers

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The aim of these lectures

- How can we understand the evolution of a parton shower? What is the underlying physics?
- I will concentrate on evolution equations.
- My analysis follows work with Zoltan Nagy.
- I will say little about computer algorithms to implement these equations.
  - In fact, the general shower evolution equation is beyond what one can efficiently implement.

What do parton shower event generators do?

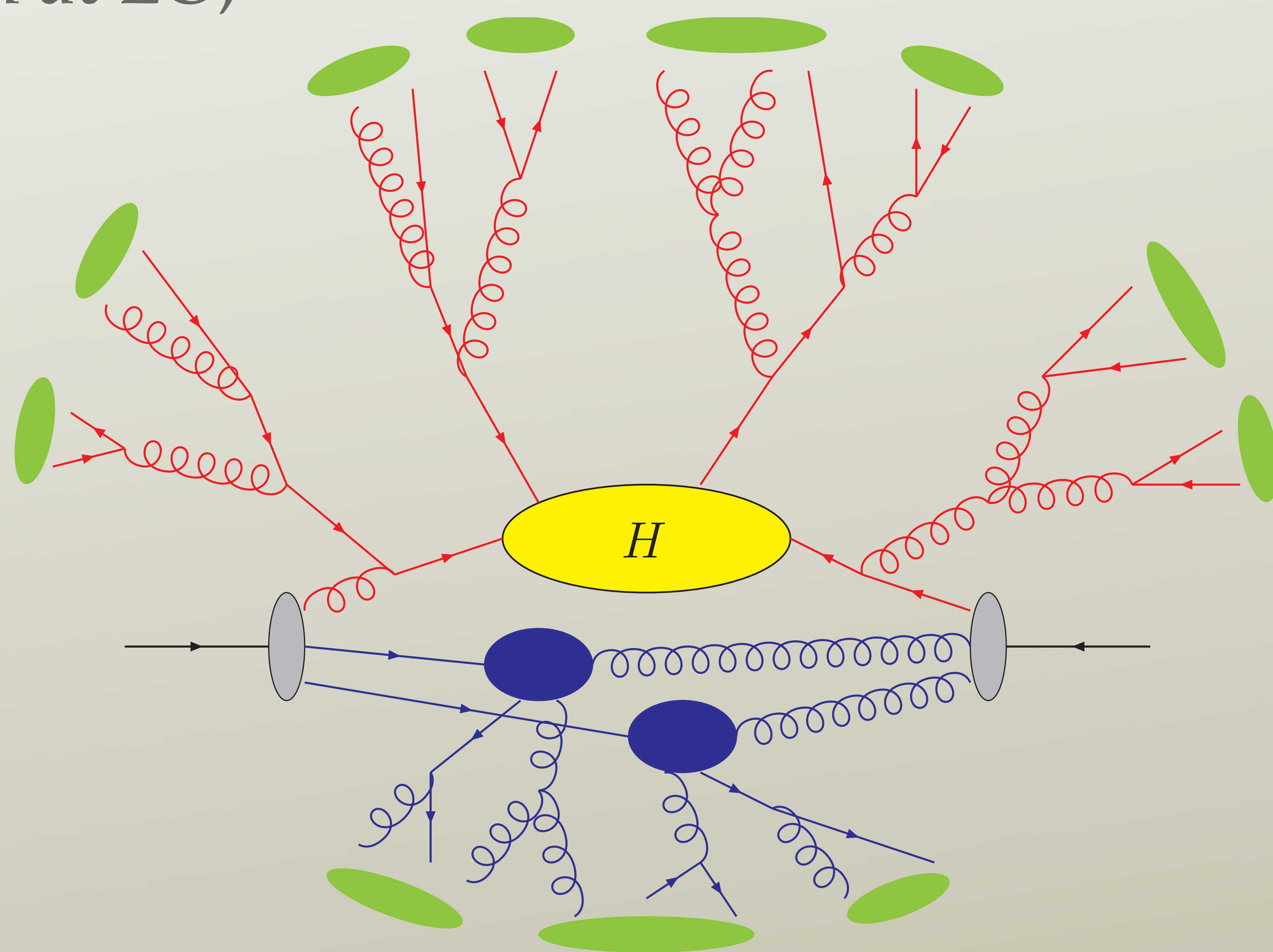
- An "event" is a list of particles (pions, protons, ...) with their momenta.
- The MCs generate events.
- The probability to generate an event is proportional to the (approximate!) cross section for such an event.
- Alternatively, cross section could be a weight given by the program times the probability to generate the event.

The description of an event is a bit tricky...

1. Incoming hadron *(gray bubbles)*
   ➡ Parton distribution function

2. Hard part of the process 

   ➡ Matrix element calculation at LO, NLO, ... level



3. Radiation *(red graphs)*
   ➡ Parton shower calculation
   ➡ Matching to the hard part

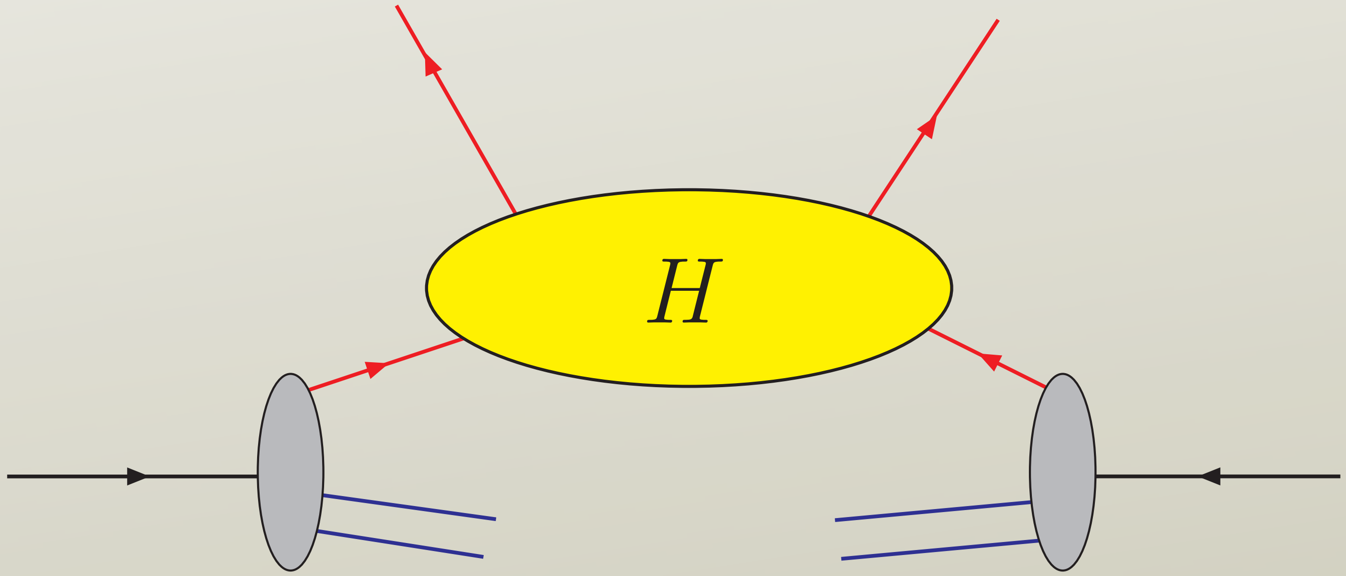
4. Underlying event *(blue graphs)*
   ➡ Models based on multiple interaction

5. Hardonization *(green bubbles)*
   ➡ Universal models

Compare this to a perturbative cross section

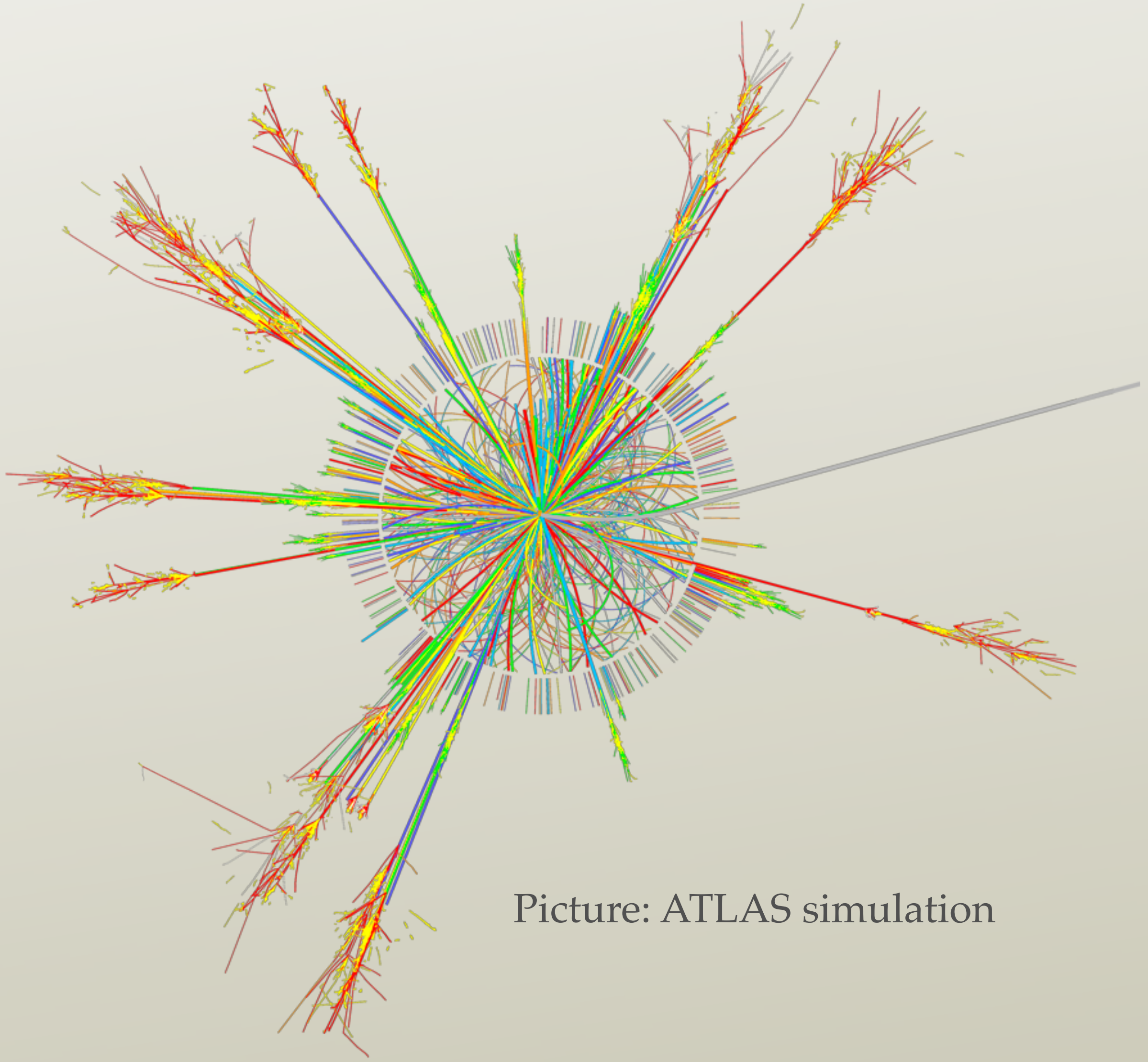
1. Incoming hadron *(gray bubbles)*
   ➡ Parton distribution function

2. Hard part of the process
   ➡ Matrix element calculation at LO, NLO, ... level



Why do we need parton showers?

- We need predictions for events at LHC and Tevatron.
- LO and NLO perturbation theory can give predictions for very inclusive cross sections.



Picture: ATLAS simulation

- We use parton showers to get predictions for the complete final state approximately right.

Matching

- One can match the parton shower calculation to exact tree level $2 \to n$ cross sections for small values of n.
- One can, with difficulty, also do this with loop level $2 \to n$ perturbative calculations.
- I omit discussion of these important topics.
- Instead, I discuss just lowest order parton showers.

A simple illustration

- Use an example in which partons carry momenta, but no flavor, color, or spin.
- ϕ^3 theory in six dimensions works for this.
- Also, just consider the evolution of the final state, as in electron-positron annihilation.

States

- For a generic description of shower MCs, use a notation adapted to classical statistical mechanics.

- State with m final state partons with momenta p

$$|\{p\}_m) = |\{p_1, p_2, \ldots, p_m\})$$

- General state $|\rho)$

- Cross section for the state to have m partons with definite momenta $(\{p\}_m|\rho)$

- Completeness relation

$$1 = \sum_m \frac{1}{m!} \int [d\{p\}_m] \; |\{p\}_m)(\{p\}_m|$$

Measurement functions

- Measurement function $(F|$
- Cross section for F

$$\begin{aligned}\sigma[F] &= \sum_m \frac{1}{m!}\int [d\{p\}_m]\ (F|\{p\}_m)(\{p\}_m|\rho) \\ &= (F|\rho)\end{aligned}$$

- Totally inclusive measurement function $(1|$

$$(1|\{p\}_m) = 1$$

Evolution

- State evolves in resolution scale t.
- $t = 0$: hard; increasing t means softer.
- Evolution follows a linear operator

$$|\rho(t)) = \mathcal{U}(t,t')|\rho(t'))$$

- Evolution does not change the cross section

$$(1|\mathcal{U}(t,t')|\rho(t')) = (1|\rho(t'))$$

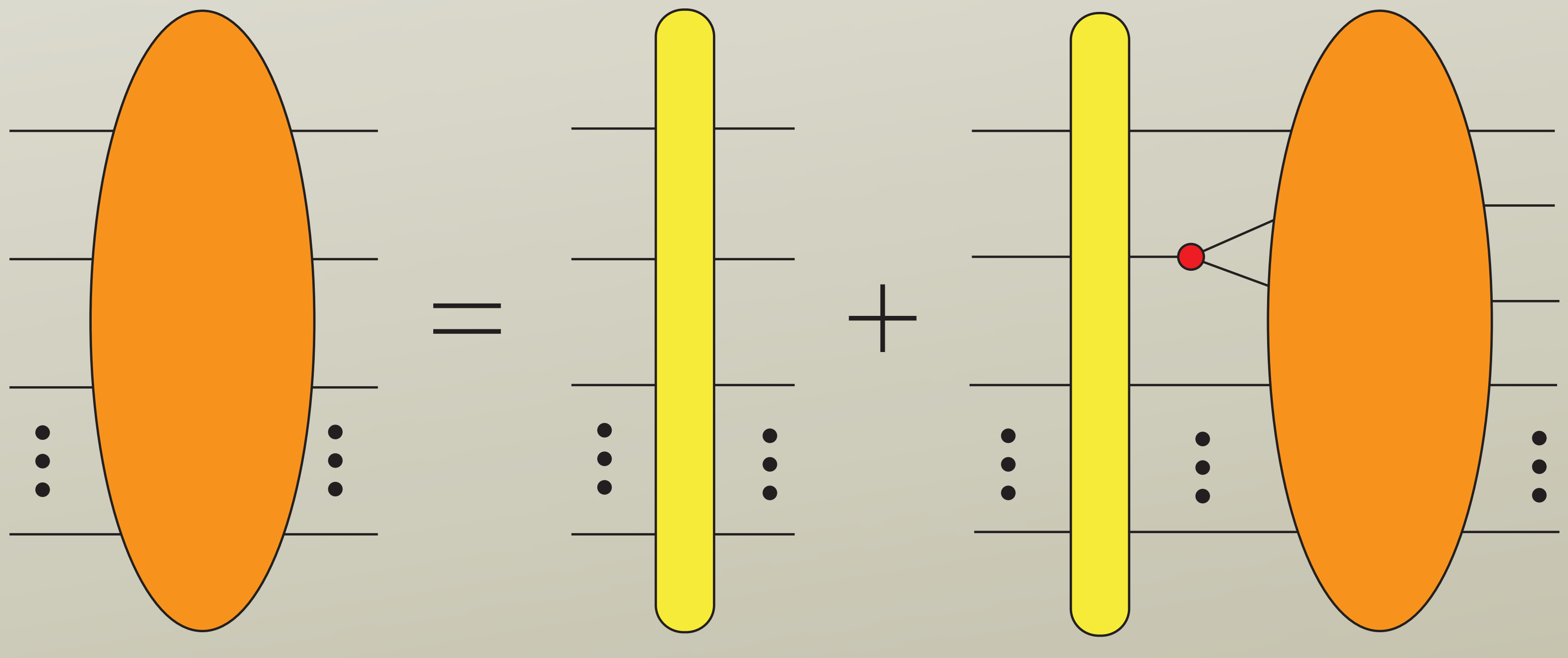
Structure of evolution

$$\mathcal{U}(t_3, t_1) = \mathcal{N}(t_3, t_1) + \int_{t_1}^{t_3} dt_2\, \mathcal{U}(t_3, t_2)\, \mathcal{H}_{\rm I}(t_2)\, \mathcal{N}(t_2, t_1)$$

$\mathcal{H}_{\rm I}(t) =$ splitting operator

$\mathcal{N}(t', t) =$ no change operator

$$\mathcal{N}(t', t)|\{p\}_m) = \Delta(t, t'; \{p\}_m)|\{p\}_m)$$



Probability conservation

$$\mathcal{U}(t_3,t_1) = \mathcal{N}(t_3,t_1) + \int_{t_1}^{t_3} dt_2\, \mathcal{U}(t_3,t_2)\, \mathcal{H}_\mathrm{I}(t_2)\, \mathcal{N}(t_2,t_1)$$

$(1|\mathcal{U}(t,t') = (1|$ **and** $\mathcal{N}(t',t)|\{p\}_m) = \Delta(t,t';\{p\}_m)|\{p\}_m)$

$$1 = \Delta(t_3,t_1;\{p\}_m) + \int_{t_1}^{t_3} dt_2\ \big(1\big|\mathcal{H}_\mathrm{I}(t_2)\big|\{p\}_m\big)\,\Delta(t_2,t_1;\{p\}_m)$$

$$\frac{d}{dt_3}\Delta(t_3,t_1;\{p\}_m) = -\big(1\big|\mathcal{H}_\mathrm{I}(t_3)\big|\{p\}_m\big)\,\Delta(t_3,t_1;\{p\}_m)$$

$$\Delta(t_3,t_1;\{p\}_m) = \exp\left(-\int_{t_1}^{t_3} d\tau\ \big(1\big|\mathcal{H}_\mathrm{I}(\tau)\big|\{p\}_m\big)\right)$$

Summary

$$\mathcal{U}(t_3,t_1) = \mathcal{N}(t_3,t_1) + \int_{t_1}^{t_3} dt_2\, \mathcal{U}(t_3,t_2)\, \mathcal{H}_{\rm I}(t_2)\, \mathcal{N}(t_2,t_1)$$

$$\mathcal{N}(t',t)\big|\{p\}_m\big) = \Delta(t,t';\{p\}_m)\big|\{p\}_m\big)$$

Inclusive probability
to split in time $d\tau$

$$\Delta(t_3,t_1;\{p\}_m) = \exp\left(-\int_{t_1}^{t_3} d\tau\ \big(1\big|\mathcal{H}_{\rm I}(\tau)\big|\{p\}_m\big)\right)$$

Probability not to split
between times t_1 and t_3

Splitting

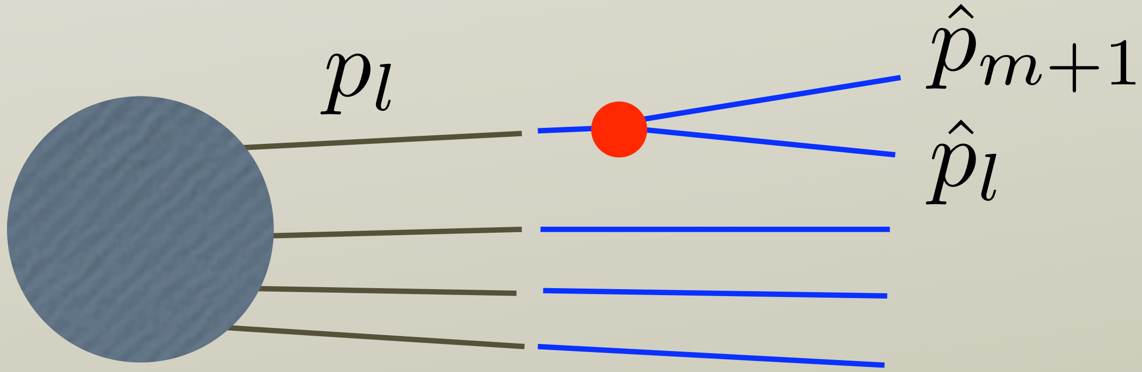
$$M(\{\hat{p}\}_{m+1}) \approx M(\{p\}_m) \times \frac{g}{2\hat{p}_l \cdot \hat{p}_{m+1}}$$

$$\big(\{\hat{p}\}_{m+1}\big|\mathcal{H}_{\rm I}(t)\big|\rho\big)$$

$$= \sum_l \delta\left(t - \log\left(\frac{Q_0^2}{2\hat{p}_l \cdot \hat{p}_{m+1}}\right)\right) \left[\frac{g}{2\hat{p}_l \cdot \hat{p}_{m+1}}\right]^2 \big(\{p\}_m\big|\rho\big)$$

Kinematics

- The details are not important, but it is important to know that there are details.
- Parton l splits into partons l and $m+1$.
- Before the splitting, momenta are p_i.
- After the splitting, momenta are $\hat{p}_i$.



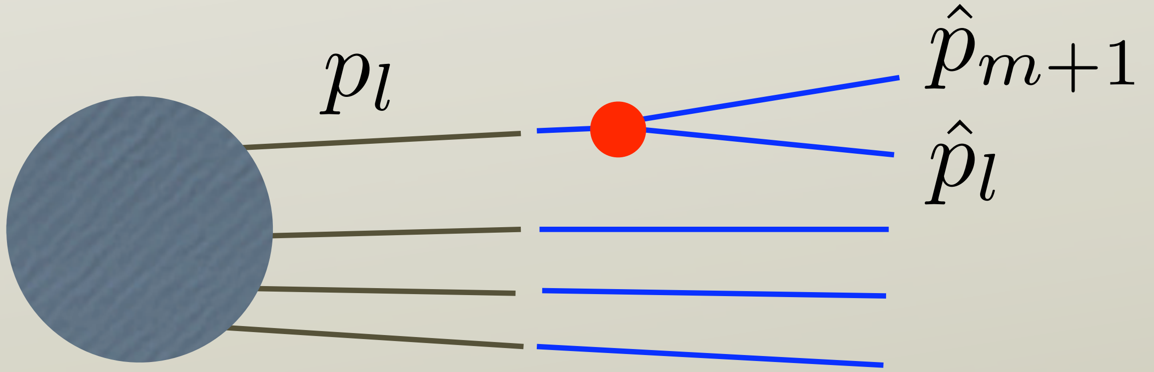
- We need $p_l^2 = 0$, but then $p_l \neq \hat{p}_l + \hat{p}_{m+1}$.

One choice

- Total momentum of final state partons Q
- Lightlike reference vector n

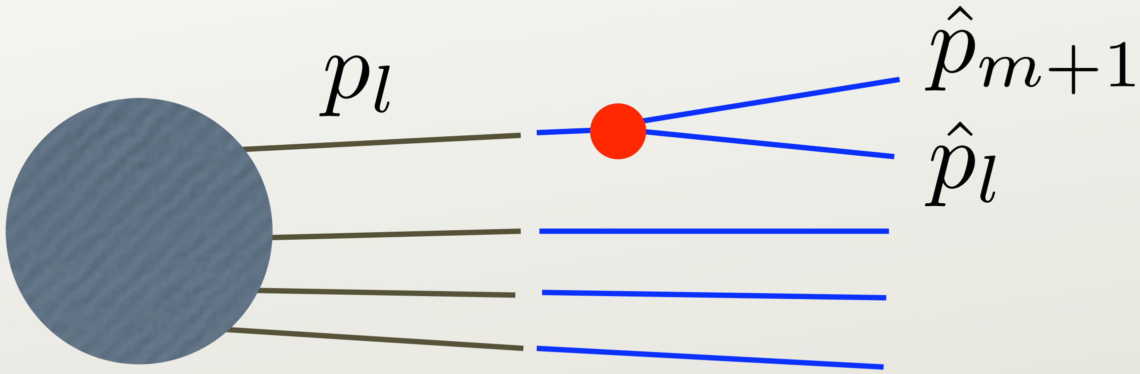
$$n = Q - \frac{Q^2}{2p_l \cdot Q}\, p_l$$

- Splitting variables:
  - ∗ Virtuality variable y
  - ∗ Momentum fraction z
  - ∗ Transverse unit vector $u_\perp$



- $$y = \frac{2\hat{p}_{m+1} \cdot \hat{p}_l}{2p_l \cdot Q}$$

- Use shorthand $\lambda = \sqrt{(1+y)^2 - 4yQ^2/(2p_l \cdot Q)}$.



- Then define $\hat{p}_{m+1}$ and $\hat{p}_l$ in terms of the splitting variables.

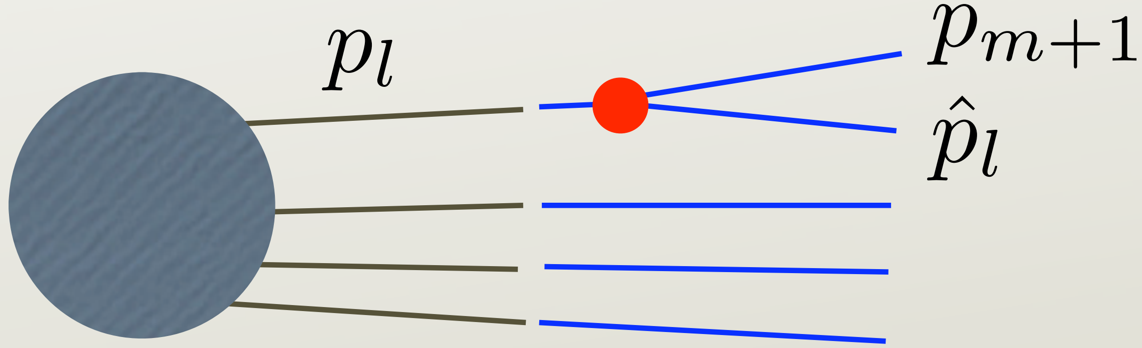
$$\hat{p}_{m+1} = z\,\frac{1+\lambda+y}{2}\,p_l + (1-z)\,\frac{2y}{1+\lambda+y}\,n_l + \sqrt{2z(1-z)y}\,u_\perp$$

$$\hat{p}_l = (1-z)\,\frac{1+\lambda+y}{2}\,p_l + z\,\frac{2y}{1+\lambda+y}\,n_l - \sqrt{2z(1-z)y}\,u_\perp$$

- Note that $\hat{p}_{m+1} + \hat{p}_l$ is not exactly p_l.
- Maintain momentum conservation with a Lorentz transformation of the spectator momenta.

$$\hat{p}_i = \Lambda p_i$$

Summary of splitting



- Using y, z, $u_\perp$,

$$\big(\{\hat{p}\}_{m+1}\big|\mathcal{H}_{\rm I}(t)\big|\rho\big)$$

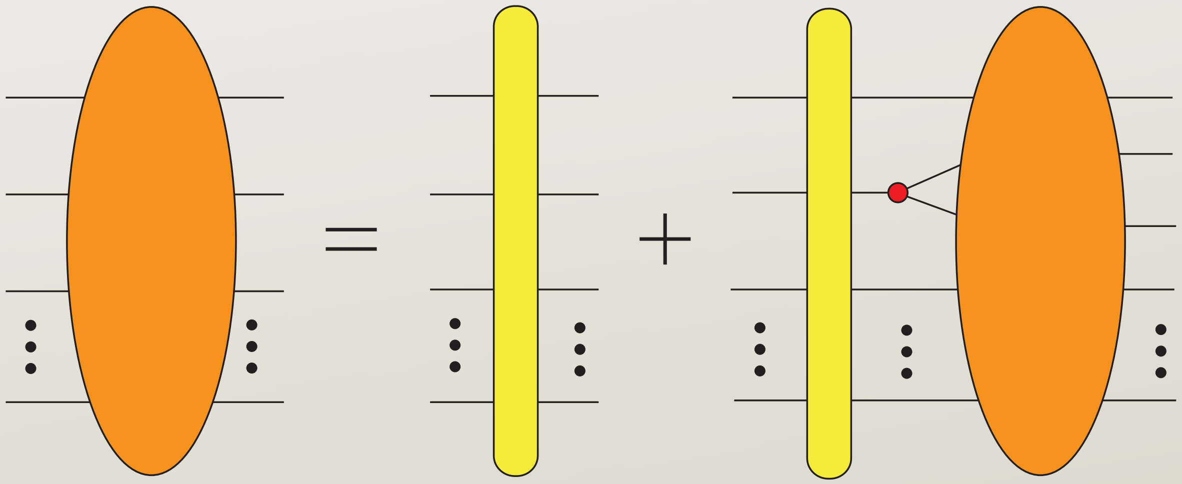
$$= \sum_l \delta\!\left(t - \log\left(\frac{Q_0^2}{y\,2p_l\cdot Q}\right)\right)\left[\frac{g}{y\,2p_l\cdot Q}\right]^2 \big(\{p\}_m\big|\rho\big)$$

- y is fixed by t.
- The $\hat{p}_i$ are given by the p_i and the splitting variables.
- The splitting probability, including a jacobian factor, is proportional to

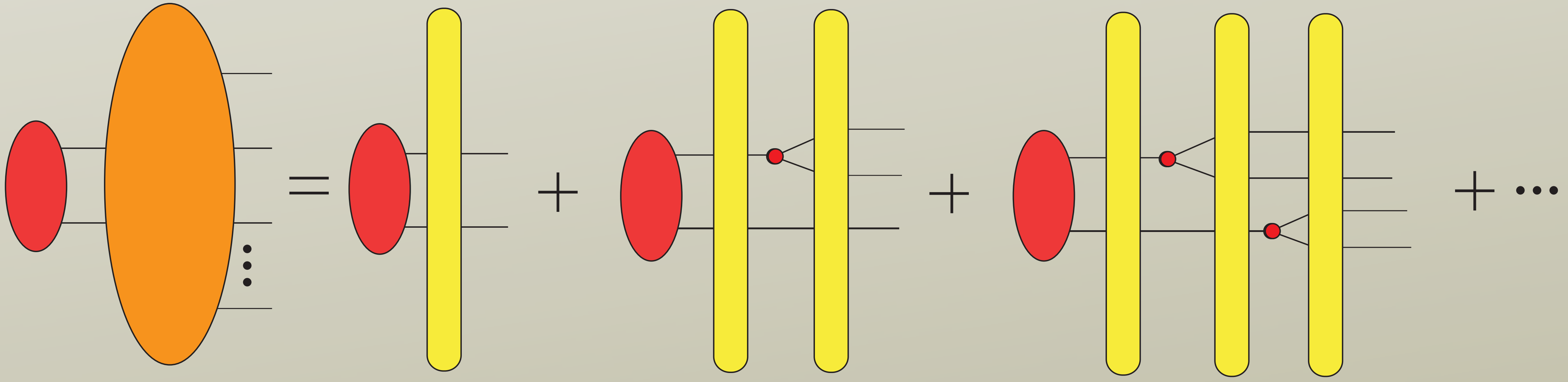
$$dt\;\, z(1-z)dz\;\, du_\perp$$

Solution of evolution

- Evolution equation



- generates (either analytically or in computer code)



Differential equation

- $\mathcal{U}(t,t')$ obeys a simple differential equation.
- Define $\mathcal{V}(t)$ by

$$\mathcal{V}(t)|\{p\}_m) = v(t,\{p\}_m)|\{p\}_m)$$

$$v(t,\{p\}_m) = (1|\mathcal{H}_{\rm I}(t)|\{p\}_m)$$

- Then

$$\frac{d}{dt}\mathcal{N}(t,t') = -\mathcal{V}(t)\,\mathcal{N}(t,t')$$

- Then

$$\frac{d}{dt}\,\mathcal{U}(t,t') = [\mathcal{H}_{\rm I}(t) - \mathcal{V}(t)]\,\mathcal{U}(t,t')$$

- Proof. Suppose that $\mathcal{U}(t,t')$ obeys this equation and define

$$\widetilde{\mathcal{U}}(t,t') = \mathcal{N}(t,t') + \int_{t'}^{t} d\tau\;\mathcal{U}(t,\tau)\,\mathcal{H}_{\rm I}(\tau)\,\mathcal{N}(\tau,t')$$

Then

$$\frac{d}{dt}\,\widetilde{\mathcal{U}}(t,t') = [\mathcal{H}_{\rm I}(t) - \mathcal{V}(t)]\,\widetilde{\mathcal{U}}(t,t')$$

Also

$$\widetilde{\mathcal{U}}(t',t') = \mathcal{U}(t',t')$$

Thus

$$\widetilde{\mathcal{U}}(t,t') = \mathcal{U}(t,t')$$

Connection with perturbation theory

$$\frac{d}{dt}\mathcal{U}(t,t') = [\mathcal{H}_{\rm I}(t) - \mathcal{V}(t)]\,\mathcal{U}(t,t')$$

implies

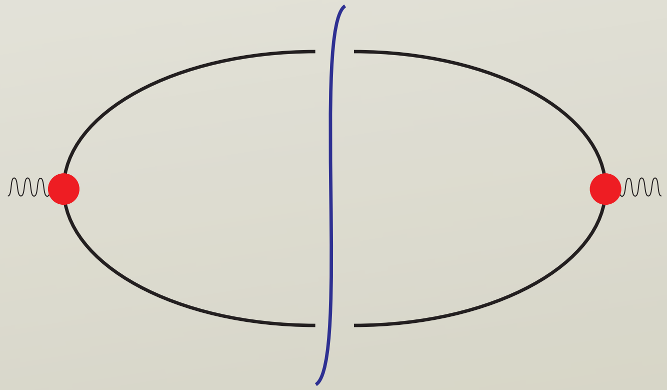
$$(F|\mathcal{U}(t_{\rm f},0)|\rho(0)) = (F|\rho(0)) + \int_0^{t_{\rm f}} dt\ (F|\mathcal{H}_{\rm I}(t) - \mathcal{V}(t)|\rho(0)) + \cdots$$

The corresponding graphs

$$(F|\mathcal{U}(t_{\rm f},0)|\rho(0)) = (F|\rho(0)) + \int_0^{t_{\rm f}} dt\ (F|\mathcal{H}_{\rm I}(t) - \mathcal{V}(t)|\rho(0)) + \cdots$$

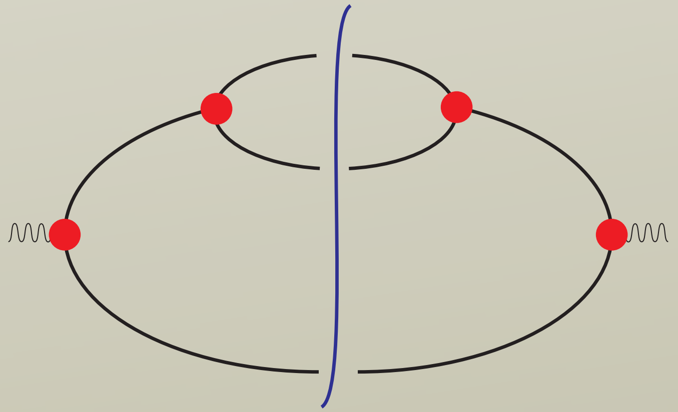
- Born hard scattering graph

$$(F|\rho(0))$$



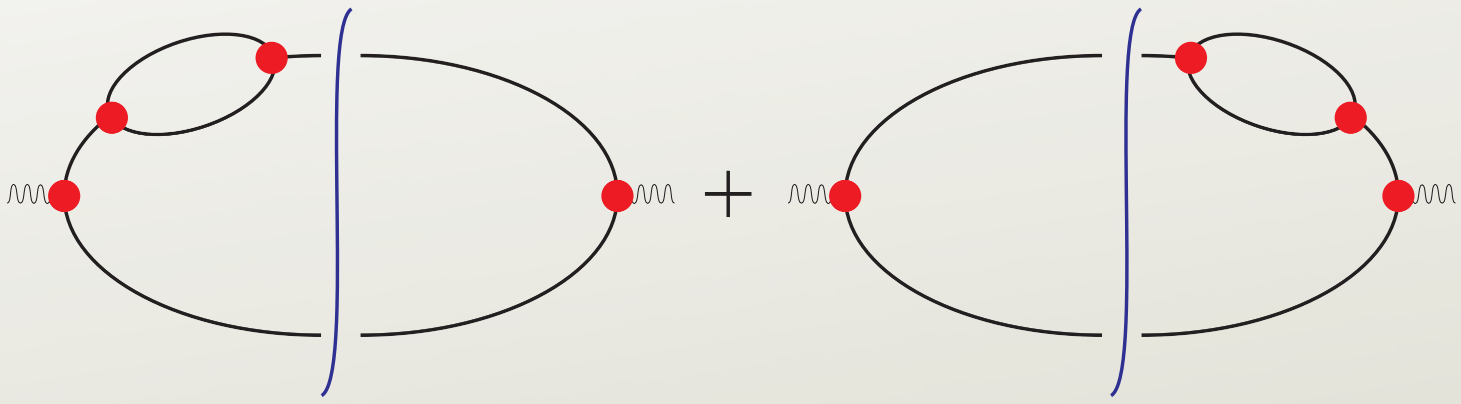
- Approximate real emission with virtuality cutoff

$$\int_0^{t_{\rm f}} dt\ (F|\mathcal{H}_{\rm I}(t)|\rho(0))$$



- Approximate virtual graphs with virtuality cutoff

$$\int_0^{t_\mathrm{f}} dt\ (F|\mathcal{V}(t)|\rho(0))$$



- The true virtual graphs obey

$$(1|\mathcal{V}_{\mathrm{true}}(t)|\rho(0)) - (1|\mathcal{H}_\mathrm{I}(t)|\rho(0)) \to 0 \quad \text{for} \quad t \to \infty.$$

- Our approximation shares this property since

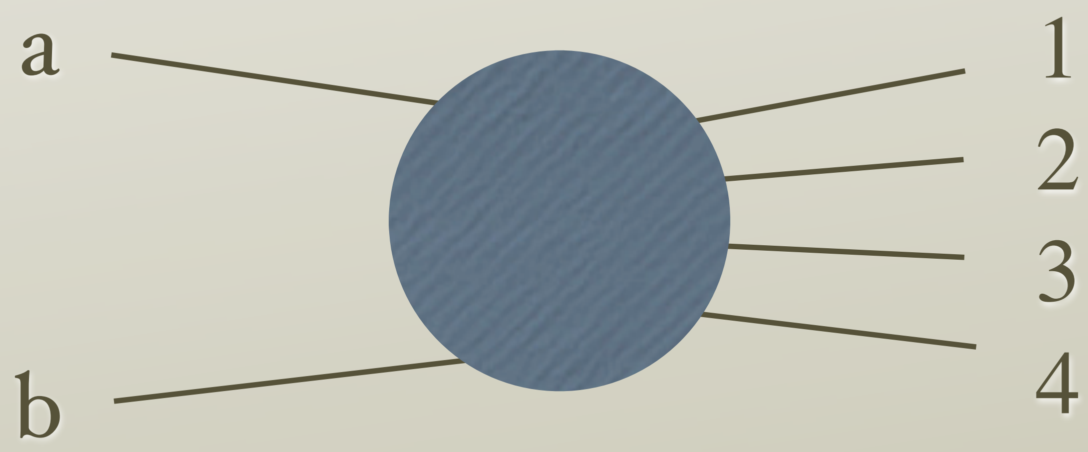
$$(1|\mathcal{V}(t)|\rho(0)) - (1|\mathcal{H}_\mathrm{I}(t)|\rho(0)) = 0$$

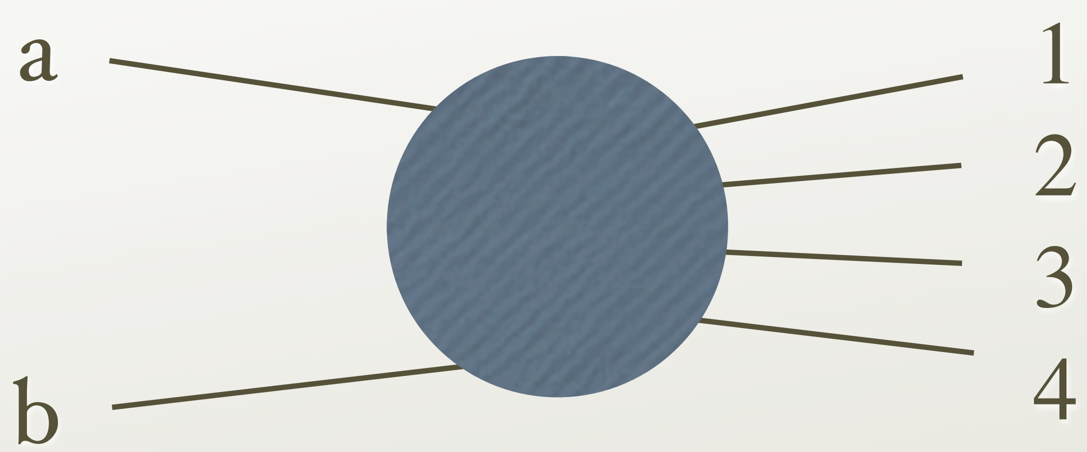
- But the approximation is not exact for finite t.
- This allows us to preserve the hard scattering cross section exactly. $\quad (1|\mathcal{U}(t,t') = (1|$

Partons in the initial state

- State with m final state partons with momenta p_i and two initial state partons with momentum fractions $\eta_{\rm a}, \eta_{\rm b}$

$$|\{p\}_m) = |\{\eta_{\rm a}, \eta_{\rm b}, p_1, p_2, \dots, p_m\})$$





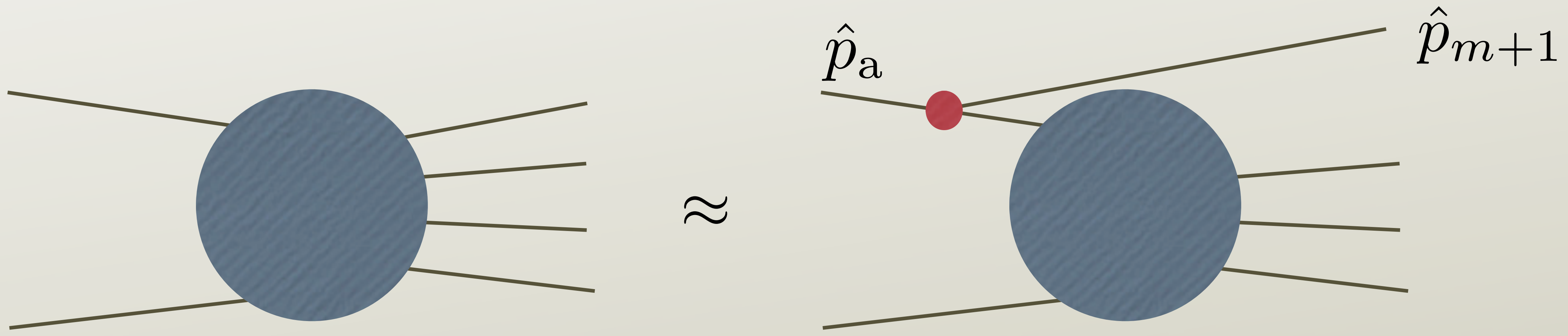
- General state $|\rho)$ so that cross section is

$$\sigma[F] = \sum_m \frac{1}{m!} \int \left[d\{p\}_m\right] (F|\{p\}_m)(\{p\}_m|\rho)$$

- $|\rho)$ includes the parton distributions

$$(\{p\}_m|\rho) = |M(\{p\}_m)|^2 \ \frac{f_A(\eta_\mathrm{a}) f_B(\eta_\mathrm{b})}{2\eta_\mathrm{a}\eta_\mathrm{b} p_\mathrm{A}\cdot p_\mathrm{B}}$$

Factorization



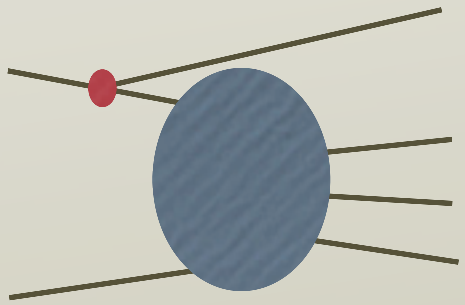
$$M(\{\hat{p}\}_{m+1}) \approx M(\{p\}_m) \times \frac{g}{-2\hat{p}_{\rm a} \cdot \hat{p}_{m+1}}$$

Splitting operator

$$(\{p\}_m|\rho) = |M(\{p\}_m)|^2 \; \frac{f_A(\eta_\mathrm{a}) f_B(\eta_\mathrm{b})}{2\eta_\mathrm{a}\eta_\mathrm{b} p_\mathrm{A}\cdot p_\mathrm{B}}$$

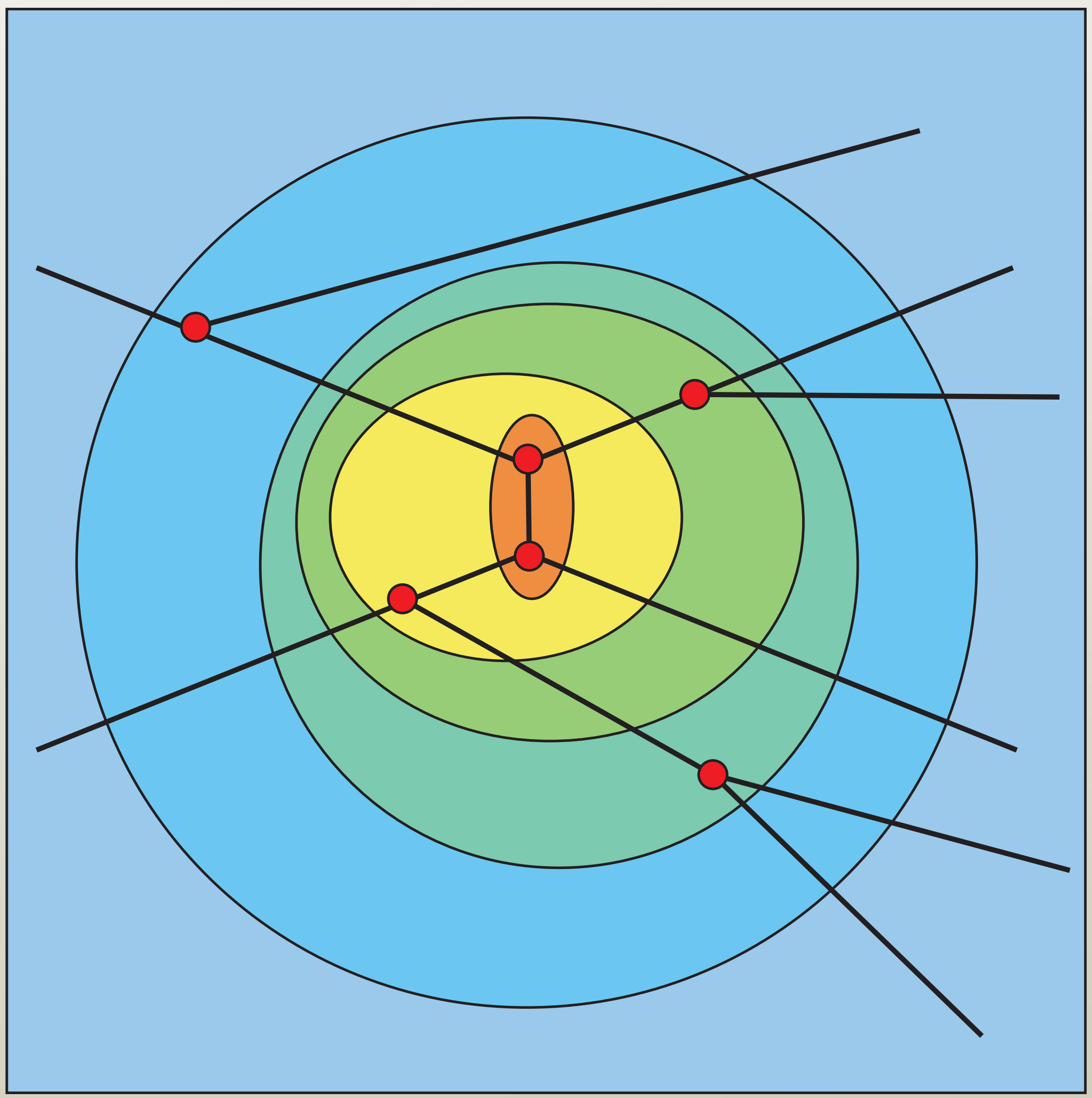
So

$$\begin{aligned}
&\big(\{\hat p\}_{m+1}\big|\mathcal{H}_\mathrm{I}(t)\big|\rho\big) \\
&= \sum_l \delta\left(t - \log\left(\frac{Q_0^2}{|2\hat p_l \cdot \hat p_{m+1}|}\right)\right) \\
&\quad \times \left[\frac{g}{2\hat p_l \cdot \hat p_{m+1}}\right]^2 \frac{\eta_\mathrm{a}\eta_\mathrm{b} f_A(\hat\eta_\mathrm{a}) f_B(\hat\eta_\mathrm{b})}{\hat\eta_\mathrm{a}\hat\eta_\mathrm{b} f_A(\eta_\mathrm{a}) f_B(\eta_\mathrm{b})} (\{p\}_m|\rho)
\end{aligned}$$

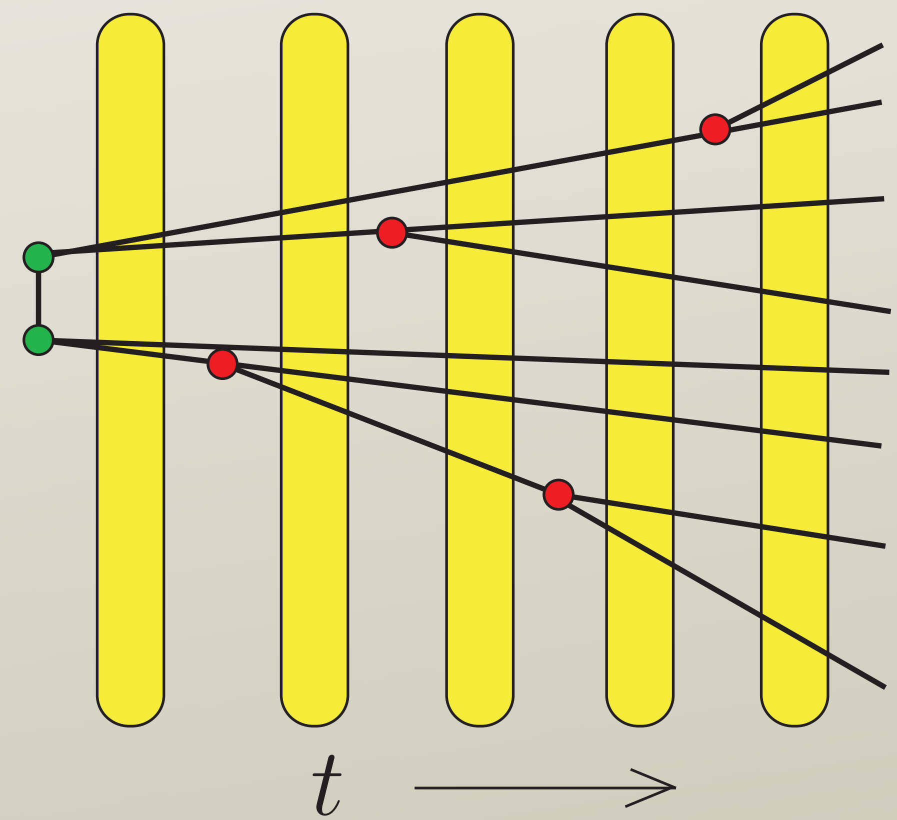


Shower time

Showers develop in “hardness” time.



Real time picture



Shower time picture

QCD

- QCD is more complicated than scalar field theory.
- In typical parton shower algorithms, the main approximation is collinear or soft splitting.
- I will first sketch the structure of evolution with just this approximation.
- Then I will describe further approximations related to color, spin, and quantum interference for soft gluons.

The matrix element

- The basic object is the quantum matrix element

$$M(\{p,f\}_m)^{c_\mathrm{a},c_\mathrm{b},c_1,\dots,c_m}_{s_\mathrm{a},s_\mathrm{b},s_1,\dots,s_m}$$

- This is a function of the momenta and flavors and carries color and spin indices. Consider it as a vector in color and spin space

$$\big|M(\{p,f\}_m)\big\rangle$$

The cross section

The cross section with a measurement function F is then

$$\sigma[F] = \sum_m \frac{1}{m!} \int \left[d\{p,f\}_m\right] \frac{f_{a/A}(\eta_{\rm a},\mu_{\rm F}^2)\, f_{b/B}(\eta_{\rm b},\mu_{\rm F}^2)}{4 n_{\rm c}(a) n_{\rm c}(b)\, 2\eta_{\rm a}\eta_{\rm b} p_{\rm A}\cdot p_{\rm B}}$$
$$\times \left\langle M(\{p,f\}_m)\middle| F(\{p,f\}_m) \middle| M(\{p,f\}_m)\right\rangle$$

- a and b are the flavors of the incoming partons.
- $f_{a/A}(\eta_{\rm a},\mu_{\rm F}^2)$ is a parton distribution function.
- $n_{\rm c}(a)$ is the number of colors for flavor a.

The density matrix

$$\sigma[F] = \sum_m \frac{1}{m!} \int \left[d\{p,f\}_m\right] \mathrm{Tr}\{\rho(\{p,f\}_m) F(\{p,f\}_m)\}$$

where

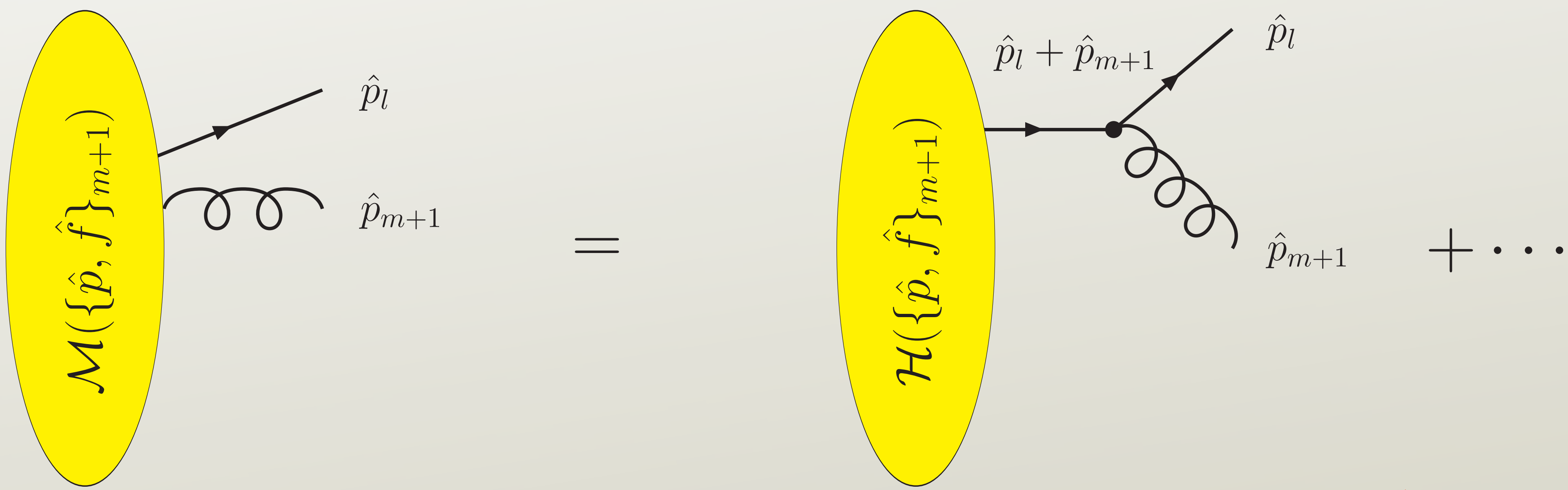
$$\rho(\{p,f\}_m)$$
$$= \big|M(\{p,f\}_m)\big\rangle \frac{f_{a/A}(\eta_\mathrm{a},\mu_F^2) f_{b/B}(\eta_\mathrm{b},\mu_F^2)}{4n_\mathrm{c}(a)n_\mathrm{c}(b)\, 2\eta_\mathrm{a}\eta_\mathrm{b} p_\mathrm{A}\cdot p_\mathrm{B}} \big\langle M(\{p,f\}_m)\big|$$
$$= \sum_{s,c}\sum_{s',c'} \big|\{s,c\}_m\big\rangle\, \rho(\{p,f,s',c',s,c\}_m)\, \big\langle\{s',c'\}_m\big|$$

Density matrix in "classical" notation

$$\rho(\{p, f, s', c', s, c\}_m) = \big(\{p, f, s', c', s, c\}_m \big| \rho\big)$$

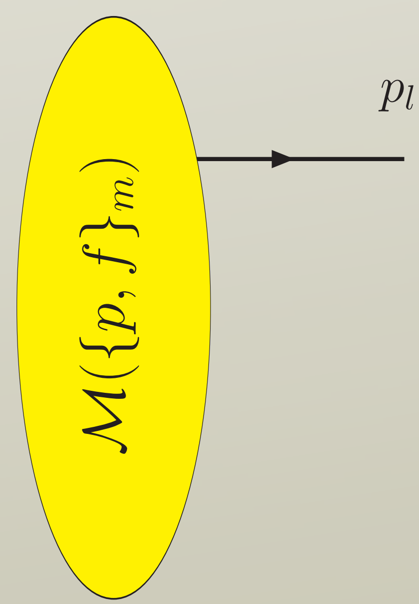
- For QCD, partons have momenta and flavors.
- Furthermore, there are two sets of spin indices and sets of color indices.
- There are lots of indices, but the general formalism is the same as sketched earlier.

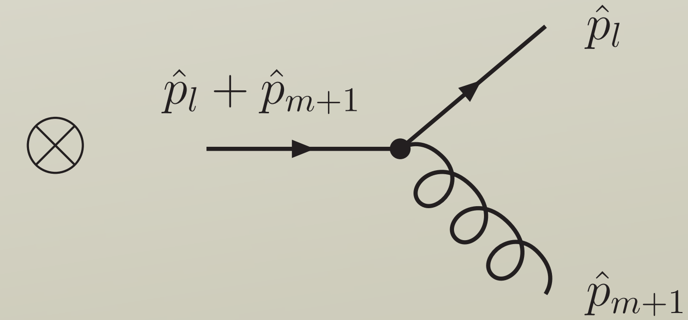
Splitting



this is an exact Feynman graph

$\approx$

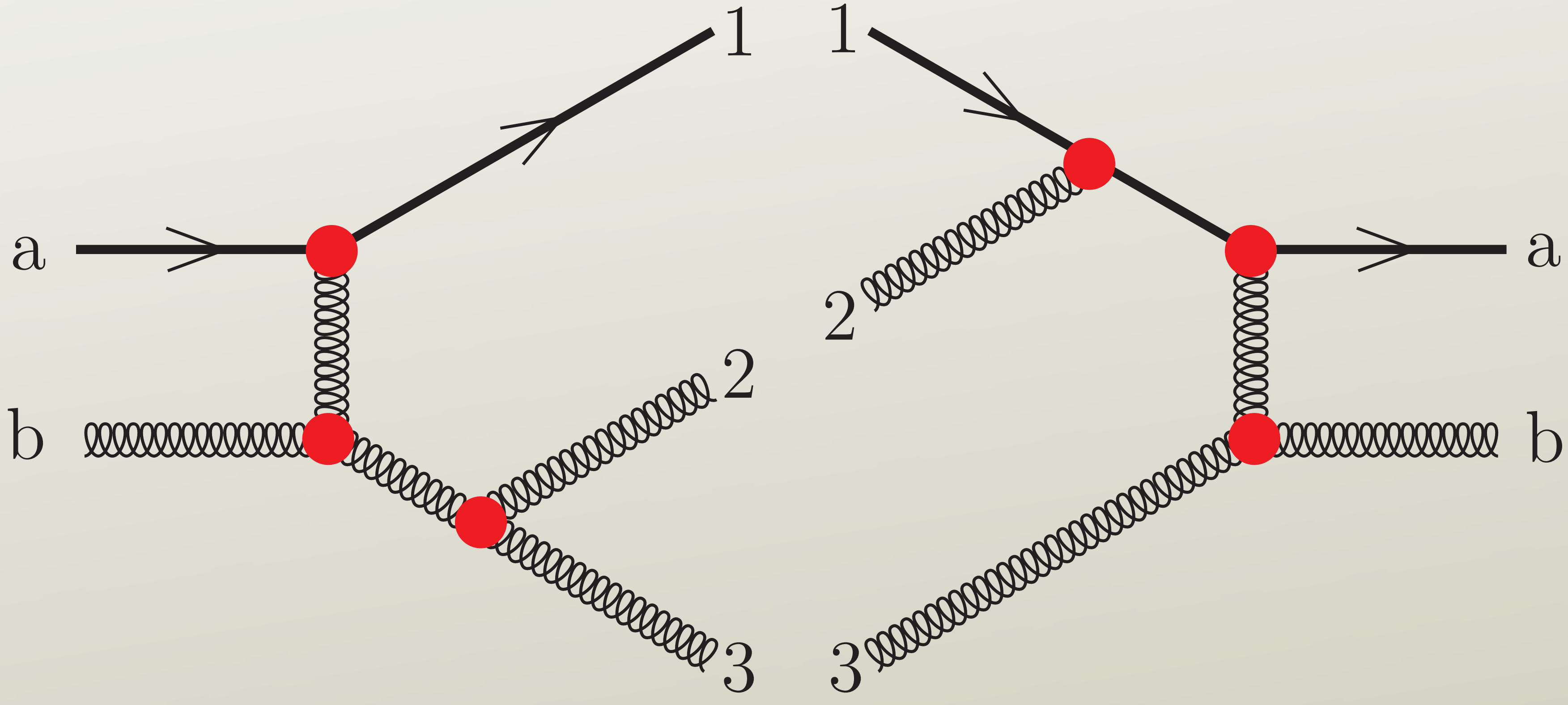




approximation is here, the kinematics is an m body configuration

Soft gluon emission

Splitting includes interference graphs.



A soft gluon approximation is used for the splitting function.

Here you may think of 1 and 3 as a “dipole” that radiates 2 coherently.

Evolution equation

- The structure of the evolution is the same as before:

$$\mathcal{U}(t_3,t_1) = \mathcal{N}(t_3,t_1) + \int_{t_1}^{t_3} dt_2\, \mathcal{U}(t_3,t_2)\,\mathcal{H}_{\rm I}(t_2)\,\mathcal{N}(t_2,t_1)$$

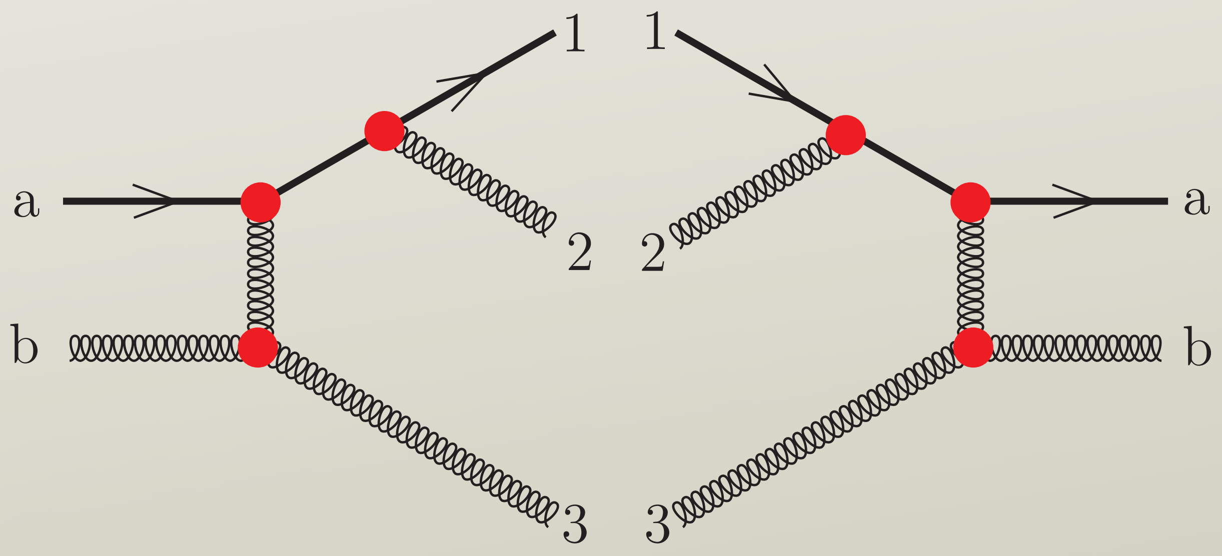
$$\frac{d}{dt}\mathcal{N}(t,t') = -\mathcal{V}(t)\,\mathcal{N}(t,t')$$

$$\big(1\big|\mathcal{V}(t) = \big(1\big|\mathcal{H}_{\rm I}(t)$$

- $\mathcal{V}(t)$ leaves the number of particles, their momenta, flavors, and spins unchanged.
- Unfortunately, it is not diagonal in color.

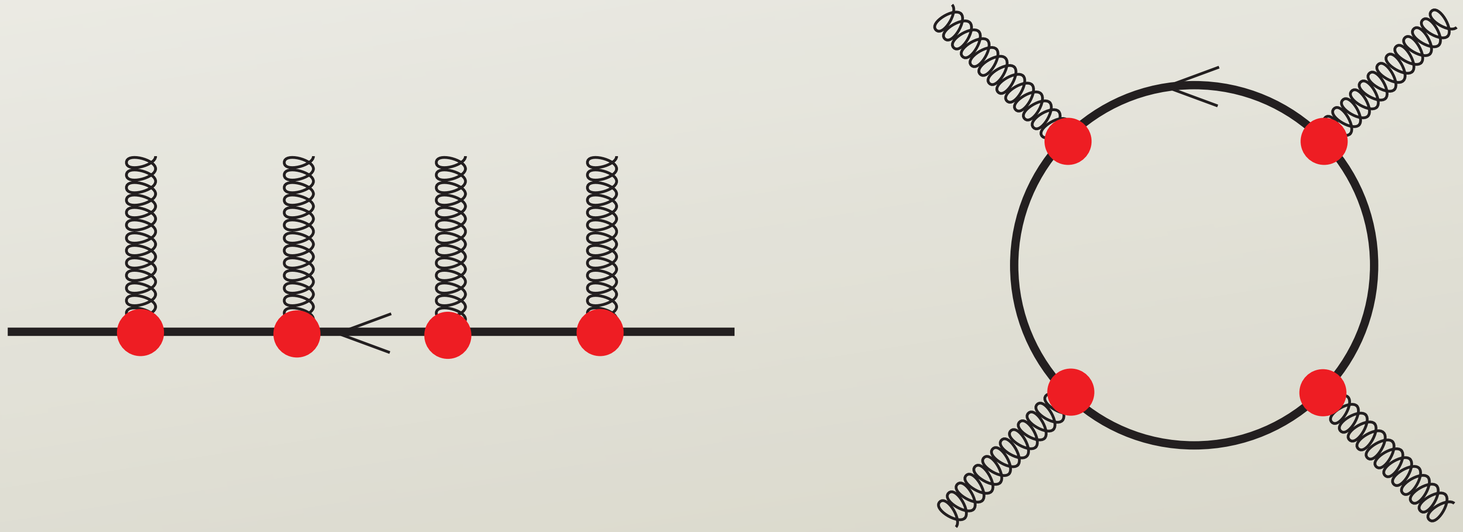
Spin approximation

- One commonly averages over the spin states of a parton that is about to split and sums over the spin states of the daughter partons.
- This eliminates angular correlations that arise from the spin states.
- For sufficiently inclusive observables, it should be a pretty good approximation.

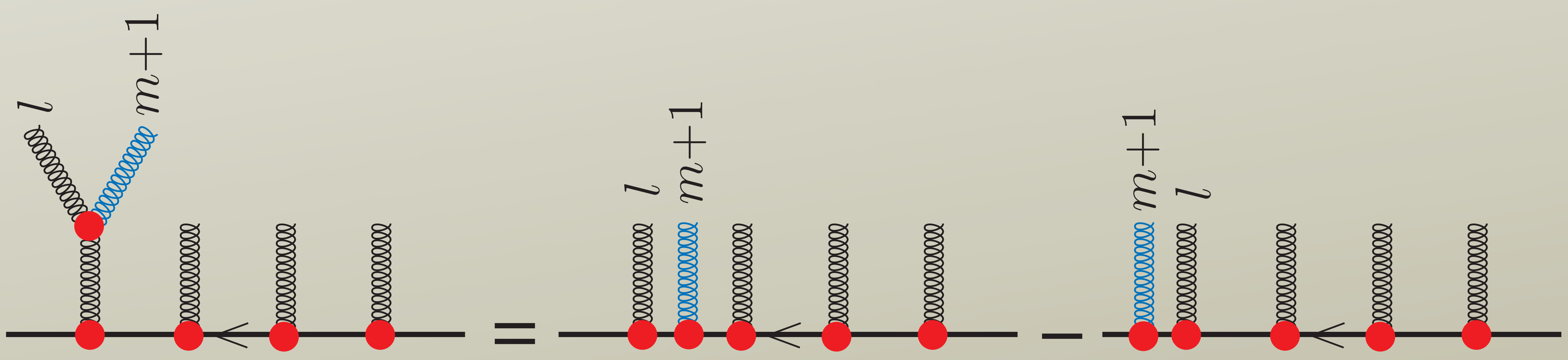


Color

- One can use a set of "string" basis states for color.

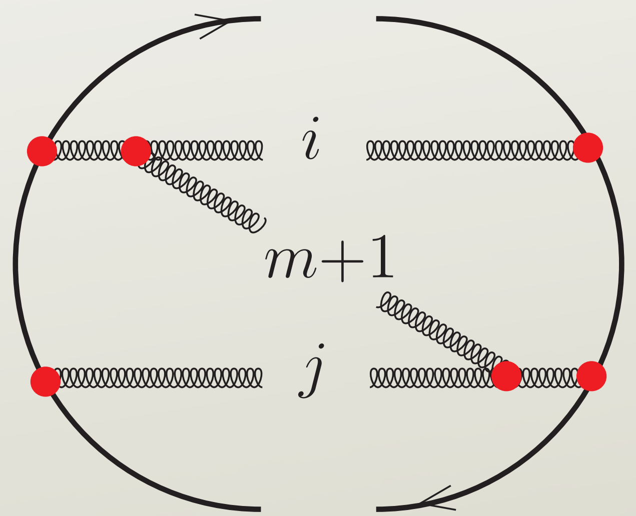


- With this basis, splitting is simple.

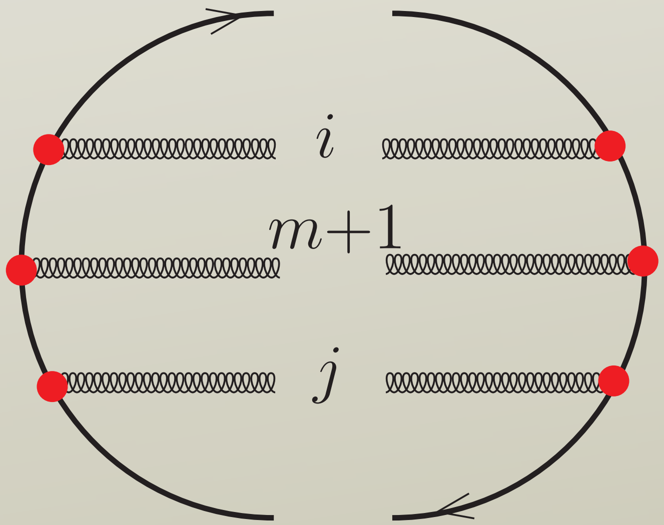


Color approximation

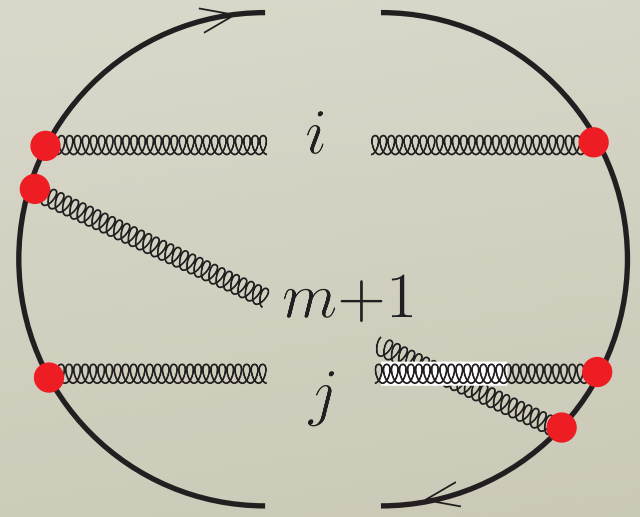
- Shower programs usually use a large N_c approximation.



An interference diagram, to be decomposed in basis states.



The leading contribution



A subleading contribution.

Simplified evolution equation

- The structure of the evolution is still:

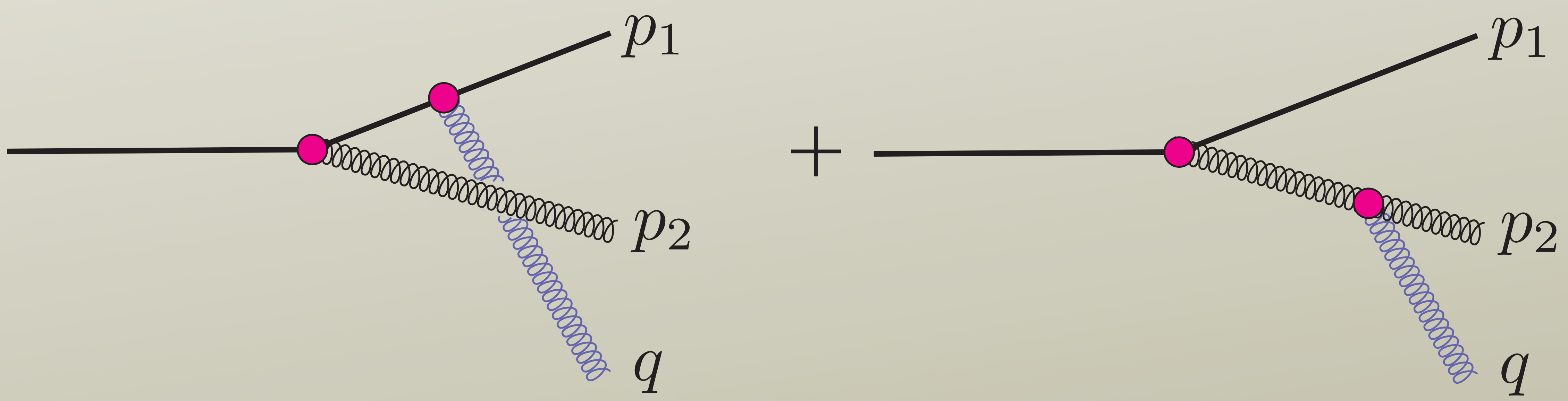
$$\mathcal{U}(t_3,t_1) = \mathcal{N}(t_3,t_1) + \int_{t_1}^{t_3} dt_2\, \mathcal{U}(t_3,t_2)\, \mathcal{H}_{\rm I}(t_2)\, \mathcal{N}(t_2,t_1)$$

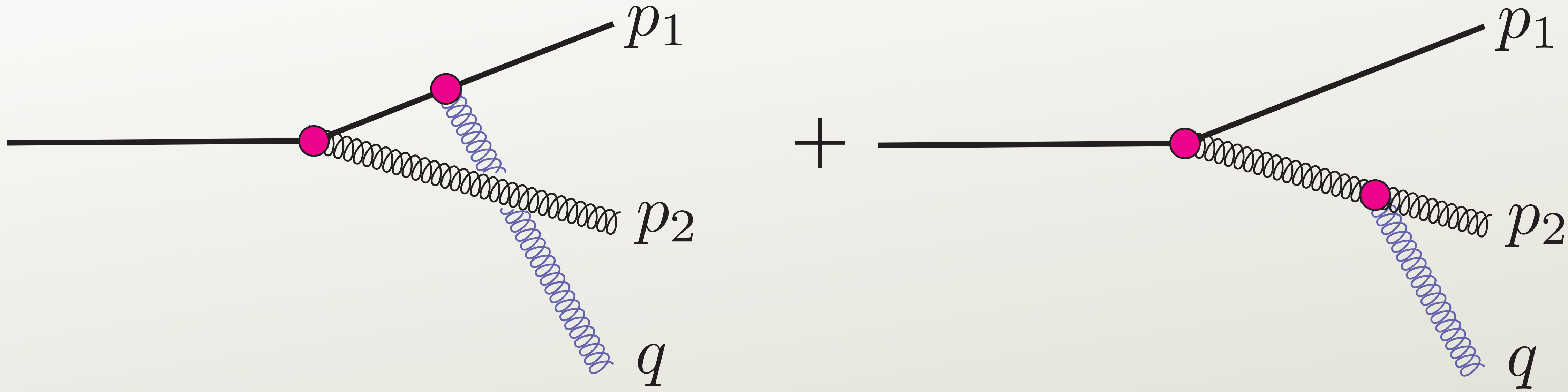
$$\frac{d}{dt}\,\mathcal{N}(t,t') = -\mathcal{V}(t)\,\mathcal{N}(t,t')$$

- $\mathcal{V}(t)$ leaves the number of particles, their momenta, flavors, and colors unchanged. Spin has been eliminated.
- This is approximately the organization of Pythia.

Angular ordering

- There is an alternative way of organizing a parton shower, used in Herwig.
- To understand it, consider the splitting of a quark into a quark + a gluon at a small angle, followed by the emission of a soft gluon from the two sister partons.

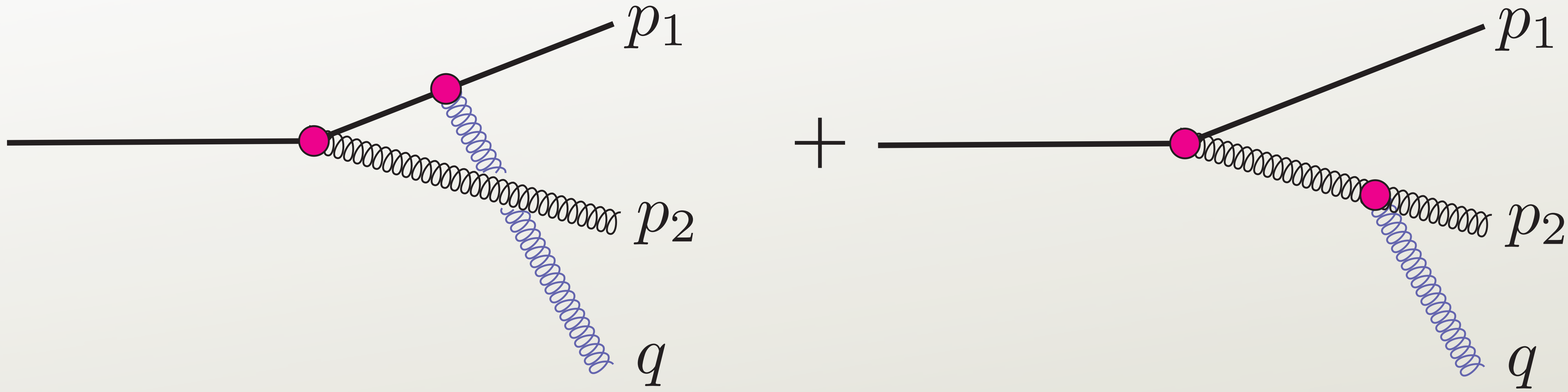




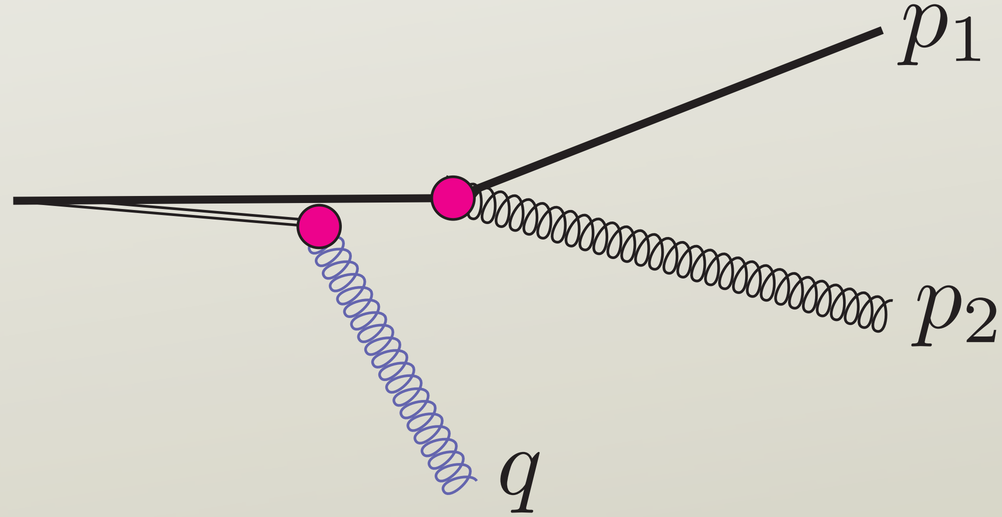
$$\frac{\vec{u}_1 \cdot \vec{\varepsilon}(\vec{u}_q)}{E_q[1 - \vec{u}_1 \cdot \vec{u}_q]} \qquad \frac{\vec{u}_2 \cdot \vec{\varepsilon}(\vec{u}_q)}{E_q[1 - \vec{u}_2 \cdot \vec{u}_q]}$$

- $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_q$ are unit three vectors $\propto \vec{p}_1$, $\vec{p}_2$, and $\vec{q}$.
- $\vec{\varepsilon}$ is the polarization vector for the soft gluon.
- If $\angle$ 1-2 is much smaller than $\angle$ $\vec{q}$-1 and $\angle$ $\vec{q}$-2 then the two factors are the same

$$\frac{\vec{u}_{12} \cdot \vec{\varepsilon}(\vec{u}_q)}{E_q[1 - \vec{u}_{12} \cdot \vec{u}_q]}$$



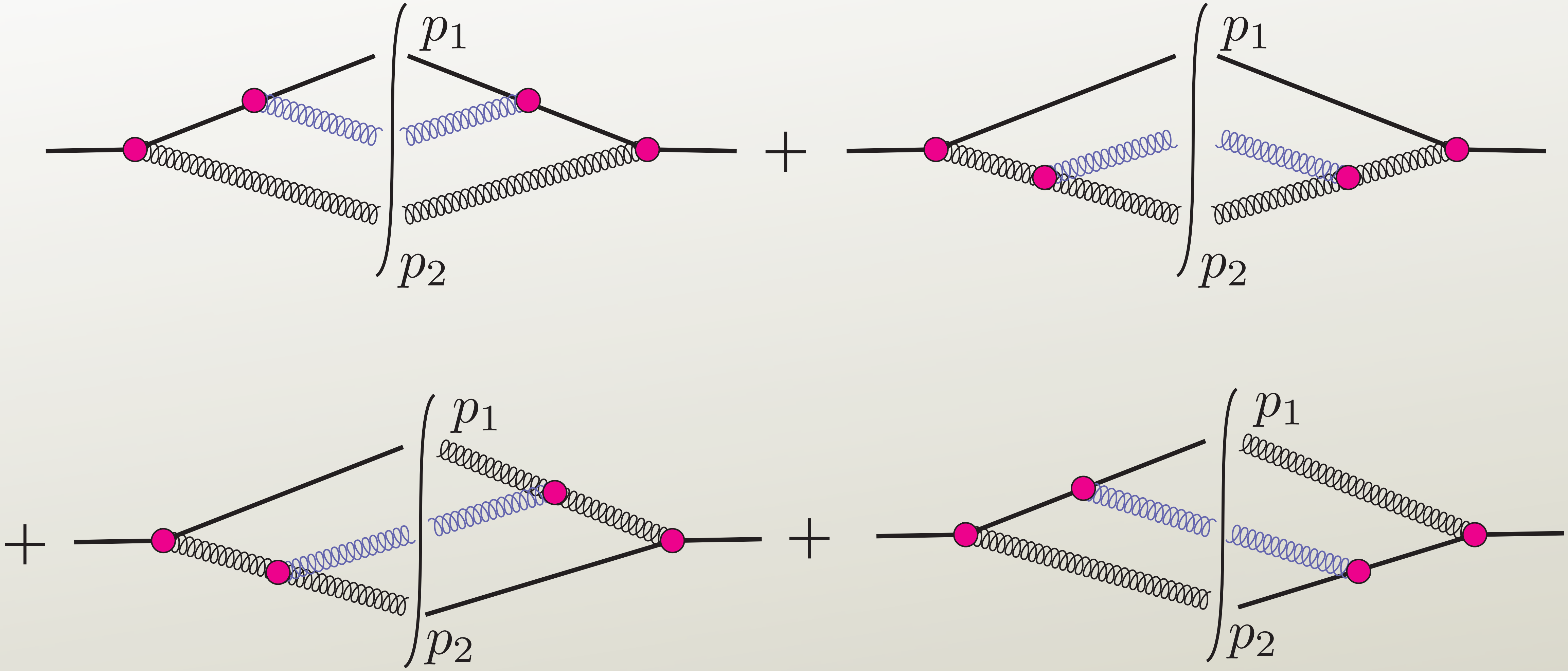
$\approx$



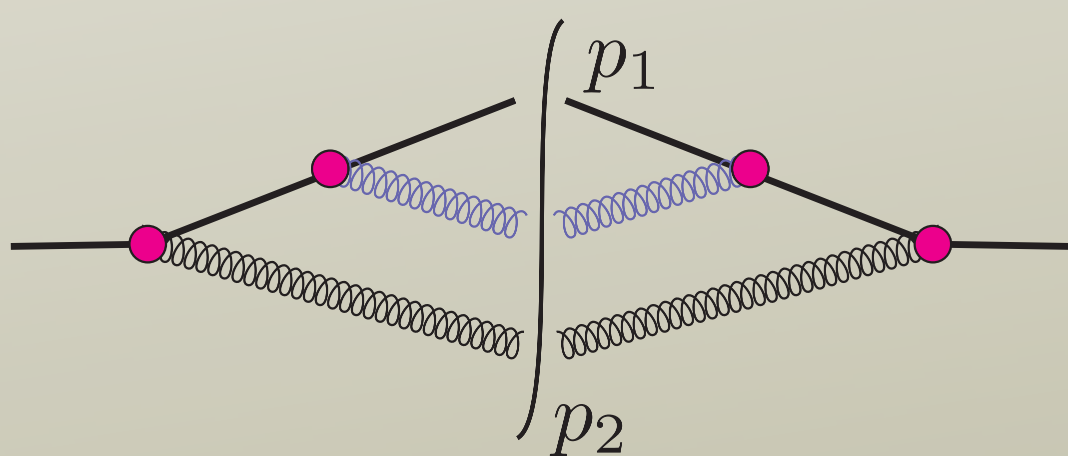
$$\frac{\vec{u}_{12} \cdot \vec{\varepsilon}(\vec{u}_q)}{E_q[1 - \vec{u}_{12} \cdot \vec{u}_q]}$$

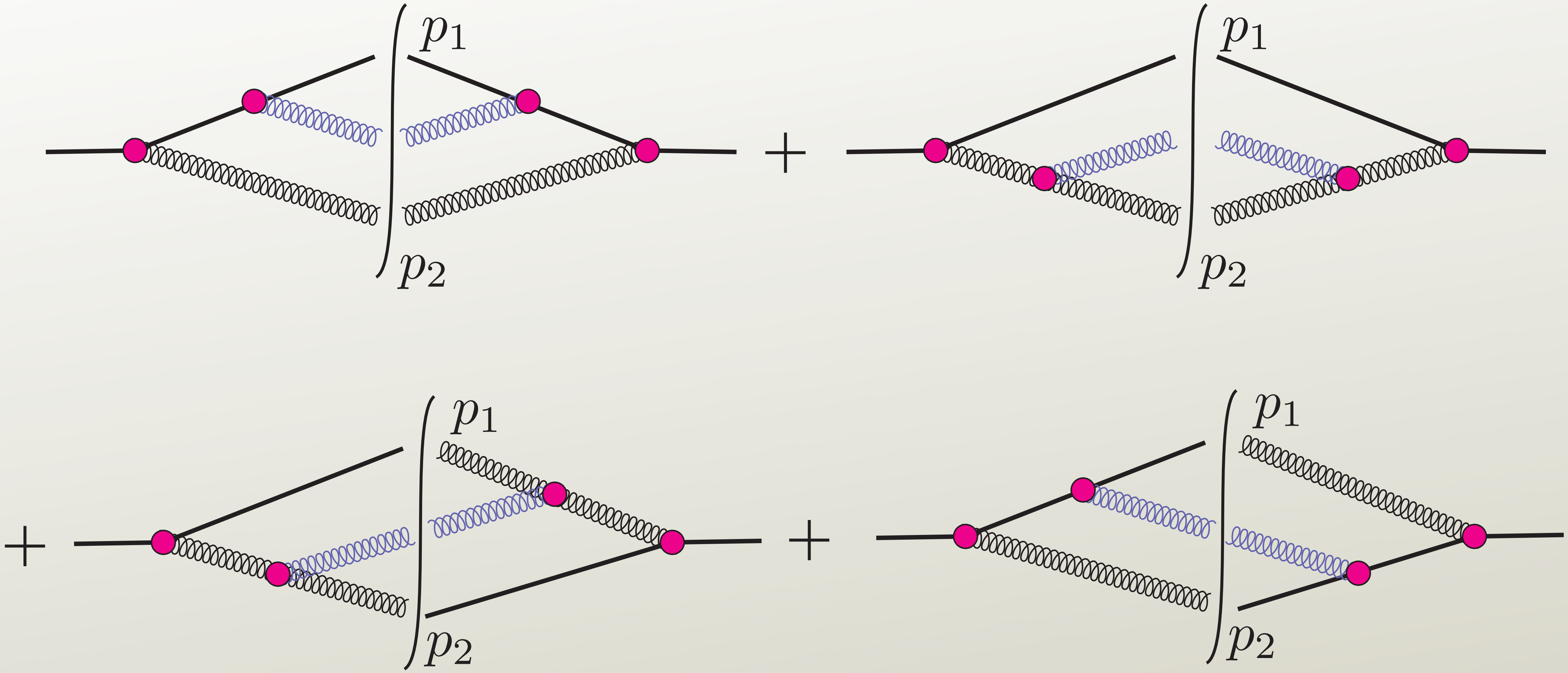
- This includes the color factor.
- It is as if the soft gluon were emitted from a lightlike line in the $\vec{p}_1 + \vec{p}_2$ direction.

- Consider the sum

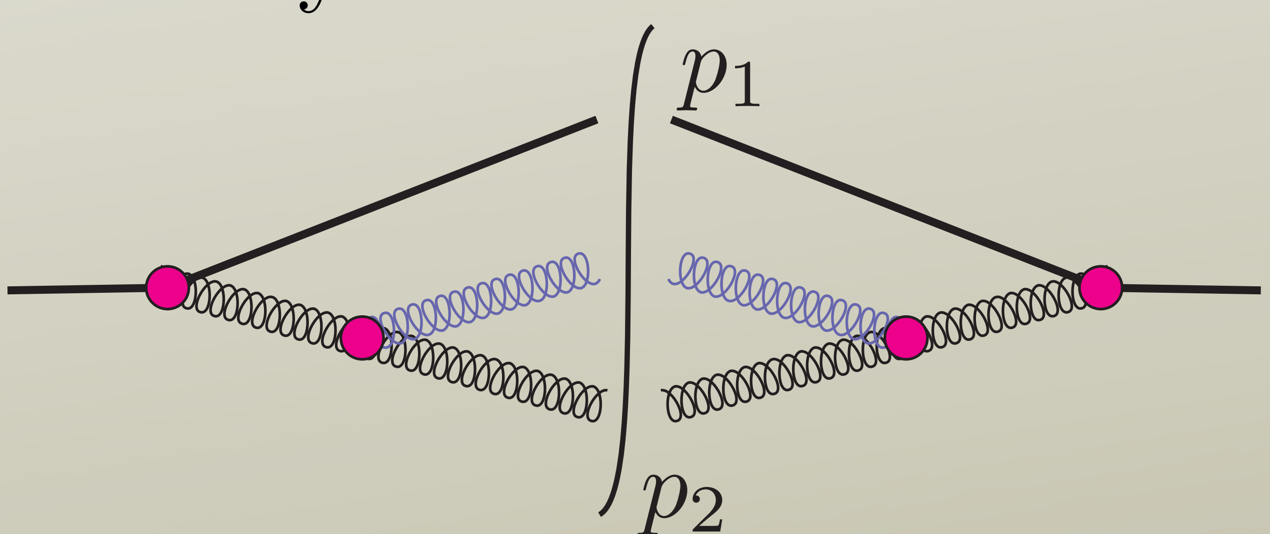


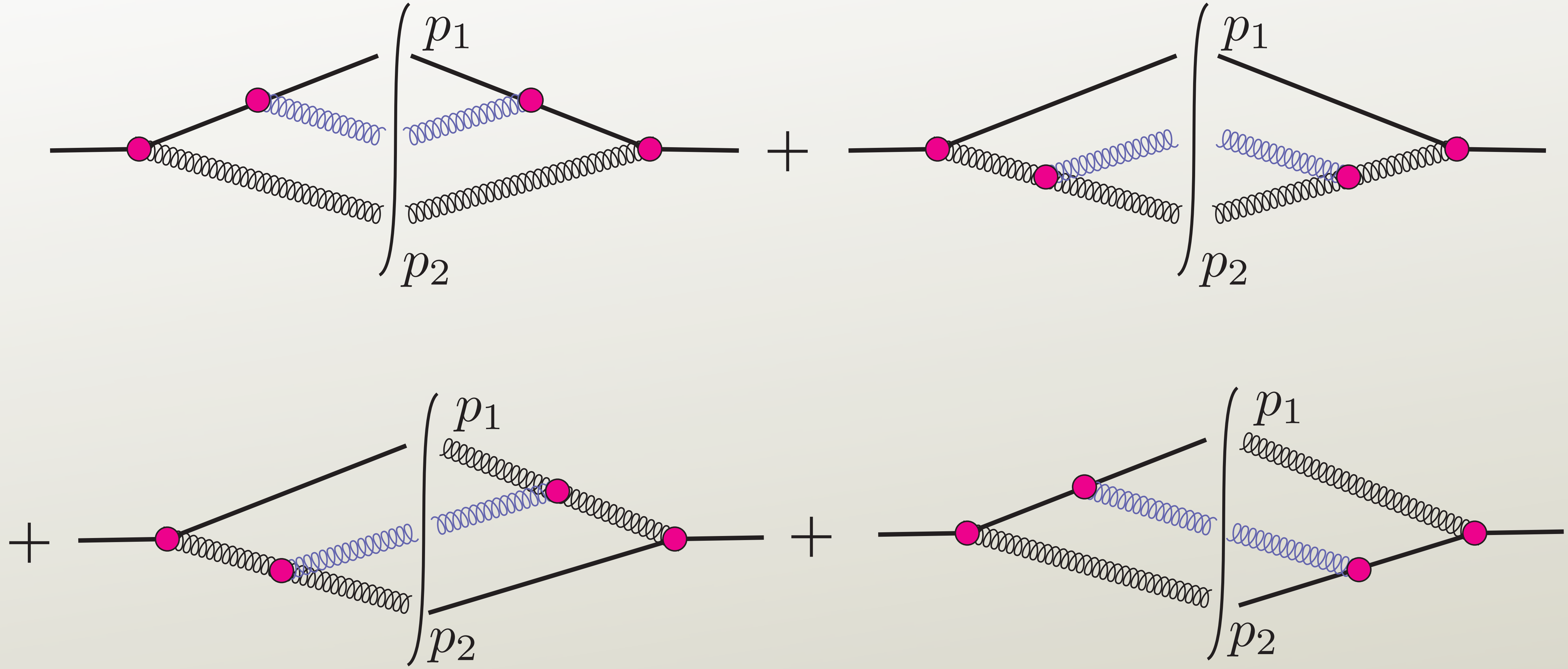
- If we add the graphs when $1 - \vec{u}_q \cdot \vec{u}_1 \ll 1 - \vec{u}_q \cdot \vec{u}_2$, we get approximately



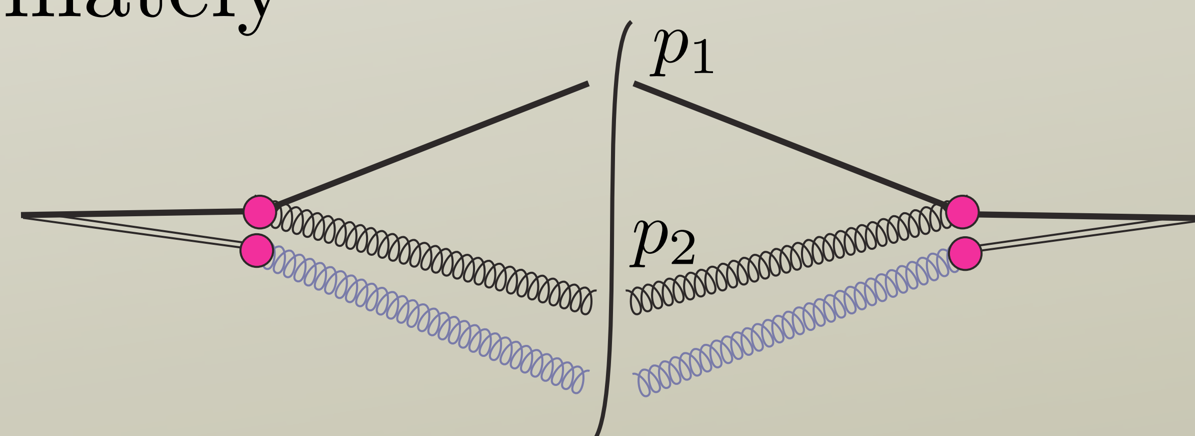


- If we add the graphs when $1 - \vec{u}_q \cdot \vec{u}_2 \ll 1 - \vec{u}_q \cdot \vec{u}_1$, we get approximately

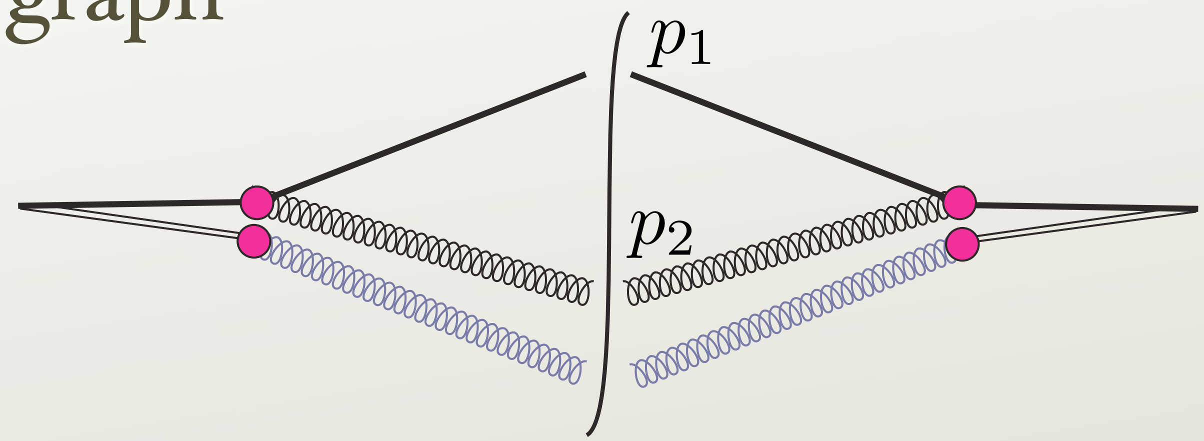




- If we add the graphs when $1 - \vec{u}_1 \cdot \vec{u}_2 \ll 1 - \vec{u}_q \cdot \vec{u}_1$, we get approximately



- For the graph



it is as if the soft, wide-angle gluon were emitted first, from an on-shell quark.

- This suggests omitting interference graphs and ordering the splittings in order of emission angles, treating daughter partons as on-shell.
- Impose lower limit on virtuality of these splittings, say 1 GeV.
- This gives an angle-ordered shower, as in Herwig.

Summary

- There are two ways to construct parton showers.
- A virtuality ordered shower puts the hardest interactions first, based on the hard-soft factorization of Feynman graphs.
  - Actually, transverse momentum is usually used in place of virtuality.
  - One needs to include interference graphs.
- Alternatively, one can skip the interference graphs and use an angle ordered shower.