Introduction to Monte Carlo Event Generators

Stefan Höche

SLAC NAL Theory Group

CTEQ-Fermilab School on QCD and Electroweak Phenomenology

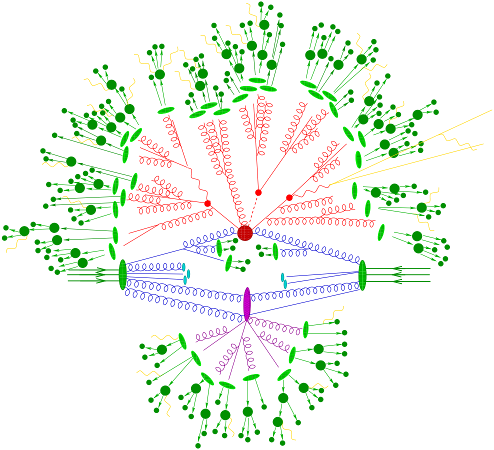
PUPC Lima, 07/30-08/09 2012

Outline of Lecture I

- The structure of Monte Carlo events
- Matrix Elements (ME)
- Parton Showers (PS)
- Matching ME and PS at NLO

The structure of MC events

- Hard interaction
- QCD evolution
- Secondary hard interactions
- Hadronization
- Hadron decays
- Higher-order QED corrections



Event generators for LHC physics

[Buckley et al.] PR504(2011)145

Herwig

- Originated in coherent shower studies → angular ordered PS
- Front-runner in development of MC@NLO and POWHEG
- Simple in-house ME generator & spin-correlated decay chains
- Original framework for cluster fragmentation

Pythia

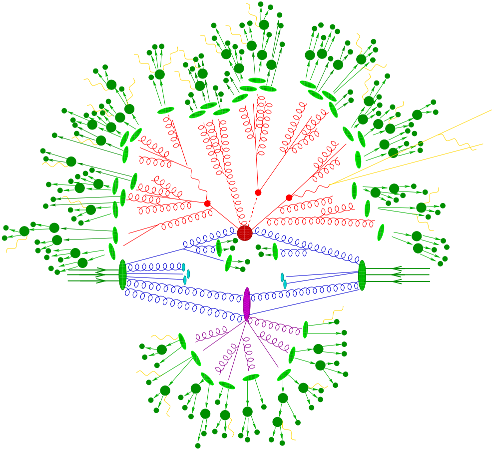
- Originated in hadronization studies → Lund string
- Leading in development of multiple interaction models
- Pragmatic attitude to ME generation → external tools
- Extensive PS development and earliest ME⊗PS matching

Sherpa

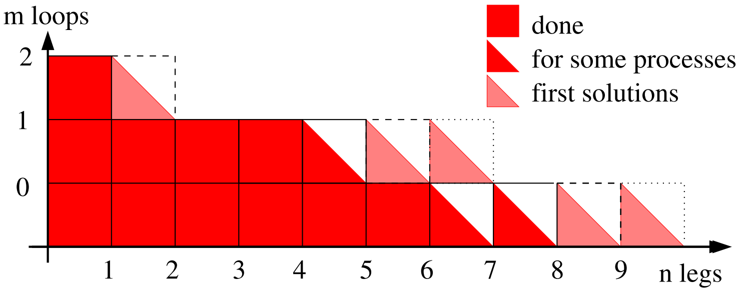
- Started with PS generator APACIC++ & ME generator AMEGIC++
- Current MPI model and hadronization pragmatic add-ons
- Leading in development of automated ME⊗PS merging
- Automated framework for NLO calculations and MC@NLO

The structure of MC events

- Hard interaction
- QCD evolution
- Secondary hard interactions
- Hadronization
- Hadron decays
- Higher-order QED corrections



Availability of parton-level calculations



- Exact n-parton perturbative calculations often best
- But typically hard to do, especially for ‘many’ loops
- Zero loops case highly automated
- One loop case getting close

Amplitude generation

- **Textbook**: Use completeness relations to square amplitudes
  sum/average over external states (helicity and color)
  Computational effort grows quadratically with number of diagrams
- **Real life**: Amplitudes are complex numbers
  first compute them, then add and square
  Effort grows linearly with number of diagrams

- Applies to dynamical degrees of freedom only
  - Consider helicity: Polarizations depend on momenta
    need to recompute for each phase-space point
  - Consider color: Mostly summed over at low multiplicity
    independent of other d.o.f. $\rightarrow$ no need to recompute

Helicity

[Dixon] hep-ph/9601359
[Dittmaier] hep-ph/9805445

- Weyl-van-der-Waerden spinors for helicity states $+/-$

$$\chi_+(p) = \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\phi_p} \end{pmatrix} \qquad \chi_-(p) = \begin{pmatrix} \sqrt{p^-}e^{i\phi_p} \\ -\sqrt{p^+} \end{pmatrix} \qquad \begin{aligned} p^\pm &= p^0 \pm p^3 \\ p_\perp &= p^1 + ip^2 \end{aligned}$$

  Basic building blocks for all amplitudes
  $+, -, \perp$ directions define "spinor gauge"

- Massive Dirac spinors in terms of WvdW spinors

$$u_+(p,m) = \frac{1}{\sqrt{2\,\bar{p}}} \begin{pmatrix} \sqrt{p_0 - \bar{p}}\,\chi_+(\hat{p}) \\ \sqrt{p_0 + \bar{p}}\,\chi_+(\hat{p}) \end{pmatrix} \qquad \bar{p} = \mathrm{sgn}(p_0)\,|\vec{p}|$$

$$u_-(p,m) = \frac{1}{\sqrt{2\,\bar{p}}} \begin{pmatrix} \sqrt{p_0 + \bar{p}}\,\chi_-(\hat{p}) \\ \sqrt{p_0 - \bar{p}}\,\chi_-(\hat{p}) \end{pmatrix} \qquad \hat{p} = (\bar{p}, \vec{p})$$

- γ^5 conveniently defined in Weyl representation

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}$$

  Projection operator $P_{R,L} = P_\pm = (1 \pm \gamma^5)/2$ identifies
  lower/upper component of Dirac spinors as right-/left-handed

Helicity

- Massless polarizations constructed from $u_\pm(p)$ and $u_\pm(k)$ with external light-like gauge vector k

$$\varepsilon^\mu_\pm(p,k) \;=\; \pm\frac{\bar{u}_\mp(k)\gamma^\mu u_\mp(p)}{\sqrt{2}\,\bar{u}_\mp(k)u_\pm(p)}\;.$$

  Defines light-like axial gauge

- For massive particles decompose momentum p using k

$$b = p - \kappa k \qquad \kappa = \frac{p^2}{2pk} \qquad \Rightarrow \qquad b^2 = 0$$

  Transverse polarizations as in massless case ($p \to b$) plus longitudinal

$$\varepsilon^\mu_0(p,k) = \frac{1}{m}\left(\,\bar{u}_-(b)\gamma^\mu u_-(b) - \kappa\,\bar{u}_-(k)\gamma^\mu u_-(k)\,\right)$$

- Vertices & propagators have simpler structure

- Building blocks for Standard model complete!

Color

[Maltoni,Stelzer,Willenbrock] hep-ph/0209271
[Duhr,SH,Maltoni] hep-ph/0607057

- QCD amplitudes can be stripped of color factors
- Fundamental representation for n-gluons

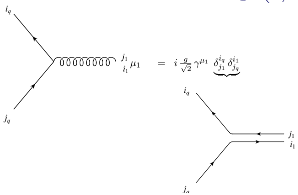
$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n)} \mathrm{Tr}(\lambda^{a_1}\lambda^{a_{\sigma_2}}\ldots\lambda^{a_{\sigma_n}})\, A(p_1,p_{\sigma_2},\ldots,p_{\sigma_n})$$

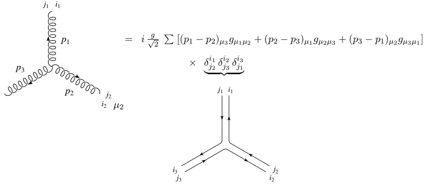
- Adjoint representation for n-gluons

$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n-1)} \left[F^{a_{\sigma_2}}\ldots F^{a_{\sigma_{n-1}}}\right]^{a_1}_{a_n} A(p_1,p_{\sigma_2},\ldots,p_{\sigma_{n-1}},p_n)$$

- Color-flow representation for n-gluons

$$\mathcal{A}_n(p_1,\ldots,p_n) = \sum_{\vec{\sigma}\in P(2,\ldots,n)} \delta^{i_1}_{j_{\sigma_2}}\delta^{i_{\sigma_2}}_{j_{\sigma_3}}\ldots\delta^{i_{\sigma_n}}_{j_1}\, A(p_1,p_{\sigma_2},\ldots,p_{\sigma_n})$$





Color

- We can sample colors just like we sample momenta
- Assign one in (r, g, b) / $(\bar{r}, \bar{g}, \bar{b})$ to each external (anti-)quark & gluon
- Average number of partial amplitudes is then smallest in color-flow basis

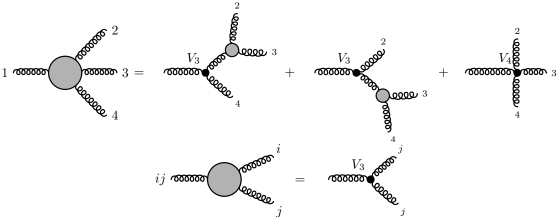
	Average # of partials		
n	Gell-Mann	Color-flow	Adjoint
4	4.83	1.28	1.15
5	15.2	1.83	1.52
6	56.5	3.21	2.55
7	251	6.80	5.53
8	1280	17.0	15.8
9	7440	48.7	56.4
10	47800	158	243

	Time [s/10^4pt]	
n	CO	CD
4	1.20	1.04
5	3.78	2.69
6	14.2	7.19
7	58.5	23.7
8	276	82.1
9	1450	270
10	7960	864

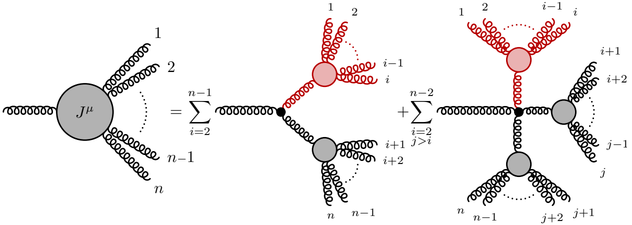
- Computational effort reduced further by not stripping amplitudes of color factors
- Evaluate dynamically at each vertex $\rightarrow$ straightforward computer algorithm
- Color dressing (CD) vs. color ordering (CO)

Amplitude construction

Example: Diagrams for
$g(1)g(2) \to g(3)g(4)$



[Berends,Giele] NPB306(1988)759



Example: Currents for
$g(1)g(2) \to g(3)g(4)$

Step 1	$J_1 = \varepsilon(1)$	$J_2 = \varepsilon(2)$	$J_3 = \varepsilon(3)$	$J_4 = \varepsilon(4)$
Step 2	J_{12}	J_{13}	J_{23}	
Step 3	J_{123}			
Step 4	$A(1,2,3,4) = J_4^* \, J_{123}$			

Phase space

[James] CERN-68-15
[Byckling,Kajantie] NPB9(1969)568

- Need to evaluate in a process-independent way

$$\mathrm{d}\Phi_n(p_a,p_b;p_1,..,p_n) = \left[\prod_{i=1}^{n}\frac{\mathrm{d}^3p_i}{(2\pi)^3\,2E_i}\right]\delta^4\left(p_a+p_b-\sum_{i=1}^{n}p_n\right)$$

- Use factorization properties of phase-space integral

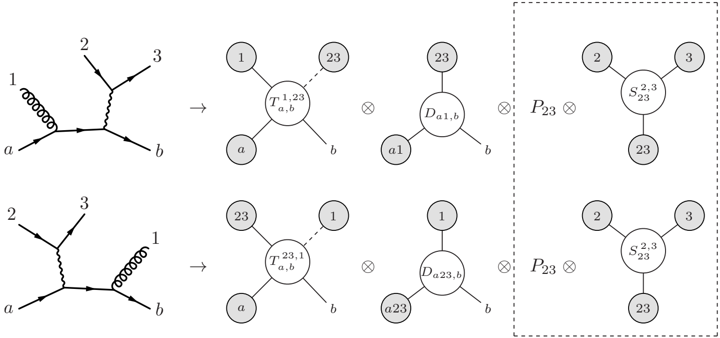
$$\mathrm{d}\Phi_n(p_a,p_b;p_1,..,p_n) = \mathrm{d}\Phi_{n-m+1}(p_a,p_b;p_{1m},p_{m+1},..,p_n) \times \frac{\mathrm{d}s_{1m}}{2\pi}\,\mathrm{d}\Phi_m(p_{1m};p_1,..,p_m)$$

- Apply repeatedly until only 2-particle phase spaces remain

$$\mathrm{d}\Phi_2 = \frac{\lambda(s_{ij},m_i^2,m_j^2)}{16\pi^2\,2s_{ij}}\mathrm{d}\cos\theta_i\mathrm{d}\phi_i$$

$\lambda^2(a,b,c) = (a-b-c)^2-4bc$ - Källen function

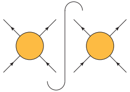
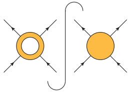
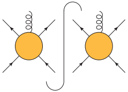
Phase space



- Construct one integrator per diagram and combine into multi-channel
- Intuitive notion of pole structure, multi-channel determines balance
- Factorial growth with number of diagrams can be tamed by recursion

Matrix element generation at NLO

NLO calculation

- Born term: $\mathrm{B} =$ 
- Virtual terms: $\mathrm{V} = \sum 2\,\mathrm{Re}\Big\{$  $\Big\}$
- Real terms: $\mathrm{R} = \sum$ 

- UV divergences in V removed by renormalization procedure
- V and R both still infrared divergent
- IR divergences cancel between V and R (KLN theorem)
- Exploit this fact to construct finite integrand for MC

Toy model for NLO

[Frixione,Webber] hep-ph/0204244

- Assume system of charged particles which radiates "photons" of fractional energy x.
- Predicting infrared-safe observables O amounts to computing expectation values

$$\langle O\rangle = \lim_{\varepsilon\to 0}\int_0^1 \mathrm{d}x\, x^{-2\varepsilon} O(x)\left[\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{B}} + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{V}} + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{R}}\right]$$

- Born, virtual and real-emission contributions given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{B,V,R}} = \quad \mathrm{B}\,\delta(x), \qquad \left(\mathrm{V}_f + \frac{\mathrm{B}C}{2\varepsilon}\right)\delta(x), \qquad \frac{\mathrm{R}(x)}{x}$$

Real-emission behaves as $\lim_{x\to 0}\mathrm{R}(x) = \mathrm{B}C$

$$\text{Virtual correction}\left\{\begin{array}{ccl} \mathrm{V}_f & - & \text{finite piece} \\ \mathrm{B}C/2\varepsilon & - & \text{singular piece}\end{array}\right.$$

Implicit: All higher-order terms proportional to coupling α

Subtraction in the toy model

- Perform NLO calculation in subtraction method

$$\langle O\rangle_R = \mathrm{B}C\,O(0)\int_0^1 \mathrm{d}x\frac{x^{-2\varepsilon}}{x} + \int_0^1 \mathrm{d}x\,\frac{\mathrm{R}(x)\,O(x) - \mathrm{B}C\,O(0)}{x^{1+2\varepsilon}}$$

- Second integral non-singular $\to$ set $\varepsilon = 0$

$$\langle O\rangle_R = -\frac{\mathrm{B}C}{2\varepsilon}\,O(0) + \int_0^1 \mathrm{d}x\,\frac{\mathrm{R}(x)\,O(x) - \mathrm{B}C\,O(0)}{x}$$

- Combine everything with Born and virtual correction

$$\langle O\rangle = \Big(\mathrm{B} + \mathrm{V}_f\Big)\,O(0) + \int_0^1 \frac{\mathrm{d}x}{x}\Big[\mathrm{R}(x)\,O(x) - \mathrm{B}C\,O(0)\Big]$$

  Both terms separately finite

- Rewrite for further reference

$$\langle O\rangle = \Big(\mathrm{B} + \mathrm{V} + \mathrm{I}\Big)\,O(0) + \int_0^1 \frac{\mathrm{d}x}{x}\Big[\mathrm{R}(x)\,O(x) - \mathrm{S}\,O(0)\Big]$$

  $\mathrm{I} = -\mathrm{B}C/2\varepsilon \to$ Integrated subtraction term
  $\mathrm{S} = \mathrm{B}C \to$ Real subtraction term

QCD subtraction

- QCD subtraction a little more cumbersome due to spin and colour correlations in R
- Basic features surviving from toy model are phase-space mapping and subtraction terms as products of Born times splitting operator
- Commonly used techniques
  - Dipole method

    [Catani,Seymour] NPB485(1997)291

    [Catani,Dittmaier,Seymour,Trocsanyi] NPB627(2002)189
  - FKS method

    [Frixione,Kunszt,Signer] NPB467(1996)399

Matrix element generators

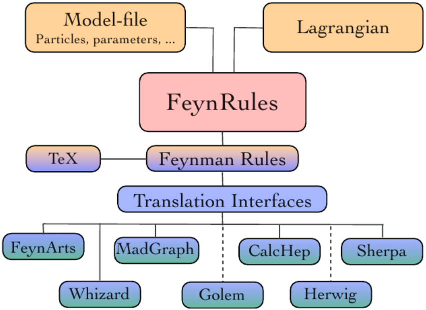
- Commonly used ME generators

	Built-in models	$2 \to$	$\|M_n\|^2$	$d\Phi_n$	NLO
ALPGEN	SM	8	recursive	Multi	-
AMEGIC	SM,MSSM,ADD	6	diagrams	Multi	sub
Comix	SM	8	recursive	Multi	sub
CompHEP	SM,MSSM	4	textbook	Single	-
HELAC	SM	8	recursive	Multi	sub+loop
MadEvent	SM,MSSM,UED	6	diagrams	Multi	sub(+loop)
Whizard	SM,MSSM,LH	8	recursive	Multi	sub

New physics models

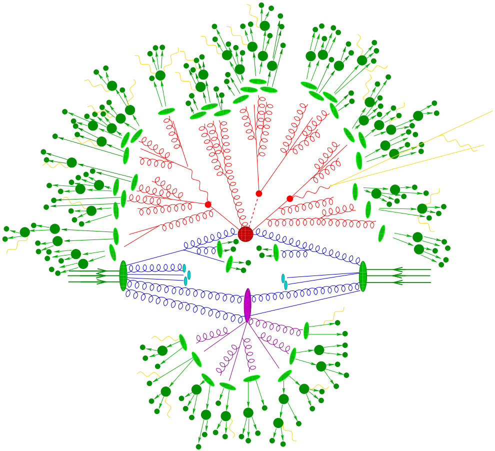
- Most ME generators suited for any physics model, but implementing Feynman rules tedious and error-prone
- Automated by FeynRules package
- Extracts vertices from Lagrangian based on minimal information about particle content
- Writes generator-specific output permitting easy cross-checks

[Christensen,Duhr] arXiv:0806.4194

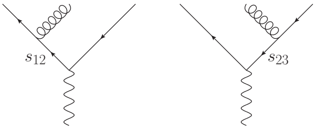


The structure of MC events

- Hard interaction
- QCD evolution
- Secondary hard interactions
- Hadronization
- Hadron decays
- Higher-order QED corrections



Parton evolution



- Consider $e^+e^- \to 3$ partons

$$\frac{1}{\sigma_{2\to 2}}\frac{\mathrm{d}\sigma_{2\to 3}}{\mathrm{d}\cos\theta\mathrm{d}z} \sim C_F\frac{\alpha_s}{2\pi}\frac{2}{\sin^2\theta}\frac{1+(1-z)^2}{z}$$

  θ - angle of gluon emission
  z - fractional energy of gluon

- Divergent in
  - Collinear limit: $\theta \to 0, \pi$
  - Soft limit: $z \to 0$
- Separate into two independent jets

$$\frac{2\mathrm{d}\cos\theta}{\sin^2\theta} = \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\theta}{1+\cos\theta} = \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\bar\theta}{1-\cos\bar\theta} \approx \frac{\mathrm{d}\theta^2}{\theta^2} + \frac{\mathrm{d}\bar\theta^2}{\bar\theta^2}$$

- Independent jet evolution

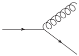
$$\mathrm{d}\sigma_3 \sim \sigma_2 \sum_{\text{jets}} C_F\frac{\alpha_s}{2\pi}\frac{\mathrm{d}\theta^2}{\theta^2}\mathrm{d}z\,\frac{1+(1-z)^2}{z}$$

Components of the parton shower

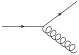
- Same equation for any variable with same limiting behavior
  - Transverse momentum $k_T^2 = z^2(1-z)^2\theta^2 E^2$
  - Virtuality $t = z(1-z)\theta^2 E^2$
- Call this the "evolution variable"

$$\frac{\mathrm{d}\theta^2}{\theta^2} = \frac{\mathrm{d}k_T^2}{k_T^2} = \frac{\mathrm{d}t}{t} \qquad \leftrightarrow \qquad \text{collinear divergence}$$

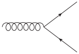
- Absorb z-dependence into flavor-dependent splitting kernel $P_{ab}(z)$



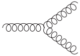
$$= C_F \frac{1+z^2}{1-z}$$



$$= C_F \frac{1+(1-z)^2}{z}$$



$$= T_R \left[z^2 + (1-z)^2 \right]$$



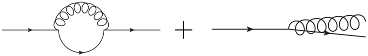
$$= C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right]$$

- Universal DGLAP evolution equation emerges

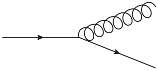
$$\mathrm{d}\sigma_{n+1} \sim \sigma_n \sum_{\text{jets}} \frac{\mathrm{d}t}{t} \mathrm{d}z \, \frac{\alpha_s}{2\pi} P_{ab}(z)$$

The Sudakov form factor

- Collinear partons not separately resolveable
- Introduce finite resolution criterion, e.g. $t > t_c$



- virtual+unresolved = finite



- resolved = finite

- Unitarity / Probability conservation $\rightarrow$ resolved $+$ unresolved $= 1$
- Must implement no-emission probability (Poisson statistics)

$$\mathrm{d}\mathcal{P}_{\mathrm{emit}}(t) = \frac{\mathrm{d}t}{t} \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} P_{ab}(z) \quad \rightarrow \quad \mathcal{P}_{\mathrm{no}}(t,t') = \exp\left\{ -\int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} P_{ab}(z) \right\}$$

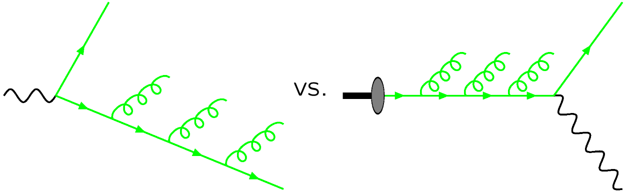
- Call $\Delta(t,t') := \mathcal{P}_{\mathrm{no}}(t,t')$ the Sudakov form factor
- Total probability for parton produced at t' to radiate at t is

$$\mathrm{d}\mathcal{P}(t) = \mathrm{d}\mathcal{P}_{\mathrm{emit}}(t)\mathcal{P}_{\mathrm{no}}(t,t') = \mathrm{d}t \, \frac{\mathrm{d}\Delta(t,t')}{\mathrm{d}t}$$

Initial-state evolution

[Sjöstrand] PLB175(1985)321

- Iteration leads to tree-like approximation of higher-order configuration
- Slight difference between final-state and initial-state evolution



- Initial-state emission probability must account for probability to resolve (different) parton at larger x

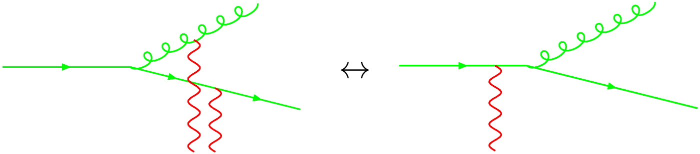
$$\mathrm{d}\mathcal{P}_{\mathrm{emit}}(x,t) = \frac{\mathrm{d}t}{t}\int\frac{\mathrm{d}z}{z}\,\frac{\alpha_s}{2\pi}P_{ab}(z)\,\frac{f_b(x/z,t)}{f_a(x,t)}$$

- Hard to implement in forward evolution (increasing t)
- Standard method is to evolve backward in initial state

Color coherence and angular ordering

[Marchesini,Webber] NPB310(1988)461

- Gluons with large wavelength not capable of resolving charges of emitting color dipole individually



- Emission occurs with combined charge of mother parton instead
- Net effect is destructive interference outside cone with opening angle defined by emitting color dipole
- Can be implemented directly by angular ordering variable or additional ordering criterion in parton showers

The MC algorithm for parton showers

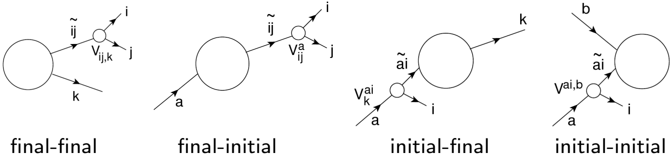
- Start with set of n partons at scale t', which evolve collectively
  Sudakov form factors factorize, schematically

$$\Delta(t,t') = \prod_{i=1}^{n} \Delta_i(t,t') \qquad\qquad \Delta_i(t,t') = \prod_{j=q,g} \Delta_{i\to j}(t,t')$$

- Use veto algorithm to find new scale t where branching occurs
  - Generate t using overestimate $\alpha_s^{\max} P_{ab}^{\max}(z)$
  - Determine “winner” parton i and select new flavor j
  - Select splitting variable according to overestimate
  - Accept point with weight $\alpha_s(k_T^2)P_{ab}(z)/\alpha_s^{\max} P_{ab}^{\max}(z)$
- Construct splitting kinematics and update color flow
- Continue until $t < t_c$

Dipole showers

- In parton showers, there is no such thing as a collinear limit
  But who absorbs recoil when a splitting parton goes off mass-shell?
- No answer in DGLAP evolution equations $\leftrightarrow$ collinear limit
  Ambiguity introduces large uncertainties, especially at large t
- Natural solution provided by $2 \to 3$ splittings
  Spectator kinematics enters splitting probability
- Basic concept of dipole showers

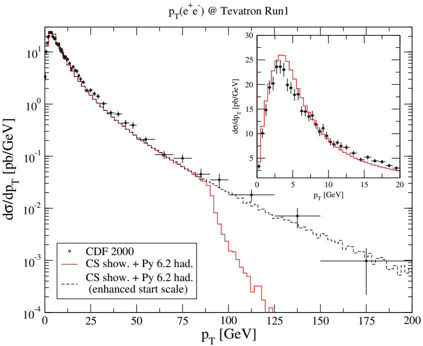


Parton-shower programs

- Publicly available generators

	Evolution variable	Splitting variable	Coherence
Ariadne	dipole-$k_\perp^2$	Rapidity	$2 \to 3$ kernel
Herwig	$E^2\theta^2$	Energy fraction	AO
Herwig++	$(t - m^2)/z(1-z)$	LC mom fraction	AO
Pythia 6.x	t	Energy fraction	Enforced
Pythia 8	$k_\perp^2$	LC mom fraction	Enforced
Sherpa 1.1.x	t	Energy fraction	Enforced
Sherpa 1.2.x	dipole-$k_\perp^2$	LC mom fraction	$2 \to 3$ kernel
Vincia	dipole-$k_\perp^2$	LC mom fraction	$2 \to 3$ kernel

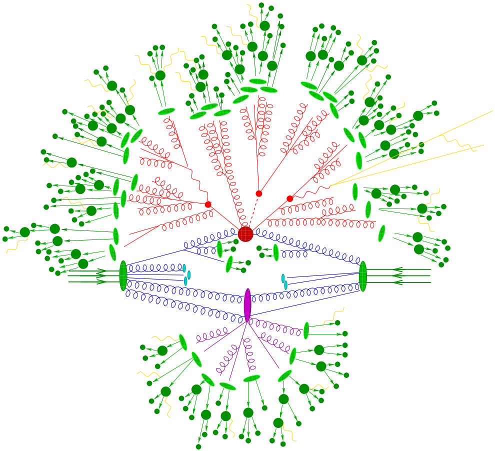
Effects of the parton shower



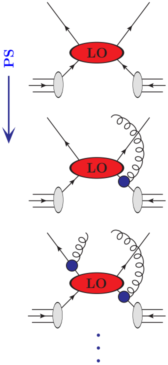
- Example: Drell-Yan lepton pair production at Tevatron
- If ME computed at leading order, then parton shower is only source of transverse momentum

The structure of MC events

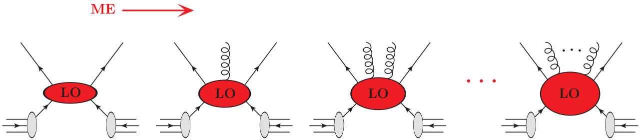
- Hard interaction
- QCD evolution
- Secondary hard interactions
- Hadronization
- Hadron decays
- Higher-order QED corrections



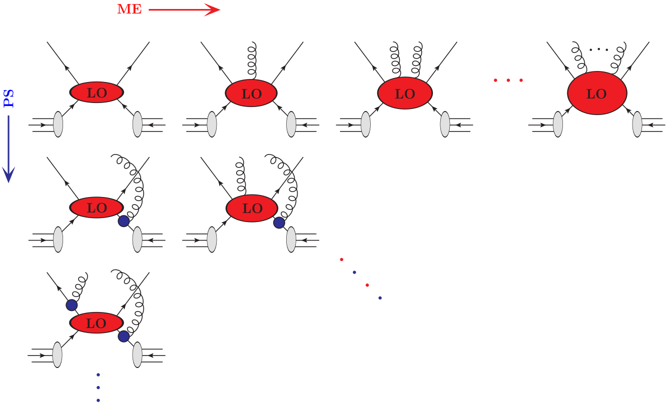
Parton showers



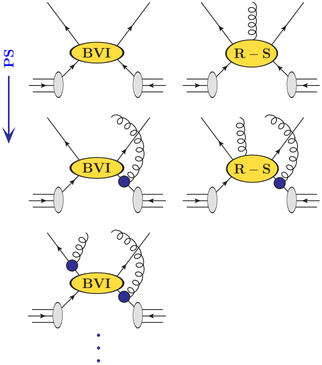
Matrix elements



Matrix elements & parton showers



NLO−PS matching



Parton showers from the fixed-order prespective

- Start with leading-order parton-level event

$$\mathrm{d}\sigma_{\mathrm{MC}} = \mathrm{d}\Phi_n \mathrm{B}_n(\Phi_n)$$

Φ_n - point in n-particle phase space
B_n - Leading order (Born) matrix element

- Generate emissions according to Sudakov form factor

$$\Delta_n^{(\mathrm{PS})}(t',t'') = \exp\left\{-\int_{t'}^{t''} \mathrm{d}\Phi_1\, \mathrm{K}_n(\Phi_1)\right\}$$

$\mathrm{d}\Phi_1 \propto \mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\phi/2\pi$ - one-emission phase space
$\mathrm{K}_n = \sum_{a,b} \frac{\alpha_s}{2\pi\, t} P_{ab}$ - sum of evolution kernels for n-particle final state

- Differential event rate up to first emission

$$\mathrm{d}\sigma_{\mathrm{MC@LO}} = \mathrm{d}\Phi_n \mathrm{B}_n(\Phi_n)\left[\Delta_n^{(\mathrm{PS})}(t_c,\mu_Q^2) + \int_{t_c}^{\mu_Q^2} \mathrm{d}\Phi_1 \mathrm{K}_n(\Phi_1)\Delta_n^{(\mathrm{PS})}(t(\Phi_1),\mu_Q^2)\right]$$

Modified subtraction

[Frixione,Webber] hep-ph/0204244

- Revisit toy model for NLO

$$\langle O\rangle = \Big(\mathrm{B}+\mathrm{V}+\mathrm{I}\Big)\,O(0) + \int_0^1 \frac{\mathrm{d}x}{x}\,\Big[\mathrm{R}(x)\,O(x) - \mathrm{S}\,O(0)\Big]$$

- In parton showers, any number of "photons" can be emitted
- Emission probability controlled by Sudakov form factor

$$\Delta(x_1,x_2) = \exp\left\{-\int_{x_1}^{x_2}\frac{\mathrm{d}x}{x}\,\mathrm{K}(x)\right\}$$

Evolution kernel behaves as $\lim\limits_{x\to 0}\mathrm{K}(x) = \mathrm{R}(0)/\mathrm{B} = C$

- Define generating functional of PS $\rightarrow$
$\mathcal{F}^{(n)}_{\mathrm{MC}}(x) \leftrightarrow$ PS starting from n emissions at x
- $\mathcal{F}^{(n)}_{\mathrm{MC}}(x)$ now replaces observable O Naively:

$O(0) \Leftrightarrow$ start MC with 0 emissions $\rightarrow \mathcal{F}^{(0)}_{\mathrm{MC}}$
$O(x) \Leftrightarrow$ start MC with 1 emission $\rightarrow \mathcal{F}^{(1)}_{\mathrm{MC}}(x)$

Modified subtraction

- Combined generating functional would be

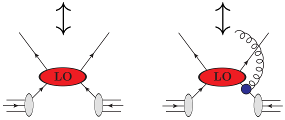
$$\left[\Big(\mathrm{B}+\mathrm{V}+\mathrm{I}\Big)-\int_0^1\frac{\mathrm{d}x}{x}\mathrm{S}\right]\mathcal{F}^{(0)}_{\mathrm{MC}}+\int_0^1\frac{\mathrm{d}x}{x}\,\mathrm{R}(x)\,\mathcal{F}^{(1)}_{\mathrm{MC}}(x)$$

- This is wrong because

$$\mathcal{F}^{(0)}_{\mathrm{MC}}=\Delta(x_c,1)+\int_{x_c}^1\frac{\mathrm{d}x}{x}\mathrm{K}(x)\Delta(x,1)$$

- So $\mathrm{B}\,\mathcal{F}^{(0)}_{\mathrm{MC}}$ generates an $\mathcal{O}(\alpha)$ term that spoils NLO accuracy

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}\right)_{\mathrm{MC}}O(x)=\mathrm{B}\left[-\frac{\mathrm{K}(x)}{x}\,O(0)+\frac{\mathrm{K}(x)}{x}\,O(x)\right]$$



Modified subtraction

- The proper MC@NLO is obtained by subtracting this $\mathcal{O}(\alpha)$ contribution

$$\mathcal{F}_{\mathrm{MC@NLO}} = \underbrace{\left[\left(\mathrm{B}+\mathrm{V}+\mathrm{I}\right) + \int_0^1 \frac{\mathrm{d}x}{x}\left(\mathrm{BK}(x)-\mathrm{S}\right)\right]}_{\text{NLO-weighted Born cross section}} \mathcal{F}^{(0)}_{\mathrm{MC}} + \int_0^1 \frac{\mathrm{d}x}{x} \underbrace{\left[\mathrm{R}(x)-\mathrm{BK}(x)\right]}_{\text{modified subtraction}} \mathcal{F}^{(1)}_{\mathrm{MC}}(x)$$

- Like at fixed order, both terms are separately finite
- We call events from the first term $\mathbb{S}$-events ($\mathbb{S}$tandard) and events from the second term $\mathbb{H}$-events ($\mathbb{H}$ard)
- For further reference, define $\mathrm{D}^{(\mathrm{K})}(x) := \mathrm{BK}(x)$ as well as

$$\bar{\mathrm{B}}^{(\mathrm{K})} = \left(\mathrm{B}+\mathrm{V}+\mathrm{I}\right) + \int_0^1 \frac{\mathrm{d}x}{x}\left(\mathrm{D}^{(\mathrm{K})}(x)-\mathrm{S}\right), \quad \mathrm{H}^{(\mathrm{K})}(x) = \mathrm{R}(x) - \mathrm{D}^{(\mathrm{K})}(x)$$

$\rightarrow$ compact notation

$$\mathcal{F}_{\mathrm{MC@NLO}} = \bar{\mathrm{B}}^{(\mathrm{K})}\,\mathcal{F}^{(0)}_{\mathrm{MC}} + \int_0^1 \frac{\mathrm{d}x}{x}\,\mathrm{H}^{(\mathrm{K})}(x)\,\mathcal{F}^{(1)}_{\mathrm{MC}}(x)$$

MC@NLO

[Frixione,Webber] hep-ph/0204244

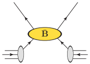
- Apply toy model to QCD, but include $1/x$-terms in coefficient functions
  Also need to sum over all flavor contributions at real-emission level

$$\bar{\mathrm{B}}_n^{(\mathrm{K})}(\Phi_n) = \Big(\mathrm{B}_n(\Phi_n) + \tilde{\mathrm{V}}_n(\Phi_n) + \mathrm{I}_n(\Phi_n)\Big) + \int \mathrm{d}\Phi_1 \Big(\mathrm{D}_n^{(\mathrm{K})}(\Phi_{n+1}) - \mathrm{S}_n(\Phi_{n+1})\Big)$$

$$\mathrm{H}_n^{(\mathrm{K})}(\Phi_{n+1}) = \mathrm{R}_n(\Phi_{n+1}) - \mathrm{D}_n^{(\mathrm{K})}(\Phi_{n+1})$$

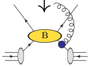
- Full differential event rate up to first emission

$$\mathrm{d}\sigma_{\mathrm{MC@NLO}} = \mathrm{d}\Phi_n \bar{\mathrm{B}}_n^{(\mathrm{K})}(\Phi_n)\Bigg[\Delta_n^{(\mathrm{PS})}(t_c, \mu_Q^2) \;\leftrightarrow$$

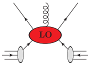


$$+ \int_{t_c}^{\mu_Q^2} \mathrm{d}\Phi_1 \mathrm{K}_n(\Phi_1) \Delta_n^{(\mathrm{PS})}(t(\Phi_1), \mu_Q^2)\Bigg] + \mathrm{d}\Phi_n \int \mathrm{d}\Phi_1\, \mathrm{H}_n^{(\mathrm{K})}(\Phi_{n+1})$$

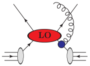
$\updownarrow$



$\updownarrow$

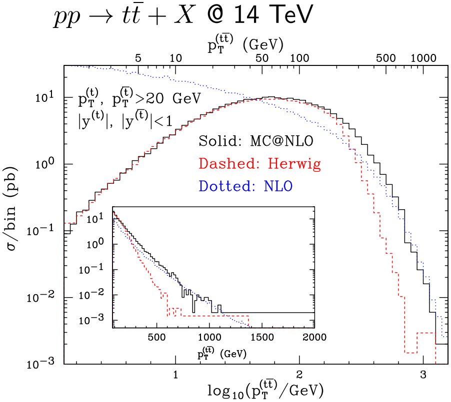


$-$



MC@NLO features

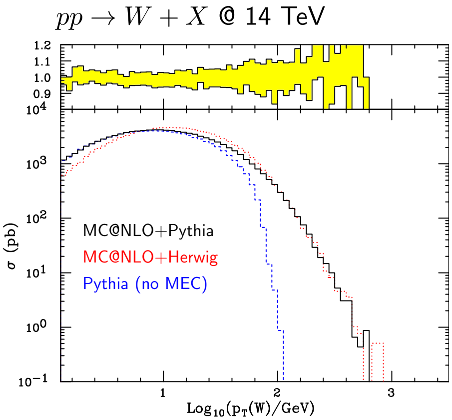
[Nason,Webber] arXiv:1202.1251



- MC@NLO interpolates smoothly between real-emission ME and PS

MC@NLO features

[Torrielli,Frixione] arXiv:1002.4293



- MC@NLO with different PS agree at high p_T $\leftrightarrow$ NLO
- Differences at low p_T due to differences in PS

POWHEG

[Nason] hep-ph/0409146

- Aim of the method: Eliminate negative weights from MC@NLO
- Set $\mathrm{D}_n^{(\mathrm{R})} := \mathrm{R}_n \to$ no $\mathbb{H}$-events $\Rightarrow$ $\bar{\mathrm{B}}_n^{(\mathrm{R})}$ positive in physical region
- Differential event rate up to first emission is

$$\mathrm{d}\sigma_{\mathrm{POWHEG}} = \mathrm{d}\Phi_n \bar{\mathrm{B}}_n^{(\mathrm{R})}(\Phi_n) \Bigg[\Delta_n^{(\mathrm{R})}(t_c, s_{\mathrm{had}}) + \int_{t_c}^{s_{\mathrm{had}}} \mathrm{d}\Phi_1 \frac{\mathrm{R}_n(\Phi_{n+1})}{\mathrm{B}_n(\Phi_n)} \Delta_n^{(\mathrm{R})}(t(\Phi_1), s_{\mathrm{had}}) \Bigg]$$

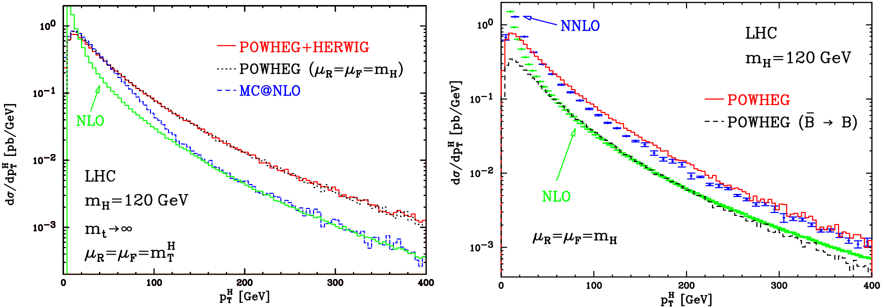
- μ_Q^2 has changed to hadronic centre-of-mass energy squared, s_{had}, as full phase space for real-emission correction, R_n, must be covered
- Absence of $\mathbb{H}$-events leads to enhancement of high-p_T region by

$$K = \frac{\bar{\mathrm{B}}_n}{\mathrm{B}_n} = 1 + \mathcal{O}(\alpha_s)$$

  Formally beyond NLO, but sizeable corrections in practice

POWHEG features

[Alioli,Nason,Oleari,Re] arXiv:0812.0578



- Large enhancement at high $p_{T,h}$
- Can be traced back to large NLO correction
- Fortunately, NNLO correction is also large $\to\sim$ agreement

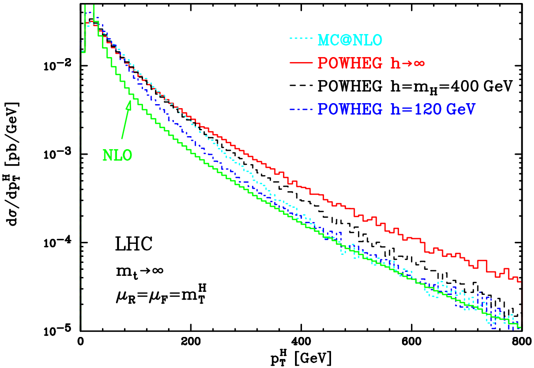
Improved POWHEG

- To avoid problems in high-p_T region, split real-emission ME into singular and finite parts as $\mathrm{R}_n = \mathrm{R}_n^s + \mathrm{R}_n^f$
- Treat singular piece in $\mathbb{S}$-events and finite piece in $\mathbb{H}$-events Similar to MC@NLO with redefined PS evolution kernels
- Differential event rate up to first emission

$$\mathrm{d}\sigma_{\mathrm{POWHEG}} = \mathrm{d}\Phi_n \bar{\mathrm{B}}_n^{(\mathrm{R}^s)}(\Phi_n)\bigg[\Delta_n^{(\mathrm{R}^s)}(t_c, s_{\mathrm{had}}) + \int_{t_c}^{s_{\mathrm{had}}} \mathrm{d}\Phi_1 \frac{\mathrm{R}_n^s(\Phi_{n+1})}{\mathrm{B}_n(\Phi_n)} \Delta_n^{(\mathrm{R}^s)}(t(\Phi_1), s_{\mathrm{had}})\bigg] + \mathrm{d}\Phi_n \int \mathrm{d}\Phi_1\, \mathrm{R}_n^f(\Phi_{n+1})$$

POWHEG features

[Alioli,Nason,Oleari,Re] arXiv:0812.0578



- Singular real-emission part here defined as

$$\mathrm{R}_n^s = \mathrm{R}\,\frac{h^2}{p_T^2 + h^2}$$

- Can “tune” NNLO contribution by varying free parameter h

Summary

- MC event generators use factorization to split simulated events into stages
- Fixed-order perturbative calculation describes production of hard objects, e.g. jets, leptons
- Parton shower describes evolution of jets and potentially production of additional jets
- Parton showers can be matched to fixed-order NLO to improve the description of Born-level observables

Stating the problem

- We want to compute expectation values of observables

$$\langle O\rangle = \sum_n \int \mathrm{d}\Phi_n \, P(\Phi_n)\, O(\Phi_n)$$

Φ_n - Point in n-particle phase-space
$P(\Phi_n)$ - Probability to produce Φ_n
$O(\Phi_n)$ - Value of observable at Φ_n

- Problem #1: Computing $P(\Phi_n)$
- Problem #2: Performing the integral

- Typically, problem #2 is harder to solve
This is where MC event generators come in

Numerical integration

- Assume one-dimensional integral

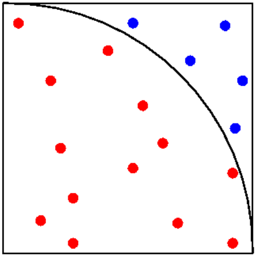
$$I = \int_a^b \mathrm{d}x\, f(x)$$

- Can be approximated numerically by sum over M rectangles of size $\Delta x = (b-a)/M$

$$I = \sum_{i=0}^{M-1} f(a + i\Delta x)\Delta x$$

- Error of numerical estimate is linear in Δx
- Trapezoids instead of rectangles reduce error $\to \propto (\Delta x)^2$
  $f(a + i\Delta x) \to [f(a + i\Delta x) + f(a + (i+1)\Delta x)]/2$
- Generic problem of the method: Number of points scales like M^n where n is the number of dimensions. Impossible to compute multi-particle observables

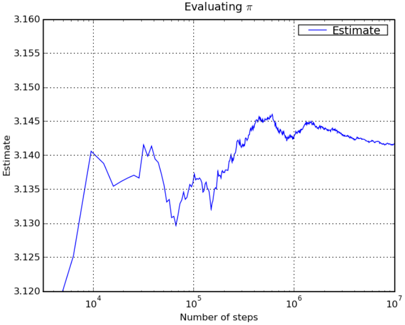
The hit-or-miss method



$$\frac{\text{Hits}}{\text{Misses} + \text{Hits}} \longrightarrow \frac{\pi}{4}$$

Throw random points (x,y),
with x, y in [0,1]

For hits: $(x^2 + y^2) < r^2 = 1$



Importance sampling

- In many cases we can approximate the integral of $f(x)$ with some known function $g(x)$ such that primitive $G(x)$ is known
- This amounts to a variable transformation

$$I = \int_a^b \mathrm{d}x\, g(x)\, \frac{f(x)}{g(x)} = \int_a^b \mathrm{d}G(x)\, w(x) \quad \text{where} \quad w(x) = \frac{f(x)}{g(x)}$$

- Integral and error estimate are

$$I = \big[G(b) - G(a)\big]\langle w\rangle \qquad \sigma = \big[G(b) - G(a)\big]\sqrt{\frac{\langle w^2\rangle - \langle w\rangle^2}{N-1}}$$

  N - Number of MC events (points)
- MC error scales as $1/\sqrt{N}$ independent of number of dimensions!
- Note that I is independent of $g(x)$, but σ is not $\rightarrow$ suitable choice of $g(x)$ can be used to minimize error

Selection from a known distribution

- Random number generators produce uniform pseudo-random numbers in $[0,1]$
- Assume we want points following the distribution $g(x)$ with known primitive $G(x)$ instead
- Probability of producing point in $[x, x+\mathrm{d}x]$ is $g(x)\,\mathrm{d}x$
- We can generate x according to

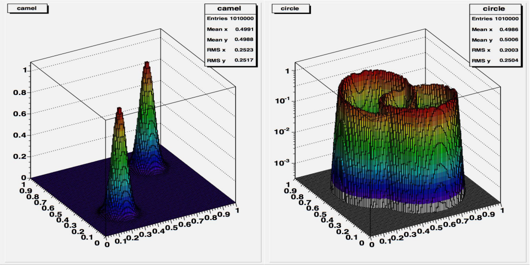
$$\int_a^x \mathrm{d}x'\, g(x') = R \int_a^b \mathrm{d}x'\, g(x')$$

where R is a uniform random number in $[0,1]$

$$x = G^{-1}\Big[G(a) + R\big(G(b) - G(a)\big)\Big]$$

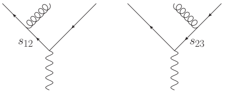
Stratified sampling

- Can divide interval $[a, b]$ into M bins
- Inside each bin, use a flat distribution g_i
- Choose $g_i = \max f(x)$ in interval i
- Problem: strongly peaked distributions
  $\rightarrow$ large variance due to inappropriate interval size
- Define intervals dynamically, such that each contributes equally to total variance $\rightarrow$ Vegas/Foam



The multi-channel method

- Assume function $f(x)$ which behaves roughly like $g_1(x) + g_2(x)$
- Practical example: Tree-level matrix element with divergences in s_{12} and s_{23}



- Define multi-channel

  $$g(x) = \sum_i \alpha_i \, g_i(x)$$

  α_i - multi-channel weight
  $g_i(x)$ - single channel
- Since all integrals $G_i(x)$ are known, event generation is easy
- Adjust weights such that each channel contributes equally to total variance $\to$ stratified sampling is a special case

Poisson distributions

- Assume nuclear decay process described by $g(x)$
- Nucleus can decay only if it has not decayed already
  Must account for survival probability $\leftrightarrow$ Poisson distribution

$$\mathcal{G}(x) \;=\; g(x)\Delta(x,b) \qquad \text{where} \qquad \Delta(x,b) \;=\; \exp\left\{-\int_x^b \mathrm{d}x'\,g(x')\right\}$$

- If $G(x)$ is known, then we also know the integral of $\mathcal{G}(x)$

$$\int_x^b \mathrm{d}x'\mathcal{G}(x') \;=\; \int_x^b \mathrm{d}x'\,\frac{\mathrm{d}\Delta(x',b)}{\mathrm{d}x'} \;=\; 1-\Delta(x,b)$$

- We can generate events by requiring $1-\Delta(x,b)=1-R$

$$x = G^{-1}\Big[G(b)+\log R\Big]$$

The veto algorithm

- Veto algorithm $\leftrightarrow$ Hit-or-miss method for Poisson distributions
  - Generate event according to $\mathcal{G}(x)$
  - Accept with $w(x) = f(x)/g(x)$
  - If rejected, continue generating from x
- Probability for immediate acceptance

$$\frac{f(x)}{g(x)}\,g(x)\exp\left\{-\int_x^b \mathrm{d}x'\,g(x')\right\}$$

- Probability for acceptance after one rejection

$$\frac{f(x)}{g(x)}\,g(x)\int_x^b \mathrm{d}x_1\exp\left\{-\int_x^{x_1}\mathrm{d}x'\,g(x')\right\}\left(1-\frac{f(x_1)}{g(x_1)}\right)g(x_1)\exp\left\{-\int_{x_1}^b \mathrm{d}x'\,g(x')\right\}$$

- For n intermediate rejections we obtain n nested integrals $\int_x^b\int_{x_1}^b\ldots\int_{x_{n-1}}^b$
- Disentangling yields $1/n!$ and summing over all possible rejections gives

$$f(x)\exp\left\{-\int_x^b \mathrm{d}x'\,g(x')\right\}\sum_{n=0}^{\infty}\frac{1}{n!}\int_x^b \mathrm{d}x'\,\left[g(x')-f(x')\right] = f(x)\exp\left\{-\int_x^b \mathrm{d}x'\,f(x')\right\}$$