Introduction to the Parton Model and Pertrubative QCD

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- **III. Factorization and Evolution**

  A. Factorization in DIS

  B. DIS at one loop

  C. Evolution

  D. Factorization in hadron-hadron scattering

  Appendices: structure of high orders in 1PI; Q_T resummation

IIIA. Factorization in DIS

- Challenge: use AF in observables σ (cross sections, also some amplitudes) that are not infrared safe

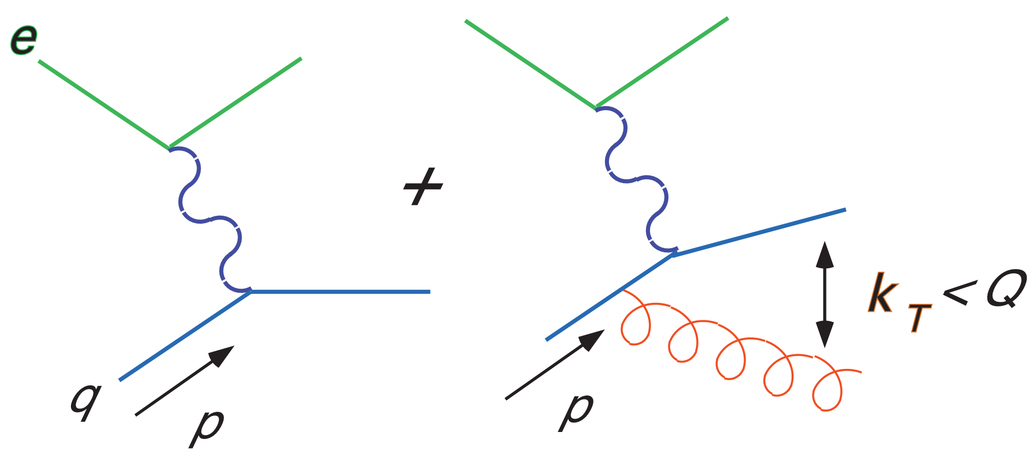
- Possible if: σ has a short-distance subprocess. Separate *IR Safe* from IR: this is factorization

- IR Safe part (short-distance) is calculable in pQCD

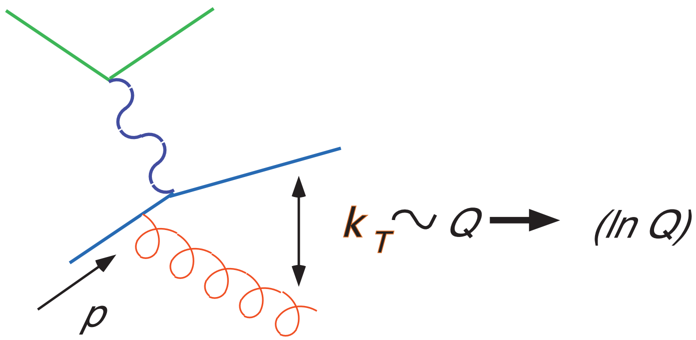
- Infrared part – example: parton distribution – measureable and universal

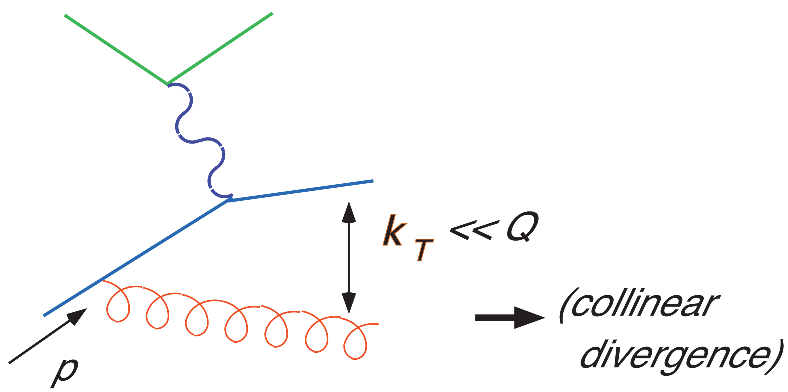
- Infrared safety – insensitive to soft gluon emission collinear rearrangements

- For DIS, will find a result ...
- Just like Parton Model except in Parton Model the infrared safe part is $\sigma_{\rm LO} \Rightarrow \phi(x)$ normalized uniquely
- In pQCD must define parton distributions more carefully: the factorization scheme
- Basic observation: virtual states are not truly frozen. Some states fluctuate on scale $1/Q$ ...



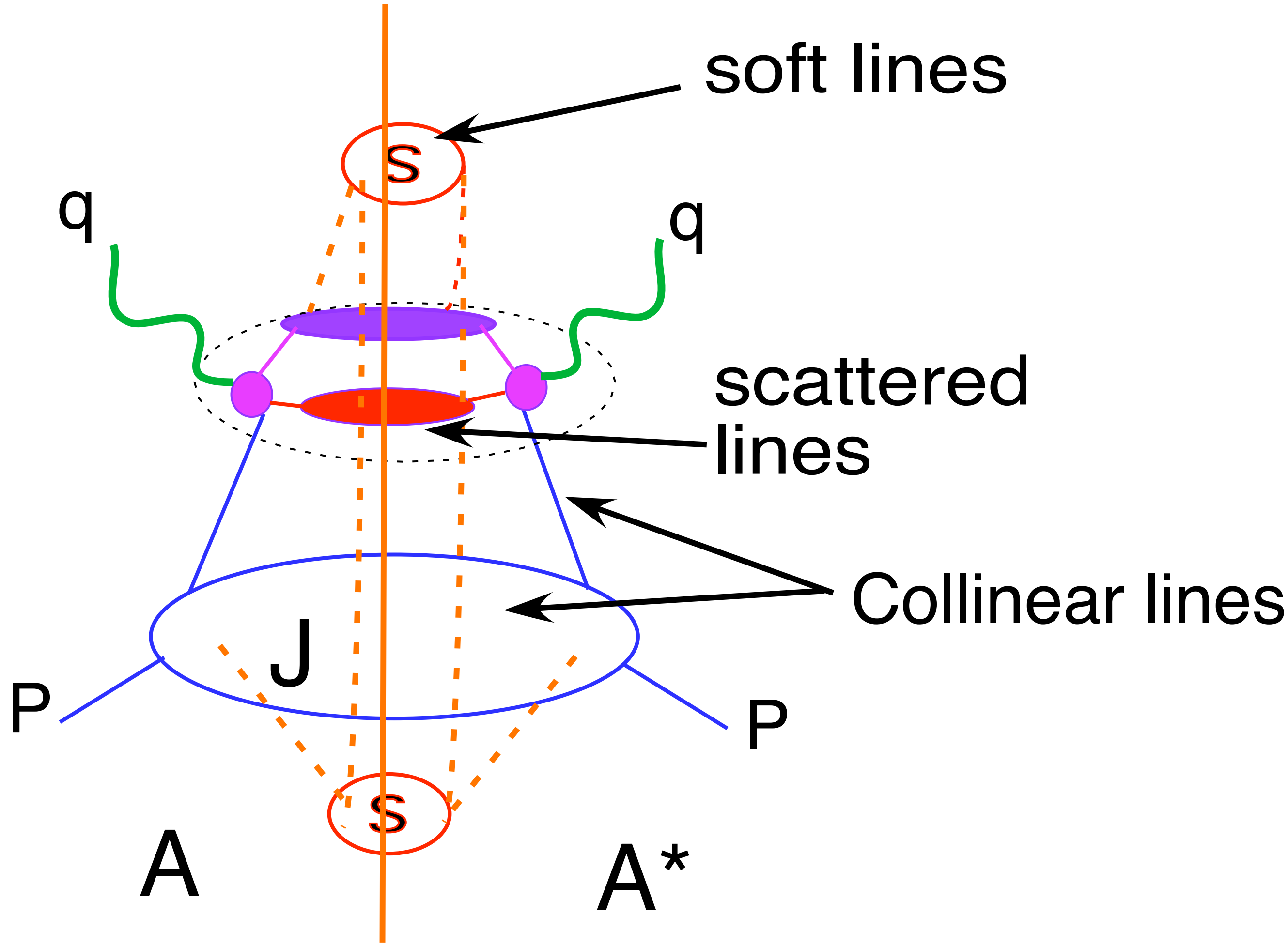
Short-lived states $\Rightarrow \ln(Q)$





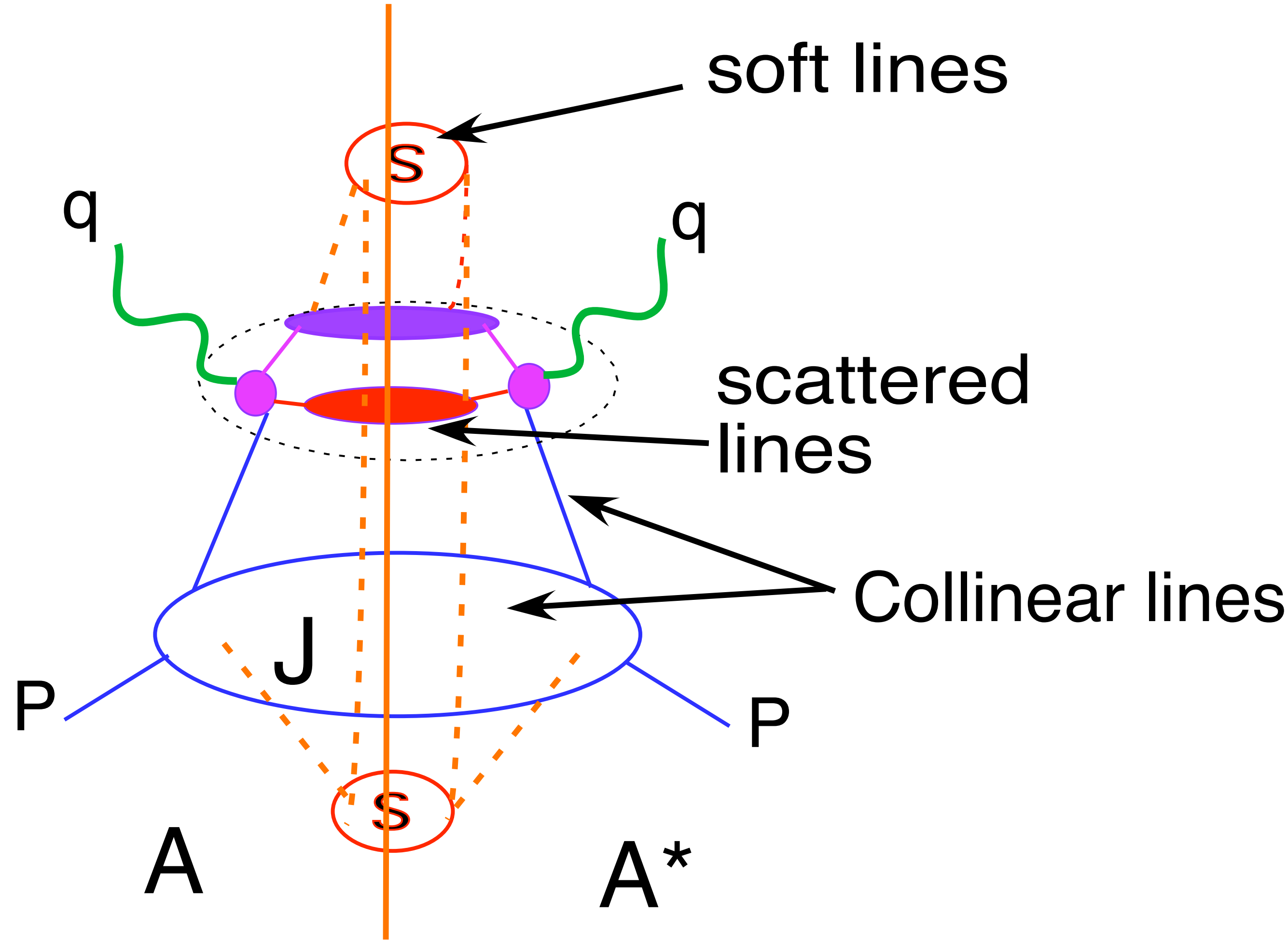
- **Longer-lived states $\Rightarrow$ Collinear Singularity (IR)**
- **How we systematize to all orders in perturbation theory . . . a taste of "all-orders" proofs in pQCD.**

- **We can generalize to all sources of mass dependence. Always from classical processes with on-shell particles.**



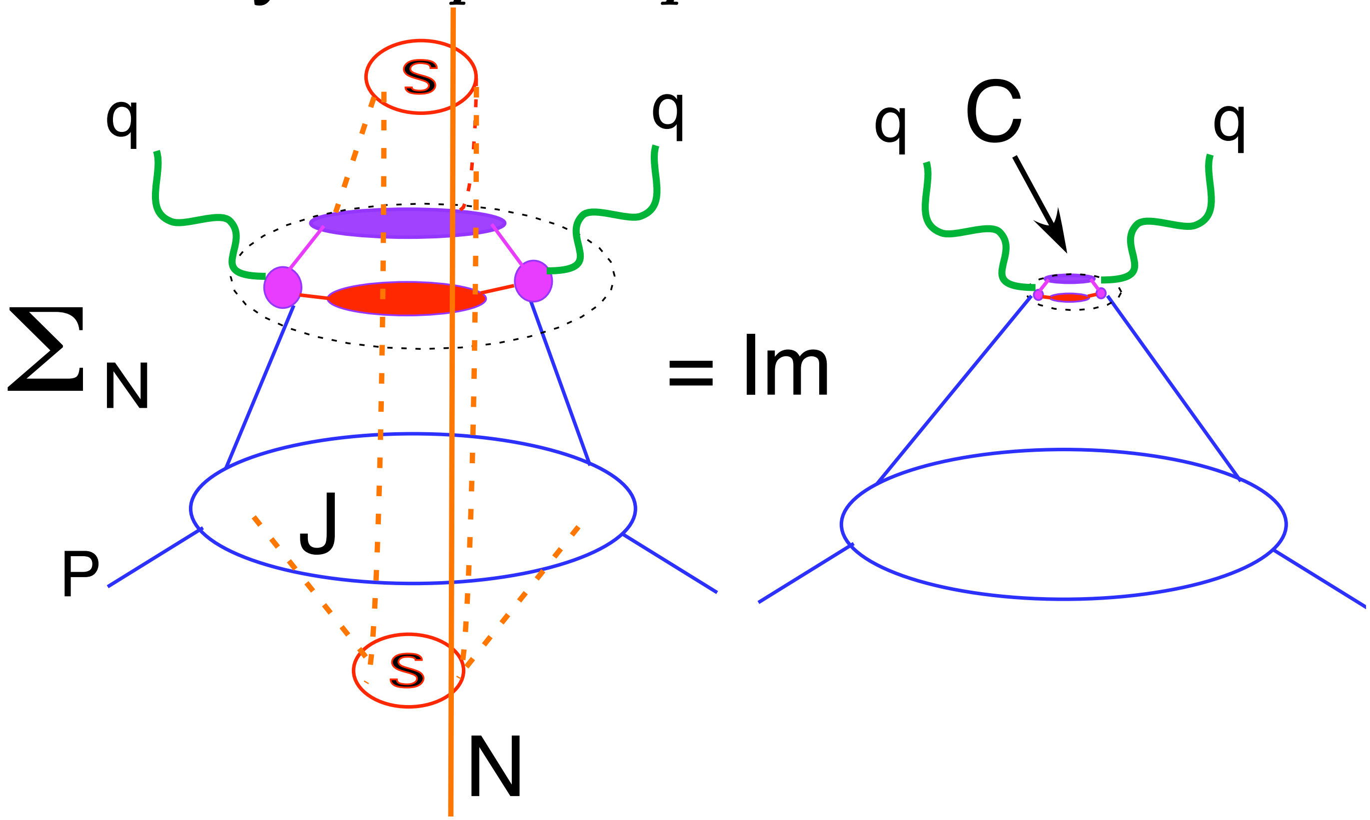
- **This is "Cut diagram notation", representing the amplitude and complex conjugate. Adding up all cut diagrams is the same as summing diagrams of A and then taking $|A|^2$.**

- **Again: the structure of on-shell lines in an arbitrary cut diagram.**



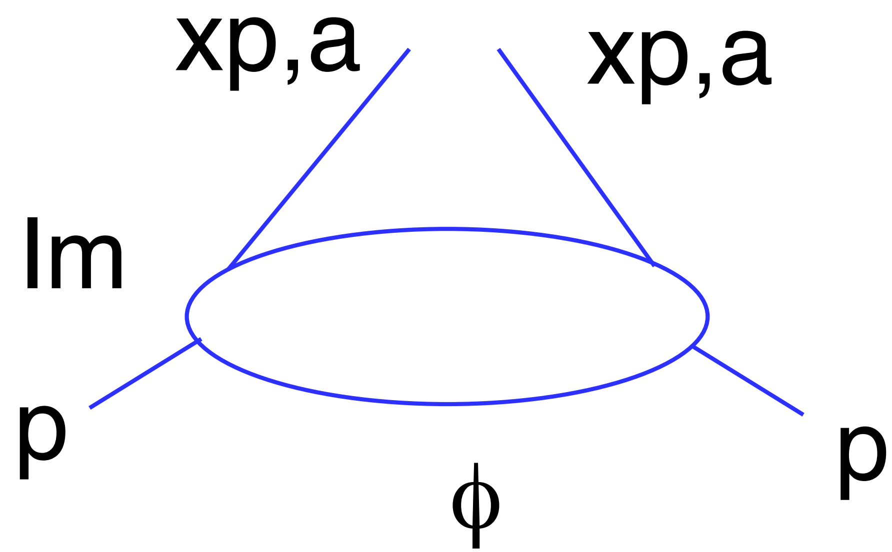
- **The story:** h **splits into collinear partons, then one of them scatters, producing jets that recede at speed of light, connected only by "infinite wavelength soft" quanta.**

- **Use of the optical theorem – relate the cut diagram to forward scattering. No classical processes are possible, because the scattered quarks must rescatter, and all interactions after the hard scattering collapse to a "short-distance" function C, that depends only on xp and q:**



- **All long-distance logs cancels because of the inclusive sum over states.**

- **The partons on each side of the short distance function $C(p,q)$ must have the same flavor and momentum fraction.**



- **Definition of parton distribution generates all the same long-distance behavior left in in the original diagrams (quark case) after the sum over hadronic final states:**

$$\phi_{a/h}(x,\mu_F) = \sum_{\text{spins }\sigma} \int \frac{dy^-}{2\pi} e^{-ixp^+y^-} \langle p,\sigma|\bar{q}(y^-)\gamma^+ q(0)|p,\sigma\rangle$$

- **This matrix element requires renormalization: thus the 'μ_F'.**

- **The result: factorized DIS**

$$F_2^{\gamma q}(x, Q^2) = \int_x^1 d\xi\, C_2^{\gamma q}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu)\right) \times \phi_{q/q}(\xi, \mu_F, \alpha_s(\mu))$$

$$\equiv C_2^{\gamma q}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \frac{\mu_F}{\mu}, \alpha_s(\mu)\right) \otimes \phi_{q/q}(\xi, \mu_F, \alpha_s(\mu))$$

- $\phi_{q/q}$ **has** $\ln(\mu_F/\Lambda_{\rm QCD})$ **...** **with** μ_F **its independent renormalization scale.**

- C **has** $\ln(Q/\mu),\ \ln(\mu_F/\mu)$

- Often pick $\mu = \mu_F$ and often pick $\mu_F = Q$. So often see:

$$F_2^{\gamma q}(x, Q^2) = C_2^{\gamma q}\left(\frac{x}{\xi}, \alpha_s(Q)\right) \otimes \phi_{q/q}(\xi, Q^2)$$

IIIB. DIS at one loop

- But we still need to specify what we *really* mean by factorization: *scheme* as well as *scale*.

- For this, compute $F_2^{\gamma q}(x, Q)$.

- Keep $\mu = \mu_F$ for simplicity.

- **“Compute quark-photon scattering”** – *What does this mean?*

  Must use an *IR-regulated* **theory**

  Extract the *IR Safe part* **then take away the regularization**

- **Let’s see how it works . . .**

- **At** *zeroth order – no interactions:*

  $$C^{\gamma q_f (0)} = e_f^2\, \delta(1 - x/\xi)$$

  (LO cross section; parton model)

  $$\phi^{(0)}_{q_f/q_{f'}}(\xi) = \delta_{ff'}\, \delta(1 - \xi)$$

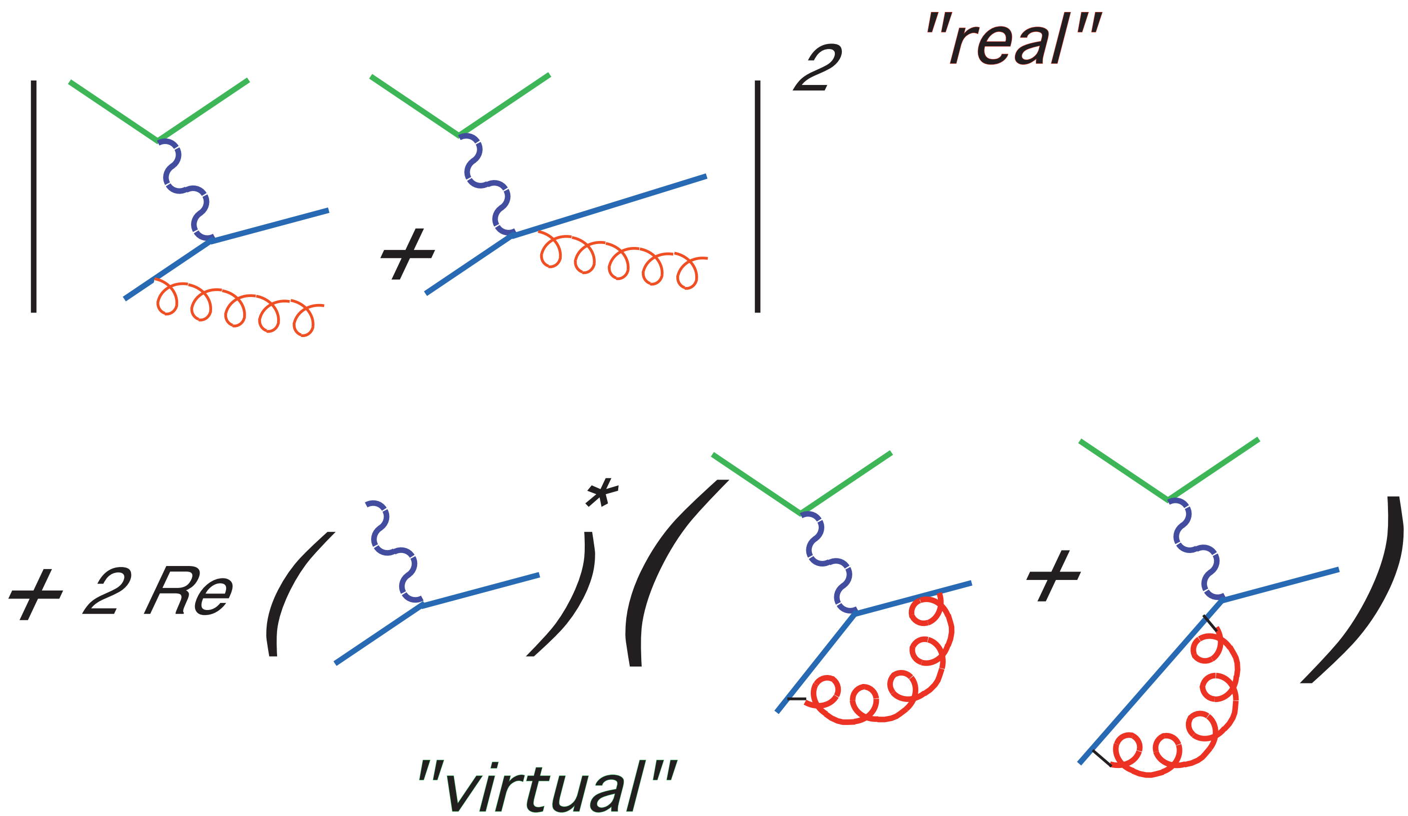
  (at zeroth order, momentum fraction conserved)

$$
\begin{aligned}
F_2^{\gamma q_f\,(0)}(x,Q^2) &= \int_x^1 d\xi\; C_2^{\gamma q_f\,(0)}\left(\frac{x}{\xi},\frac{Q}{\mu},\frac{\mu_F}{\mu},\alpha_s(\mu)\right) \\
&\quad \times\; \phi^{(0)}_{q_f/q_f}(\xi,\mu_F,\alpha_s(\mu)) \\
&= e_f^2\; \int_x^1 d\xi\; \delta(1-x/\xi)\;\delta(1-\xi) \\
&= e_f^2\; x\,\delta(1-x)
\end{aligned}
$$

- **On to one loop . . .**

- $F^{\gamma q}$ **at one loop: factorization schemes**

- **Start with** F_2 **for a** *quark***:**



Have to combine final states with different phase space . . .

- “Plus Distributions”:

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx\, \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx\, f(x)\left(\frac{\ln(1-x)}{1-x}\right)_+ \equiv \int_0^1 dx\, \left(\, f(x)-f(1)\,\right) \frac{\ln(1-x)}{(1-x)}$$

and so on . . . where

- $f(x)$ will be parton distributions
- $f(x)$ term: real gluon, with momentum fraction $1-x$
- $f(1)$ term: virtual, with elastic kinematics

- DGLAP “evolution kernel” = “splitting function”

$$P_{qq}^{(1)}(x) = C_F\, \frac{\alpha_s}{\pi} \left[\frac{1+x^2}{1-x}\right]_+$$

- α_s **Expansion:**

$$F_2^{\gamma q}(x,Q^2) = \int_x^1 d\xi \; C_2^{\gamma q}\left(\frac{x}{\xi},\frac{Q}{\mu},\frac{\mu_F}{\mu},\alpha_s(\mu)\right) \times \phi_{q/q}(\xi,\mu_F,\alpha_s(\mu))$$

$$F_2^{\gamma q_f}(x,Q^2) = C_2^{(0)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\,C^{(1)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\,C^{(0)}\;\phi^{(1)} + \ldots$$

- **And result:**

$$F_2^{\gamma q_f}(x,Q^2) = e_f^2 \left\{ x\,\delta(1-x) \right.$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[\frac{1+x^2}{1-x} \left(\frac{\ln(1-x)}{x} \right) + \frac{1}{4}(9-5x) \right]_+$$

$$\left. + \frac{\alpha_s}{2\pi} C_F \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \left[\frac{1+x^2}{1-x} \right]_+ \right\} + \ldots$$

$$F_1^{\gamma q_f}(x,Q^2) = \frac{1}{2x} \left\{ F_2^{\gamma q_f}(x,Q^2) - C_F\,\alpha\,\frac{\alpha_s}{\pi^2}\,2x \right\}$$

- **Factorization Schemes**

$\overline{\mathrm{MS}}$ (Corresponds to matrix element above.)

$$\phi^{(1)}_{q/q}(x,\mu^2) = \frac{\alpha_s}{\pi^2}\, P_{qq}(x) \int_0^{\mu^2} \frac{dk_T^2}{k_T^2}$$

With k_T-integral "IR regulated".

Advantage: technical simplicity; not tied to process.

$C^{(1)}(x)_{\overline{\mathrm{MS}}} = (\alpha_s/2\pi)\, P_{qq}(x) \ln(Q^2/\mu^2) + \mu$-independent

DIS:

$$\phi_{q/q}(x,\mu^2) = \frac{\alpha_s}{\pi^2}\, F^{\gamma q_f}(x,\mu^2)$$

Absorbs all uncertainties in DIS into a PDF.

Closer to experiment for DIS.

$C^{(1)}(x)_{\overline{DIS}} = (\alpha_s/2\pi)\, P_{qq}(x) \ln(Q^2/\mu^2) + 0$

- **Using the Regulated Theory to Get Parton Distributions for Real Hadrons . . .**

IR-regulated QCD is not $REAL$ **QCD**

BUT it only differs at low momenta

THUS we can use it for IR Safe functions: $C_2^{\gamma q}$**, etc.**

THIS enables us to get PDFs from experiment.

- **Compute $F_2^{\gamma q}$, $F_2^{\gamma G}$ ...**

  Define factorization scheme; find IR Safe C's

  Use factorization in the full theory

$$F_2^{\gamma N} = \sum_{a=q_f,\bar{q}_f,G} C^{\gamma a} \otimes \phi_{a/N}$$

  Measure F_2; then use the known C's to derive $\phi_{a/N}$

  NOW HAVE $\phi_{a/N}(\xi, \mu^2)$ AND CAN USE IT IN ANY OTHER PROCESS THAT FACTORIZES.

- **Multiple flavors and cross sections complicate technicalities; not logic (Global Fits)**

- **IIIC. Evolution: Q^2-dependence**

- **In general, Q^2/μ^2 dependence still in $C_a\left(x/\xi, Q^2/\mu^2, \alpha_s(\mu)\right)$**

  Choose $\mu = Q$

$$F_2^{\gamma A}(x, Q^2) = \sum_a \int_x^1 d\xi\; C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right)\; \phi_{a/A}(\xi, Q^2)$$

  $Q \gg \Lambda_{\rm QCD} \to$ *compute* C*'s in PT*.

$$C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right) = \sum_n \left(\frac{\alpha_s(Q)}{\pi}\right)^n C_2^{\gamma a(n)}\left(\frac{x}{\xi}\right)$$

  But still need PDFs at $\mu = Q$: $\phi_{a/A}(\xi, Q^2)$ for different Q's.

- **How evolution works . . .**

- **A remarkable consequence of factorization.**

- *Can use* $\boldsymbol{\phi_{a/A}(x, Q_0^2)}$ **to determine**

  $\boldsymbol{\phi_{a/A}(x, Q^2)}$ *and hence* $\boldsymbol{F_{1,2,3}(x, Q^2)}$ *for any* $\boldsymbol{Q}$

- **So long at $\alpha_s(Q)$ is still small.**

- **Let's see how it works explicitly in an example.**

- **The 'nonsinglet' distribution**

$$F_a^{\gamma \mathrm{NS}} = F_a^{\gamma p} - F_a^{\gamma n}$$

$$F_2^{\gamma \mathrm{NS}}(x, Q^2) = \int_x^1 d\xi \; C_2^{\gamma \mathrm{NS}}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)\right) \; \phi_{\mathrm{NS}}(\xi, \mu^2)$$

Gluons, antiquarks cancel

At one loop: $C_2^{\mathrm{NS}} = C_2^{\gamma N}$

- **Basic tool:**

- **'Mellin' Moments and Anomalous Dimensions**

$$\bar{f}(N) = \int_0^1 dx\; x^{N-1}\; f(x)$$

- **Reduces convolution to a product**

$$f(x) = \int_x^1 dy\; g\left(\frac{x}{y}\right)\; h(y) \rightarrow \bar{f}(N) = \bar{g}(N)\; \bar{h}(N+1)$$

- **Moments applied to NS structure function:**

$$\bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2) = \bar{C}_2^{\gamma \mathrm{NS}}\left(N, \frac{Q}{\mu}, \alpha_s(\mu)\right) \bar{\phi}_{\mathrm{NS}}(N, \mu^2)$$

(Note $\phi_{\mathrm{NS}}(N, \mu^2) \equiv \int_0^1 d\xi \xi^N f(\xi, \mu^2)$ **here.)**

- $\bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2)$ **is Physical**

$$\Rightarrow \quad \mu \frac{d}{d\mu} \bar{F}_2^{\gamma \mathrm{NS}}(N, Q^2) = 0$$

- **‘Separation of variables’**

$$\mu\frac{d}{d\mu}\ln\bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_{\rm NS}(N,\alpha_s(\mu))$$

$$\gamma_{\rm NS}(N,\alpha_s(\mu)) = \mu\frac{d}{d\mu}\ln\bar{C}_2^{\gamma{\rm NS}}(N,\alpha_s(\mu))$$

- **Because α_s is the only variable held in common.**

- $\gamma_{\rm NS}$ **an “anomalous dimension”, which controls the logarithmic μ dependence.**

$$\mu \frac{d}{d\mu} \ln \bar{\phi}_{\mathrm{NS}}(N, \mu^2) = -\gamma_{\mathrm{NS}}(N, \alpha_s(\mu))$$

$$\gamma_{\mathrm{NS}}(N, \alpha_s(\mu)) = \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma^{\mathrm{NS}}}(N, \alpha_s(\mu))$$

- **Only need to know C's $\Rightarrow \gamma_N$ from IR regulated theory!**

$$\Downarrow$$

Q-DEPENDENCE DETERMINED BY PT

EVOLUTION

THIS WAS HOW WE FOUND OUT QCD IS 'RIGHT'

AND THIS IS HOW QCD PREDICTS PHYSICS AT NEW SCALES

• $\gamma_{\rm NS}$ **at one loop** **(5th line is an exercise.)**

$$
\begin{aligned}
\gamma_{\rm NS}(N,\alpha_s) &= \mu\frac{d}{d\mu}\,\ln \bar{C}_2^{\gamma\rm NS}\,(N,\alpha_s(Q)) \\
&= \mu\frac{d}{d\mu}\,\left\{\,(\alpha_s/2\pi)\,\bar{P}_{qq}(N)\ln(Q^2/\mu^2) + \mu \text{ indep.}\right\} \\
&= -\frac{\alpha_s}{\pi}\,\int_0^1 dx\; x^{N-1}\; P_{qq}(x) \\
&= -\frac{\alpha_s}{\pi}C_F\,\int_0^1 dx\,\left[\left(x^{N-1}-1\right)\frac{1+x^2}{1-x}\right] \\
&= -\frac{\alpha_s}{\pi}C_F\,\left[\,4\sum_{m=2}^{N}\frac{1}{m} - 2\frac{2}{N(N+1)} + 1\,\right] \\
&\equiv -\frac{\alpha_s}{\pi}\,\gamma_{\rm NS}^{(1)}
\end{aligned}
$$

Hint: $(1-x^2)/(1-x) = 1 + x \ldots (1-x^k)/(1-x) = \sum_{i=0}^{k-1} x^k$

- **Solution and scale breaking.**

$$\mu \frac{d}{d\mu}\ \bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_{\rm NS}(N,\alpha_s(\mu))\ \bar{\phi}_{\rm NS}(N,\mu^2)$$

$$\bar{\phi}_{\rm NS}(N,\mu^2) = \bar{\phi}_{\rm NS}(N,\mu_0^2) \times \exp\left[-\frac{1}{2} \int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2}\ \gamma_{\rm NS}(N,\alpha_s(\mu)) \right]$$

$\Downarrow$

$$\bar{\phi}_{\rm NS}(N,Q^2)\ = \bar{\phi}_{\rm NS}(N,Q_0^2)\ \left(\frac{\ln(Q^2/\Lambda^2_{\rm QCD})}{\ln(Q_0^2/\Lambda^2_{\rm QCD})} \right)^{-2\gamma_N^{(1)}/\beta_0}$$

Hint:

$$\alpha_s(Q) = \frac{4\pi}{\beta_0\ \ln(Q^2/\Lambda^2_{\rm QCD})}$$

So also: $\bar{\phi}_{\rm NS}(N,Q^2)\ = \bar{\phi}_{\rm NS}(N,Q_0^2)\ \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{-2\gamma_N^{(1)}/\beta_0}$

Qualitatively,

$$\bar{\phi}_{\rm NS}(N,Q^2) \;=\; \bar{\phi}_{\rm NS}(N,Q_0^2)\;\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-2\gamma_N^{(1)}/\beta_0}$$

- **Is 'mild' scale breaking, to be contrasted to**
- **Case of $\alpha_s \to \alpha_0 \neq 0$, get a power Q-dependence:**

$$\left(Q^2\right)^{\gamma^{(1)}\frac{\alpha_s}{2\pi}}$$

- **$\Rightarrow$ QCD's consistency with the Parton Model (73-74)**

- **Inverting the Moments.**

$$\mu \frac{d}{d\mu}\ \bar{\phi}_{\mathrm{NS}}(N,\mu^2) = -\gamma_N(\alpha_s(\mu))\ \bar{\phi}_{\mathrm{NS}}(N,\mu^2)$$

$$\Downarrow$$

$$\mu \frac{d}{d\mu}\ \phi_{qq}(x,\mu^2) = \int_x^1 \frac{d\xi}{\xi}\ P_{\mathrm{NS}}(x/\xi,\alpha_s(\mu))\ \phi_{\mathrm{NS}}(\xi,\mu^2)$$

Splitting function $\leftrightarrow$ Anomalous dimensions

$$\int_0^1 dx\ x^{N-1}\ P_{qq}(x,\alpha_s) = \gamma_{NS}(N,\alpha_s)$$

- **Singlet (Full) Evolution**

$$\mu \frac{d}{d\mu} \phi_{b/A}(x, \mu^2) = \sum_{b=q,\bar{q},G} \int_x^1 \frac{d\xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu)) \phi_{b/A}(\xi, \mu^2)$$

- **The Physical Context of Evolution**

  – **Parton Model:** $\phi_{a/A}(x)$ **density of parton** a **with momentum fraction** x**, assumed independent of** Q

  – **PQCD:** $\phi_{a/A}(x, \mu)$**: same density, but with transverse momentum** $\leq \mu$

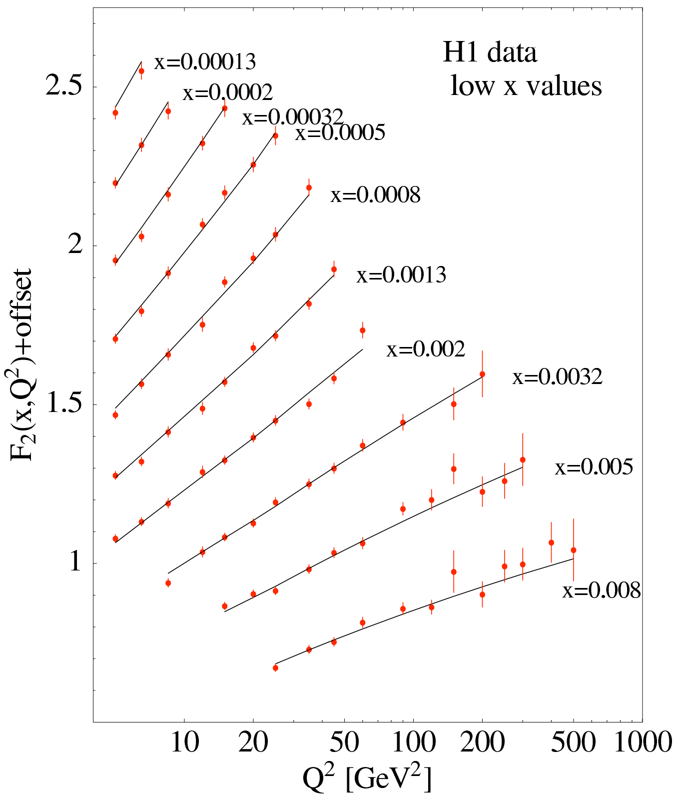
- **If there** *were* **a maximum transverse momentum** Q_0**, each** $\phi_{a/h}(x, Q_0)$ **would freeze for** $\mu \geq Q_0$**.**

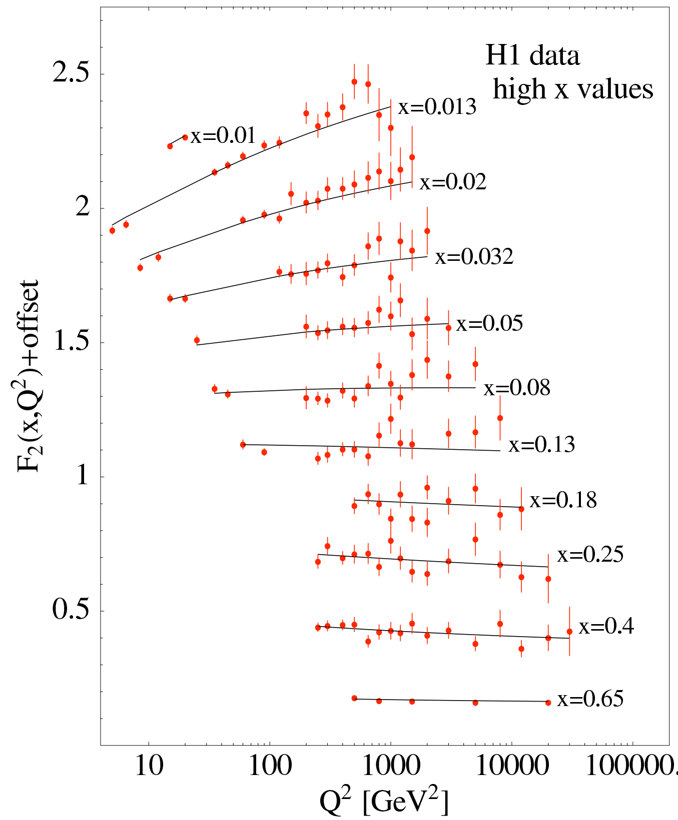
- *Not so* **in renormalized PT.**

- **Scale breaking measures the change in the density as maximum transverse momentum increases.**

- **Cross sections we compute still depend on our choice of** μ **through uncomputed "higher orders" in** C **and evolution.**

- **Evolution in DIS (with CTEQ6 fits)**





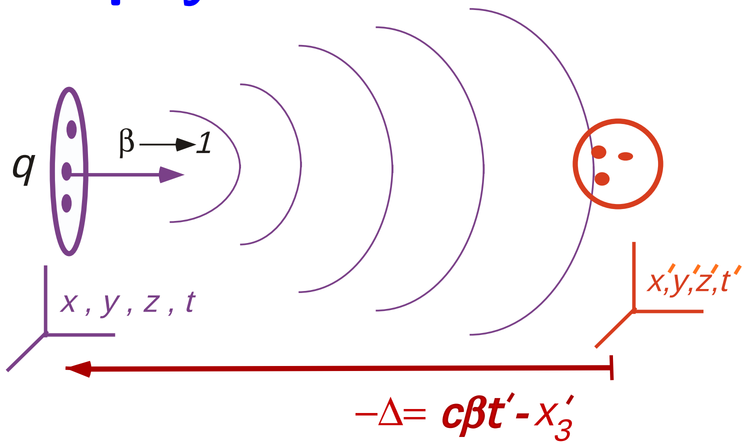
IIID. Factorization in hadron-hadron scattering

- General relation for hadron-hadron scattering for a hard, inclusive process with momentum transfer M to produce final state $F + X$:

$$d\sigma_{\mathrm{H_1H_2}}(p_1, p_2, M) = \sum_{a,b} \int_0^1 d\xi_a \, d\xi_b d\hat{\sigma}_{ab\to F+X}\left(\xi_a p_1, \xi_b p_2, M, \mu\right) \times \phi_{a/H_1}(\xi_a, \mu)\, \phi_{b/H_2}(\xi_b, \mu),$$

- Factorization proofs justify of the universality of the parton distributions.
- Also underly a range of generalizations of evolution: resummations (see appendix slides for an example).

- **The physical basis: classical fields**



$$\Delta \equiv x_3' - \beta c t'$$

- **Why a classical picture isn't far-fetched . . .**

The correspondence principle is the key to to IR divergences.

An accelerated charge must produce classical radiation,

and an infinite numbers of soft gluons are required to make a classical field.

Transformation of a scalar field:

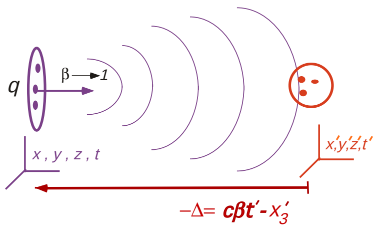
$$\phi(x) = \frac{q}{(x_T^2 + x_3^2)^{1/2}} = \phi'(x') = \frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$$

From the Lorentz transformation:

$$x_3 = -\gamma(\beta c t' - x_3') \equiv \gamma\Delta.$$

Closest approach is at $\Delta = 0$, i.e. $t' = \frac{1}{\beta c} x_3'$.

The scalar field transforms "like a ruler": At any fixed $\Delta \neq 0$, the field decreases like $1/\gamma = \sqrt{1 - \beta^2}$.



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
gauge (0)	$A^0(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^0(x') = \frac{-q\gamma}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{q}{\lvert\vec{x}\rvert^2}$	$E_3'(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}}$
Gauge fields :	$E_3 \sim \gamma^0,$	$E_3 \sim \gamma^{-2}$

- **The "gluon" $\vec{A}$ is enhanced, yet is a total derivative:**

$$A^\mu = q\frac{\partial}{\partial x'_\mu}\ \ln\left(\Delta(t', x'_3)\right) + \mathcal{O}(1-\beta) \sim A^-$$

- **The "large" part of A^μ can be removed by a gauge transformation!**

- **The "force" $\vec{\mathrm{E}}$ field of the incident particle does not overlap the "target" until the moment of the scattering.**

- **"Advanced" effects are corrections to the total derivative:**

$$1 - \beta \sim \frac{1}{2}\left[\sqrt{1-\beta^2}\right]^2 \sim \frac{m^2}{2E^2}$$

- **Power-suppressed! These are corrections to factorization.**

- **At the same time, a gauge transformation also induces a phase on charged fields:**

$$q(x) \Rightarrow q(x)\, e^{i \ln(\Delta)}$$

Cancelled if the fields are well-localized $\Leftrightarrow \sigma$ inclusive

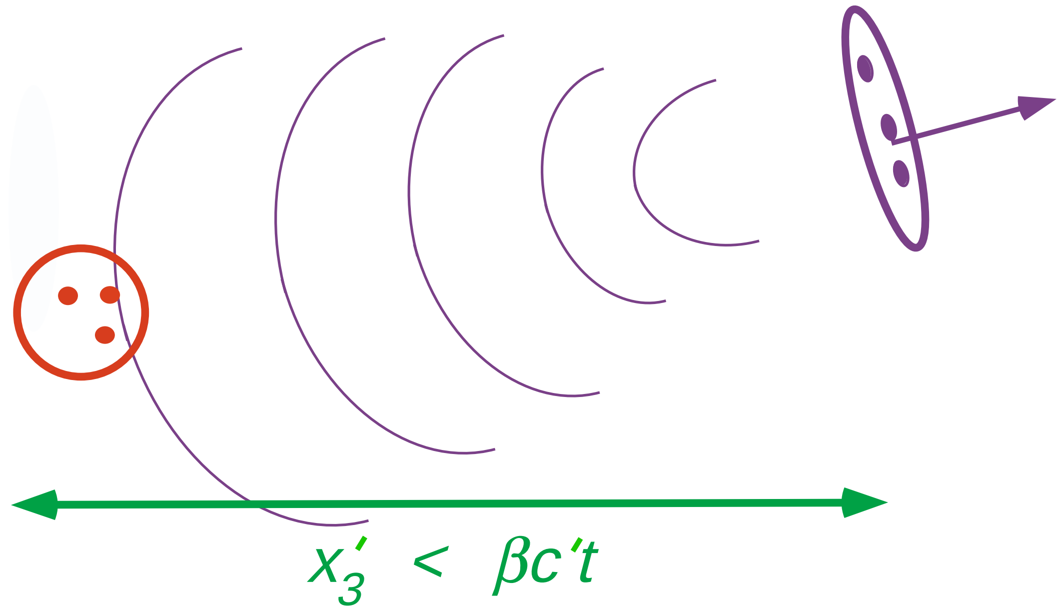
- **Initial-state interactions decouple from hard scattering**
- **Summarized by multiplicative factors: the parton distributions.**

  ⇒ Cross section for inclusive hard scattering is IR safe, with power-suppressed corrections.

- **Factorizing dynamics at short and long distance can be built into effective field theories based on the QCD Lagrangian: in particular "soft-collinear effective field theory" (SCET) can streamline many applications.**

- **What about cross sections where we observe specific particles in the final state? Single hadrons, dihadron correlations, etc?**

- **Much of the same reasoning holds:**

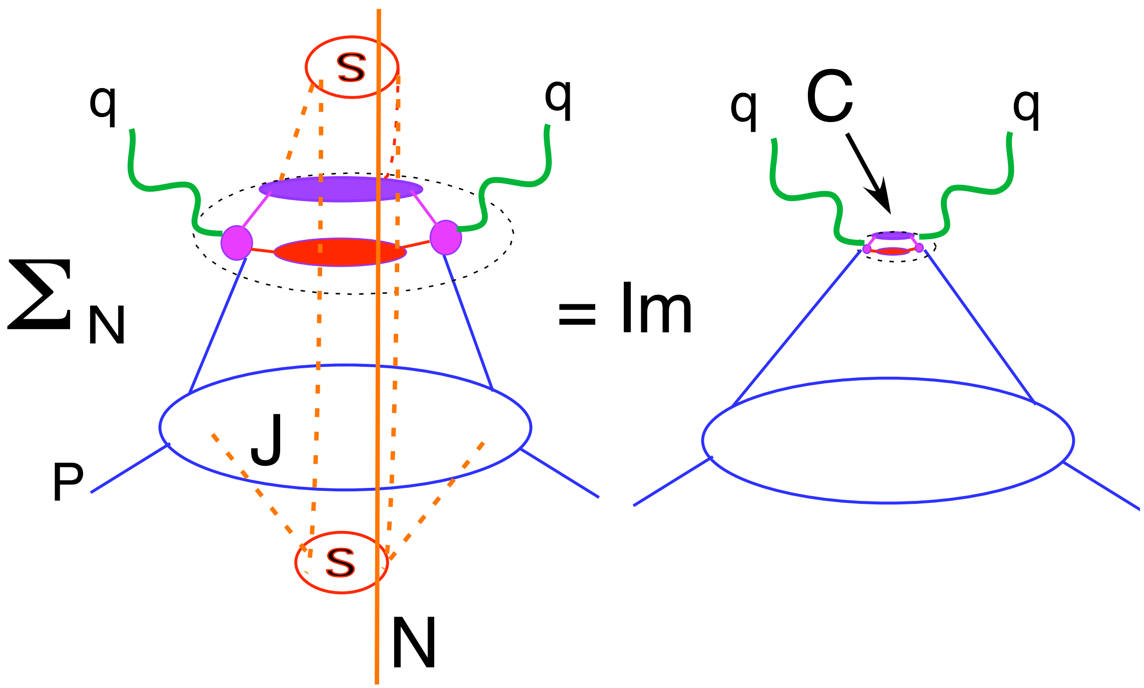


- **For single-particle inclusive . . .**

  Interactions after the scattering are too late to affect large momentum transfer, creation of heavy particle, etc.

  The fragmentation of partons to jets is too slow to know details of the hard scattering: factorization of fragmentation functions.

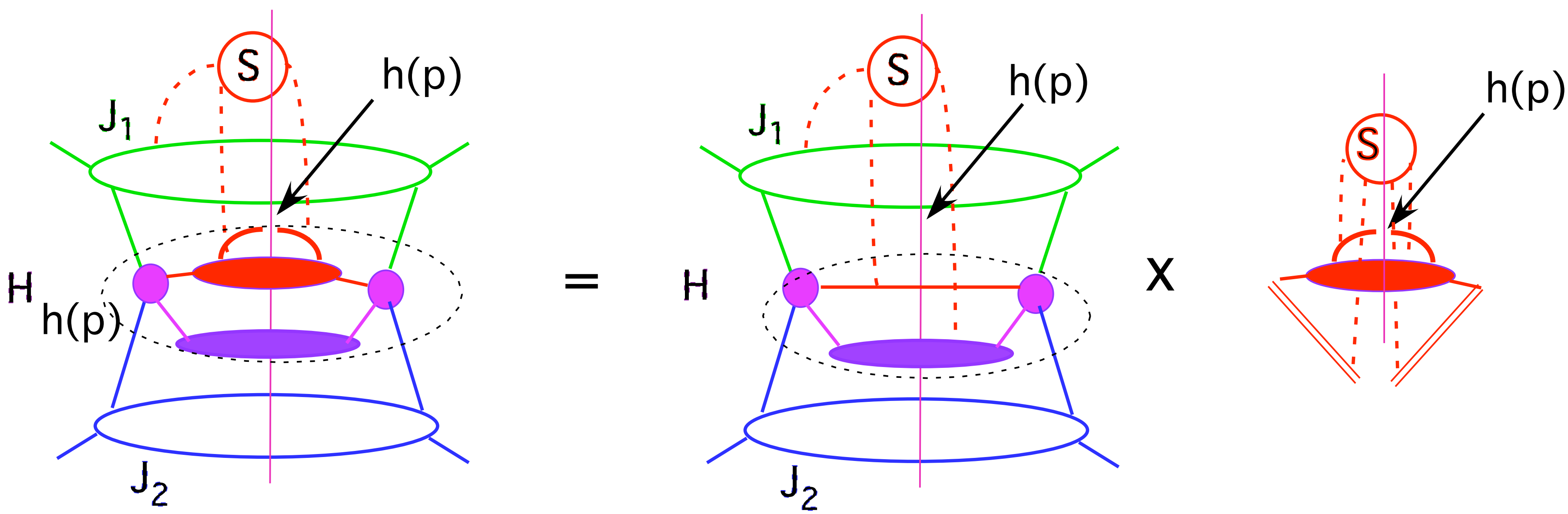
- **Conclude with a few comments . . .**
- **Factorization, although powerful, is brittle. To apply it, we must define our cross sections to be "sufficiently inclusive". We have to be able to apply an analog of the optical theorem as in DIS, recall:**



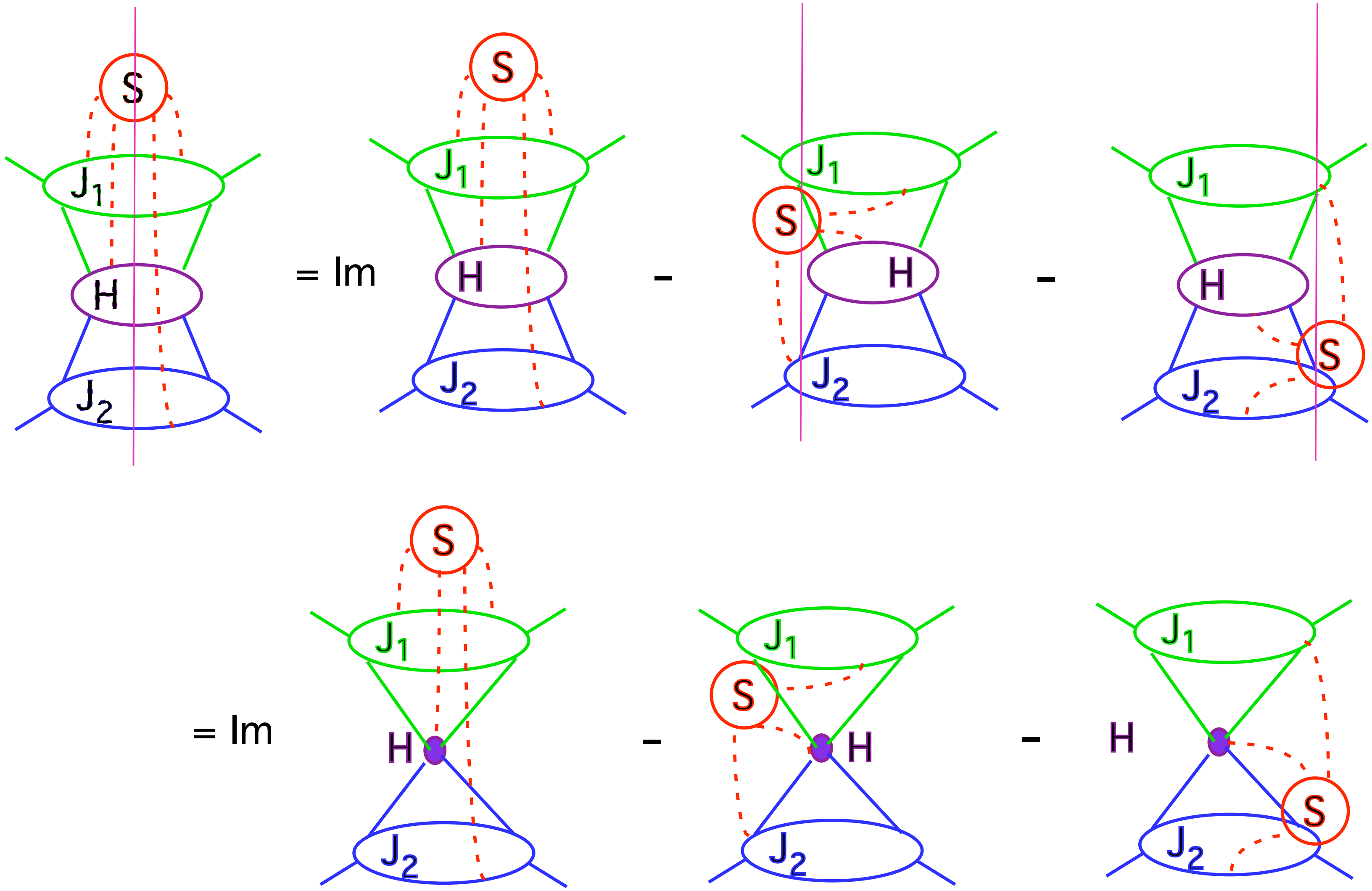
- **How this works out for 1PI cross sections is sketched in the "appendix" slides. Also in appendix – basics of Q_T resummation from a factorization point of view.**
- **Event generators for showering depend on the physics of factorization: each sequential branching (gluon emission, pair creation) is independent. A series of "mini-factorizations".**
- **The key to applications of perturbative QCD is to avoid uncontrolled dependence of long-distance physics. It must either cancel or be factorized from calculable quantities.**
- **pQCD will give sensible answers if you ask the right questions.**

Appendix III.1: high orders in factorization proofs for 1PI cross sections

- How it works in pQCD, with pictures as in DIS:
- Separation of soft quanta from fragmenting partons because soft radiation cannot resolve collinear-moving particles.



- **The all-orders cancellation of soft singularities that connect initial and final states for single-particle inclusive and other short-distance cross sections in hadron-hadron scattering:**



- **all terms on RHS are power-suppressed**

Appendix III.2: Resummation: the Classic Case: Q_T

- **Start with the Drell-Yan transverse momentum distribution at order α_s**

$$q(p_1) + \bar{q}(p_2) \rightarrow \gamma^*(Q) + g(k)$$

- **Treat this $2 \rightarrow 2$ process at lowest order (α_s) "LO" in factorized cross section, so that $\mathrm{k} = -\mathrm{Q}_T$**

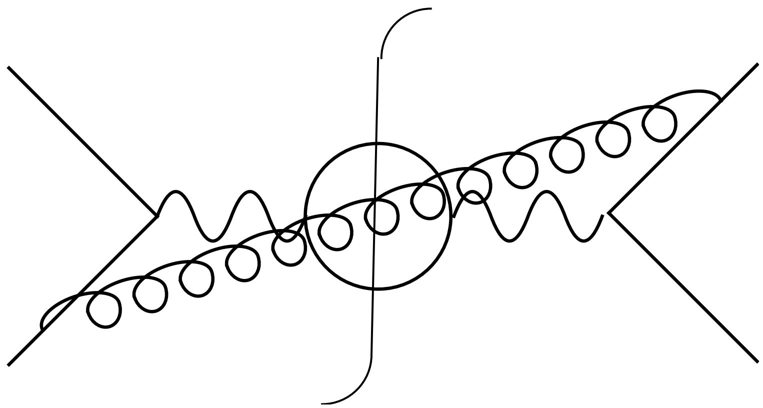
- **Factorized cross section at fixed Q_T:**

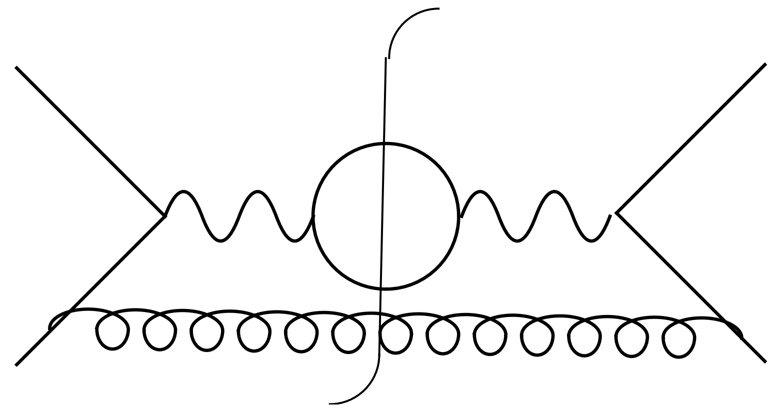
$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2d^2Q_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,Q_T,\xi_1 p_1,\xi_2 p_2,\mu)}{dQ^2d^2Q_T} \times f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

- μ **is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.**

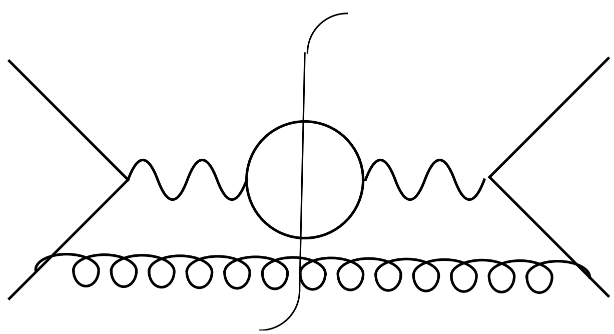
- **The diagrams at order α_s. Finite for $\mathrm{Q}_T \neq 0$ . . .**

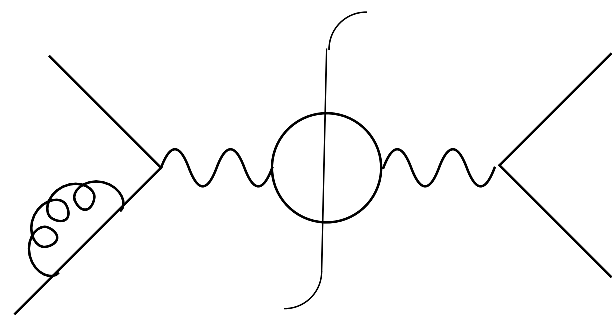
  Gluon emission contributes at $Q_T \neq 0$





Virtual corrections contribute only at $Q_T = 0$





$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathrm{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathrm{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2} \times \left[\frac{1}{\mathrm{Q}_T^2} \frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

OK as long as $\mathrm{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

The Q_T integral $\to \frac{\ln(1-z)}{1-z}$; z integral $\to \frac{\ln \mathrm{Q}_T^2}{\mathrm{Q}_T^2}$.

The leading singularity in Q_T

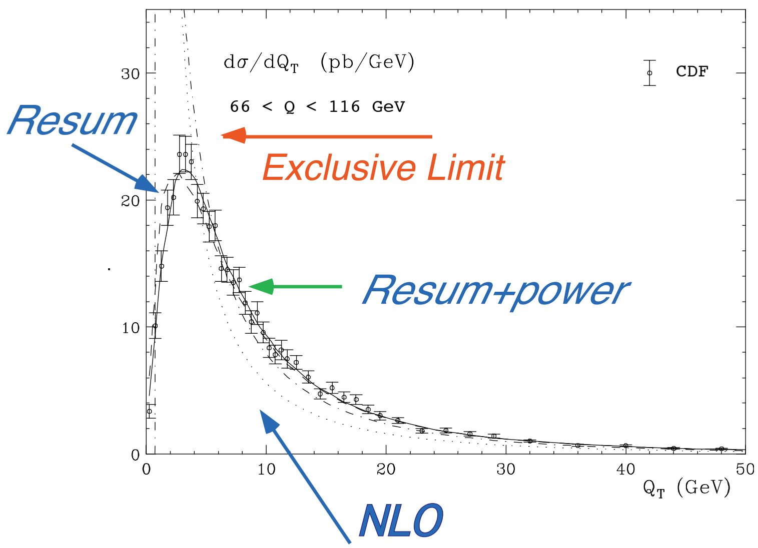
- z **integral:** If Q^2/S not too big, PDFs nearly constant:

$$\frac{1}{Q_T^2}\int_{1-Q^2/S}^{1-Q_T^2/Q^2}\frac{dz}{1-z}=\frac{1}{Q_T^2}\ln\left[\frac{Q^2}{Q_T^2}\right]$$

$\Rightarrow$ Prediction for Q_T dependence:

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,Q_T)}{dQ^2d^2Q_T}=\frac{\alpha_sC_F}{\pi}\frac{1}{Q_T^2}\ln\left[\frac{Q^2}{Q_T^2}\right]$$
$$\times\sum_{a=q\bar{q}}\int_{\xi_1\xi_2}\frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}$$
$$\times f_{a/N}(\xi_1,\mu)\,f_{\bar{a}/N}(\xi_2,\mu)$$

- **Compare to: Z p_T (from Kulesza, G.S., Vogelsang (2002))**



- $\ln Q_T/Q_T$ **works pretty well for large** Q_T
- **But at smaller** Q_T **reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)**
- **Most events are at "low"** $Q_T \ll Q = m_Z$.

- **Getting to $Q_T \ll Q$: Transverse momentum resummation**

  (Logs of Q_T)/Q_T to all orders

  How? Variant factorization and separation of variables

  q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

  q and $\bar{q}$ radiate independently (fields don't overlap!).

  Final-state QCD radiation too late to affect cross section

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q, \mathrm{Q}_T)}{dQ^2 d^2\mathrm{Q}_T}$$

Summarized by: Q_T-factorization:

$$\frac{d\sigma_{NN\to QX}}{dQ\,d^2Q_T} = \int d\xi_1 d\xi_2\; d^2\mathrm{k}_{1T} d^2\mathrm{k}_{2T} d^2\mathrm{k}_{sT}$$
$$\times\; H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})$$
$$\times U_{a\bar{a}}(k_{sT}, n)\, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$

The $\mathcal{P}'s$: new Transverse momentum-dependent PDFs

Also need U: "soft function" for wide-angle radiation

Symbolically:

$$\frac{d\sigma_{NN\to QX}}{dQ\,d^2Q_T} = H \times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T}) \otimes_{\xi_i, k_{iT}} U_{a\bar{a}}(k_{sT}, n)$$

We will solve for the k_T dependence of the $\mathcal{P}$'s.

New factorization variables: n^μ apportions gluons k:

$$p_i \cdot k < n \cdot k \;\Rightarrow\; k \in \mathcal{P}_i$$

$$p_a \cdot k, p_{\bar{a}} \cdot k > n \cdot k \;\Rightarrow\; k \in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

The factorized cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ} = \int d\xi_1 d\xi_2\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \quad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

Method of Collins and Soper, and Sen (1981)

Change in jet must cancel change in (UV) H and (IR) U:

$$p \cdot n \frac{\partial}{\partial p \cdot n} \ln \mathcal{P}(p \cdot n/\mu, b\mu) = G(p \cdot n/\mu) + K(b\mu)$$

G matches H, K matches U. Renormalization indep. of n^μ:

$$\mu \frac{\partial}{\partial \mu} \left[G(p \cdot n/\mu) + K(b\mu) \right] = 0$$

$$\mu \frac{\partial}{\partial \mu} G(p \cdot n/\mu) \; = \; A(\alpha_s(\mu)) \; = \; - \mu \frac{\partial}{\partial \mu} K(b\mu)$$

Solve this one first. μ in α_s varies (& α_s need not be small).

$$G(p\cdot n/\mu) + K(b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n)$$

The consistency equation for the jet becomes

$$p\cdot n \frac{\partial}{\partial p\cdot n} \ln \mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n)$$
$$- \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A(\alpha_s(\mu'))$$

Integrate $p\cdot n$ and get double logs in $b \to \alpha_s^n \frac{\ln^{2n-1}(Q/Q_T)}{Q_T}$.

Transformed solution back to Q_T: all the (Logs of Q_T)/Q_T,
Which fits the data; (viz. RESBOS; Yuan, Nadolsky et al.)

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}}\, e^{E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)}$$
$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}}{dQ^2}\, f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

"Sudakov" exponent links large and low virtuality:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T))\right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)