Introduction to Monte Carlos

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MCnet

PHYSICS AT THE TERA SCALE
Helmholtz Alliance

Outline

- Part I — Basics
  - Introduction
  - Monte Carlo techniques
- Part II — Perturbative physics
  - Hard scattering
  - Parton showers
- Part III — Non–perturbative physics
  - Hadronization
  - Hadronic decays
  - Comparison to data

Thanks

Thanks to my colleagues

Frank Krauss, Leif Lönnblad, Steve Mrenna, Peter Richardson, Mike Seymour, Torbjörn Sjöstrand.

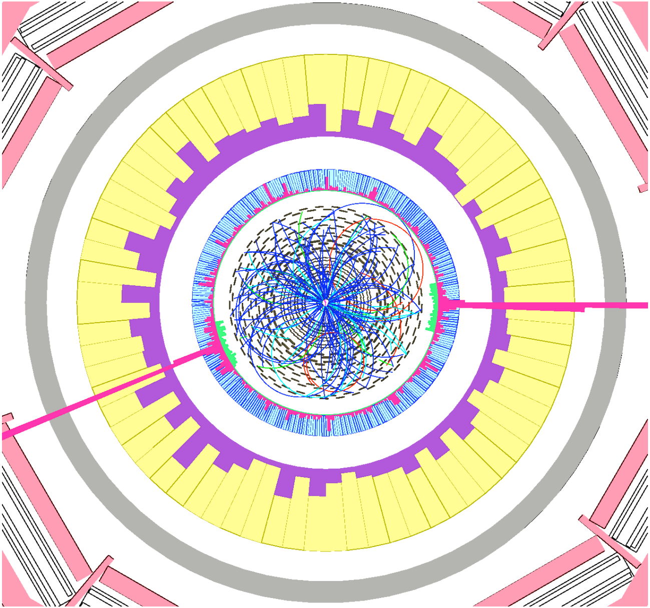
Introduction

Why Monte Carlos?

We want to understand

$$\mathscr{L}_{\text{int}} \longleftrightarrow \text{Final states}\ .$$

Can you spot the Higgs?



Why Monte Carlos?

LHC experiments require
sound understanding of signals and *backgrounds*.

$\uparrow$

Full detector simulation.

$\uparrow$

Fully exclusive hadronic final state.

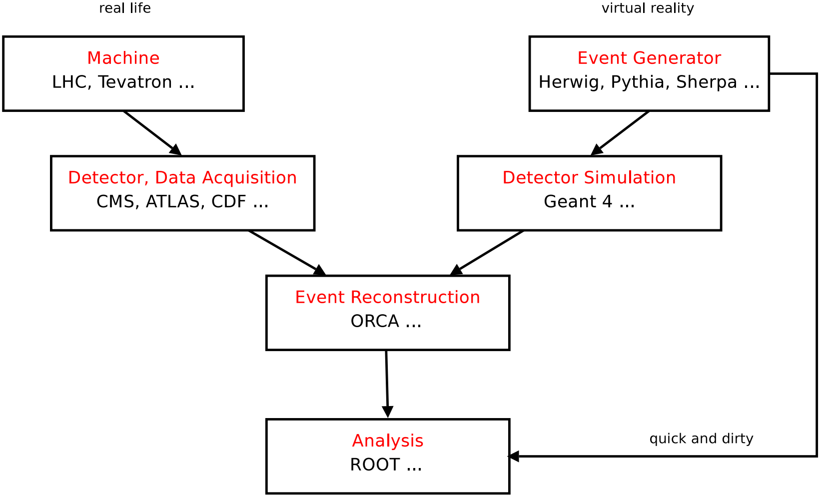
$\uparrow$

Monte Carlo event generator with
parton shower, hadronization model, decays of unstable particles.

$\uparrow$

Parton level computations.

Experiment and Simulation



Monte Carlo Event Generators

- Complex final states in full detail (jets).
- Arbitrary observables and cuts from final states.
- Studies of new physics models.

- Rates and topologies of final states.
- Background studies.
- Detector Design.
- Detector Performance Studies (Acceptance).

- *Obvious* for calculation of observables on the quantum level

$$|A|^2 \longrightarrow \text{Probability.}$$

pp Event Generator

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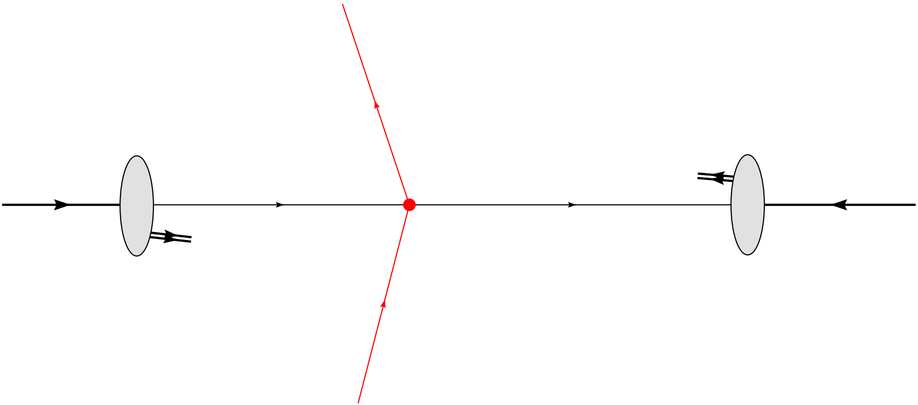

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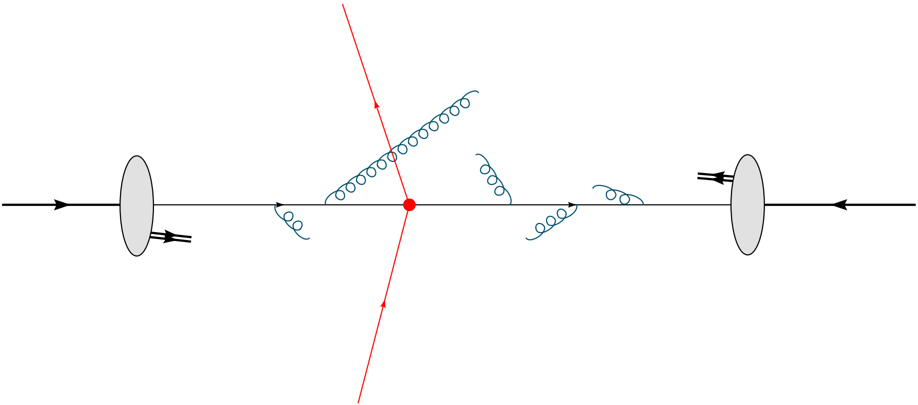

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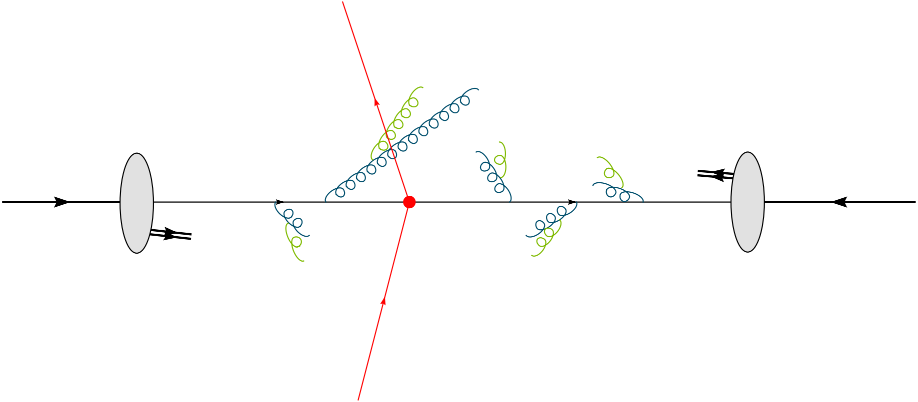

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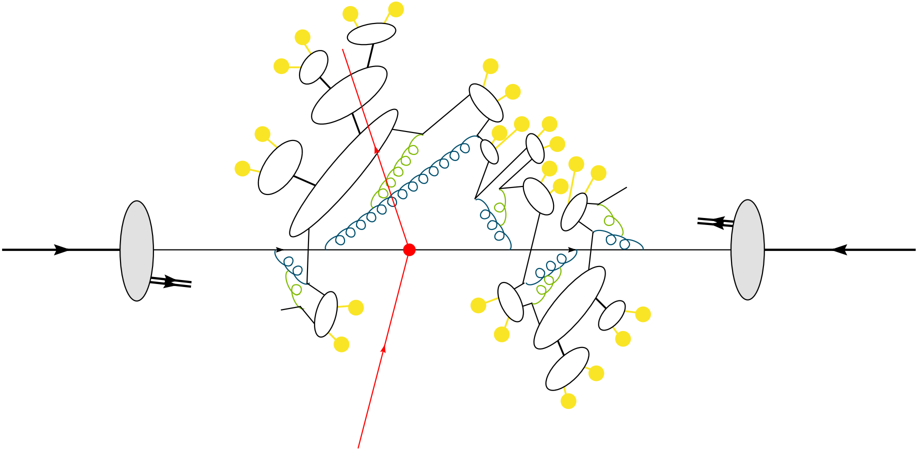

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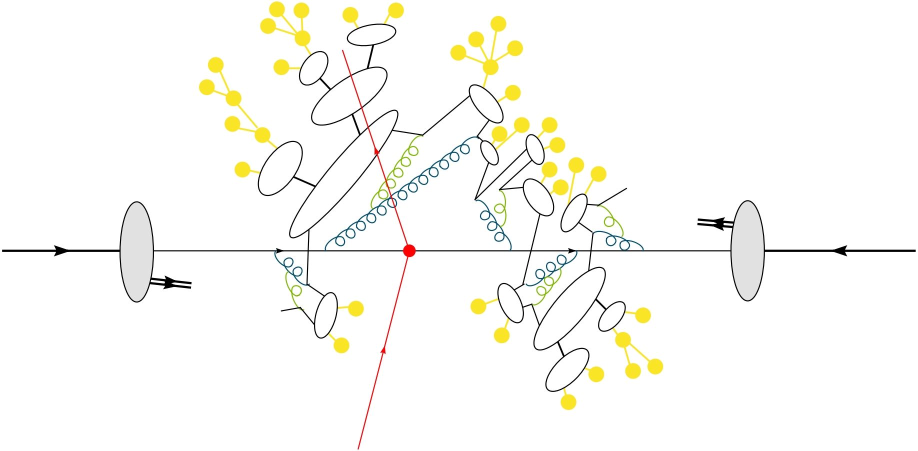

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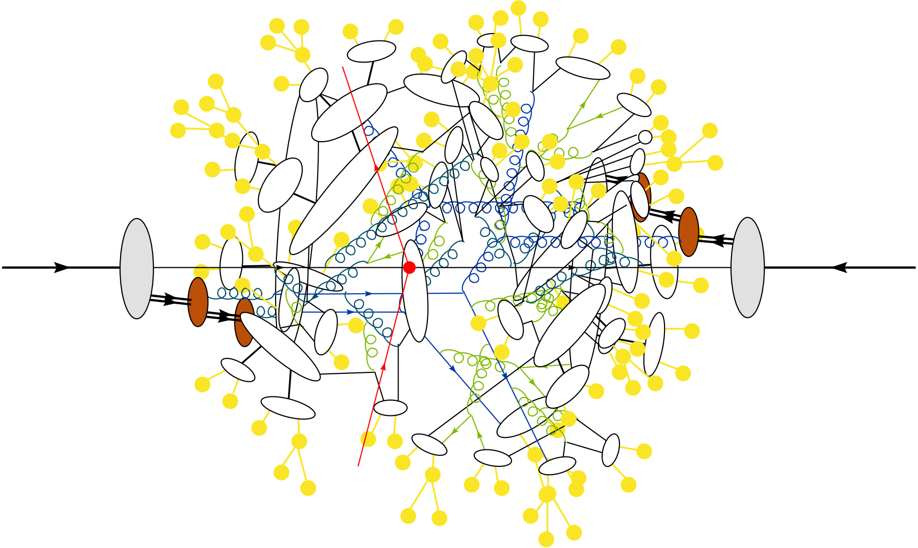

Divide and conquer

Partonic cross section from Feynman diagrams

$$\mathrm{d}\sigma = \mathrm{d}\sigma_{\mathrm{hard}}\mathrm{d}P(\mathrm{partons} \to \mathrm{hadrons})$$

Note, that

$$\int \mathrm{d}P(\mathrm{partons} \to \mathrm{hadrons}) = 1\ ,$$

- σ remains unchanged
- introduce realistic fluctuations into distributions.

Divide and conquer

Partonic cross section from Feynman diagrams

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Note, that

$$\int dP(\text{partons} \to \text{hadrons}) = 1 \; ,$$

- σ remains unchanged
- introduce realistic fluctuations into distributions.

Simulation steps governed by different scales
$\longrightarrow$ separation into ($Q_0 \approx 1\,\text{GeV} > \Lambda_{\text{QCD}}$)

$$\begin{aligned} dP(\text{partons} \to \text{hadrons}) = \; & dP(\text{resonance decays}) && [\Gamma > Q_0] \\ & \times dP(\text{parton shower}) && [\text{TeV} \to Q_0] \\ & \times dP(\text{hadronisation}) && [\sim Q_0] \\ & \times dP(\text{hadronic decays}) && [O(\text{MeV})] \end{aligned}$$

Divide and conquer

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Quite complicated integration.

Divide and conquer

$$
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\end{aligned}
$$

Quite complicated integration.

Monte Carlo is the only choice.

Monte Carlo Methods

Introduction to the most important MC sampling
(= integration) techniques.

1. Hit and miss.
2. Simple MC integration.
3. (Some) methods of variance reduction.
4. Multichannel.

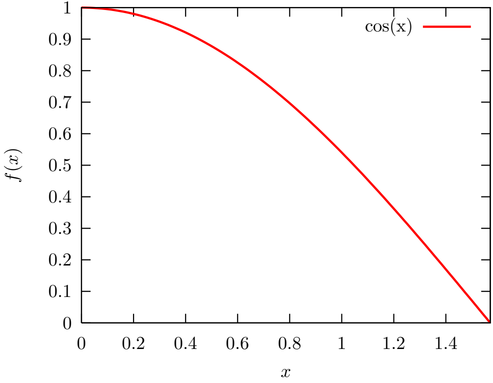
Probability

Example: $f(x) = \cos(x)$.

Probability density:

$$dP = f(x)\,dx$$

is probability to find value x.



Probability $\sim$ *Area*

Probability

Example: $f(x) = \cos(x)$.

Probability density:

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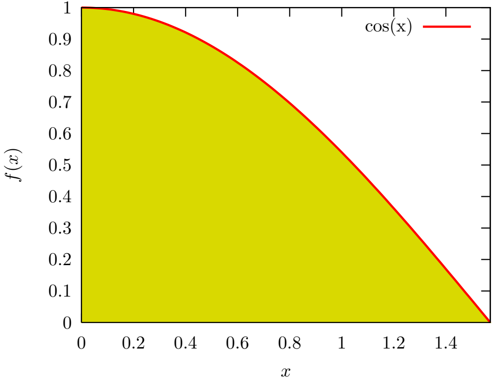
is probability to find value x.

$$F(x) = \int_{x_0}^{x} f(x)\,dx$$

is called *probability distribution*.



Probability $\sim$ *Area*

Hit and Miss

Hit and miss method:

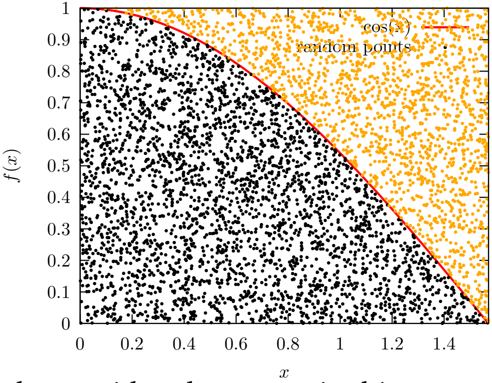
- throw N random points (x, y) into region.
- Count hits N_{hit}, i.e. whenever $y < f(x)$.

Then

$$I \approx V \frac{N_{\text{hit}}}{N}.$$

approaches 1 again in our example.

Example: $f(x) = \cos(x)$.



Every accepted value of x can be considered an event in this picture. As $f(x)$ is the 'histogram' of x, it seems obvious that the x values are distributed as $f(x)$ from this picture.

Hit and Miss

This method is used in many event generators. However, it is not sufficient as such.

- Can handle any density $f(x)$,
  however wild and unknown it is.
- $f(x)$ should be bounded from above.
- Sampling will be very *inefficient* whenever $\mathrm{Var}(f)$ is large.

Improvements go under the name variance reduction as they improve the error of the crude MC at the same time.

Simple MC integration

Mean value theorem of integration:

$$I = \int_{x_0}^{x_1} f(x)\,dx$$
$$= (x_1 - x_0)\langle f(x)\rangle$$
$$\approx (x_1 - x_0)\frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

(Riemann integral).

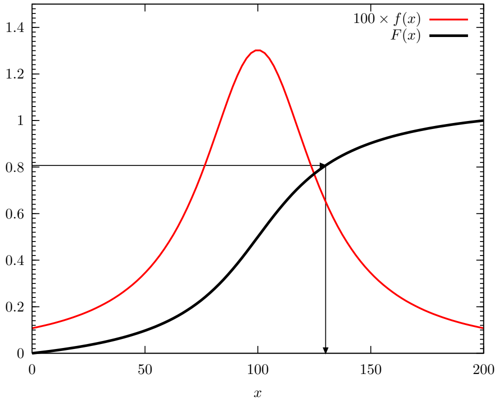
Sum doesn't depend on ordering
$\longrightarrow$ randomize x_i.

Yields a flat distribution of events x_i,
but weighted with *weight* $f(x_i)$ ($\rightarrow$ unweighting).

Inverting the Integral

- Probability density $f(x)$. Not necessarily normalized.
- Integral $F(x)$ known,
- $P(x < x_s) = F(x_s)$.
- Probability = 'area', distributed evenly,

$$\int_{x_0}^{x} dP = r \cdot \text{area}$$



Sample x according to $f(x)$ with

$$x = F^{-1}\Big[F(x_0) + r\big(F(x_1) - F(x_0)\big)\Big] .$$

Inverting the Integral

Sample x according to $f(x)$ with

$$x = F^{-1}\Big[F(x_0) + r\big(F(x_1) - F(x_0)\big)\Big] \, .$$

Optimal method, but we need to know

- The integral $F(x) = \int f(x)\,\mathrm{d}x$,
- It's inverse $F^{-1}(y)$.

That's rarely the case for real problems.

But very powerful in combination with other techniques.

Importance sampling

Error on Crude MC $\sigma_{MC} = \sigma/\sqrt{N}$.
$\Longrightarrow$ Reduce error by reducing variance of integrand.

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Error on Crude MC $\sigma_{MC} = \sigma/\sqrt{N}$.
$\Longrightarrow$ Reduce error by reducing variance of integrand.

Idea: *Divide out the singular structure.*

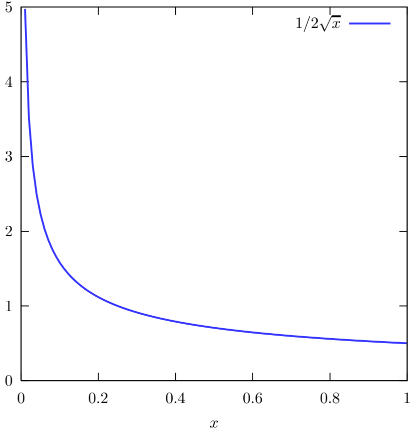
$$I = \int f \, \mathrm{d}V = \int \frac{f}{p} p \mathrm{d}V \approx \left\langle \frac{f}{p} \right\rangle \pm \sqrt{\frac{\langle f^2/p^2 \rangle - \langle f/p \rangle^2}{N}} \, .$$

where we have chosen $\int p \, \mathrm{d}V = 1$ for convenience.

Note: need to sample flat in $p \, \mathrm{d}V$, so we better know $\int p \mathrm{d}V$ and it's inverse.

More interesting for divergent integrands, eg

$$\frac{1}{2\sqrt{x}} \ ,$$



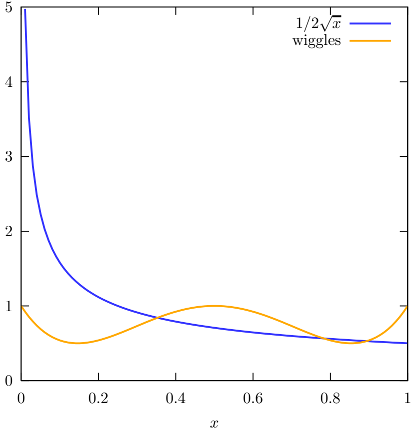
Importance sampling — better example

More interesting for divergent integrands, eg

$$\frac{1}{2\sqrt{x}} \,,$$

with some wiggles,

$$p(x) = 1 - 8x + 40x^2 - 64x^3 + 32x^4 \,.$$



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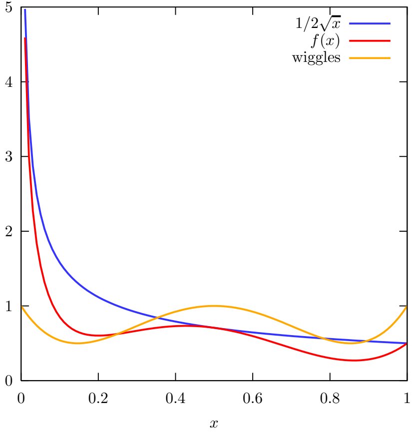
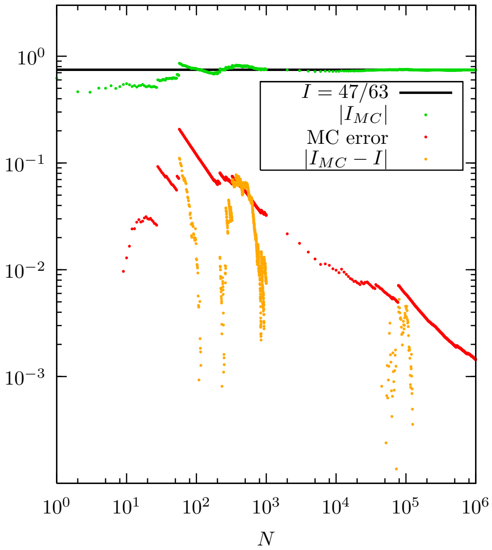
i.e. we want to integrate

$$f(x) = \frac{p(x)}{2\sqrt{x}}\ .$$



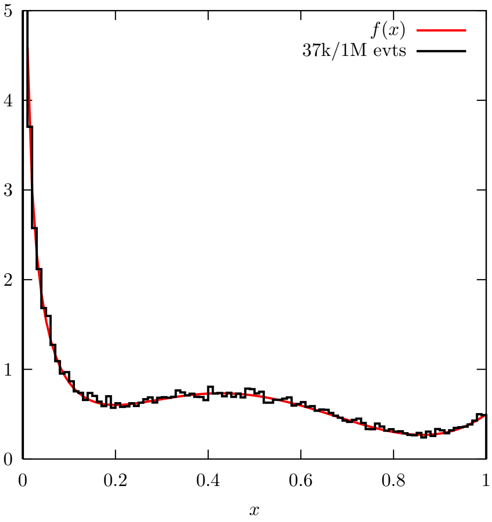
Importance sampling — better example

- Crude MC gives result in reasonable 'time'.
- Error a bit unstable.
- Event generation with maximum weight $w_{\text{max}} = 20$. (that's arbitrary.)
- hit/miss/events with $(w > w_{\text{max}}) =$ 36566/963434/617 with 1M generated events.



Want events:
use hit+mass variant here:

- Choose new random number r
- $w = f(x)$ in this case.
- if $r < w/w_{\max}$ then "hit".
- MC efficiency = hit$/N$.
- Efficiency for MC events only 3.7%.
- Note the wiggly histogram.



Importance sampling — example

Now importance sampling, i.e. divide out $1/2\sqrt{x}$.

$$\int_0^1 \frac{p(x)}{2\sqrt{x}}\,dx = \int_0^1 \left(\frac{p(x)}{2\sqrt{x}}\middle/\frac{1}{2\sqrt{x}}\right)\frac{dx}{2\sqrt{x}}$$
$$= \int_0^1 p(x)\,d\sqrt{x}$$
$$= \int_0^1 p(x(\rho))\,d\rho$$
$$= \int_0^1 1-8\rho^2+40\rho^4-64\rho^6+32\rho^8\,d\rho$$

so,

$$\rho = \sqrt{x}, \qquad d\rho = \frac{dx}{2\sqrt{x}}$$

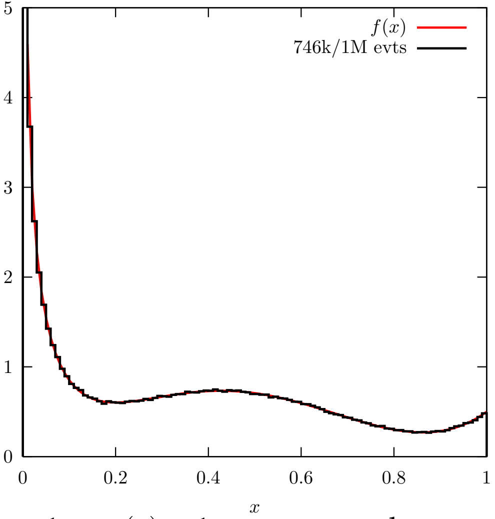
x sampled with *inverting the integral* from flat random numbers ρ, $x=\rho^2$.

Importance sampling — example

$$\int_0^1 \frac{p(x)}{2\sqrt{x}}\,dx = \int_0^1 p(x(\rho))\,d\rho$$

with

$$\rho = \sqrt{x}, \qquad d\rho = \frac{dx}{2\sqrt{x}}$$



Events generated with $w_{\max} = 1$, as $p(x) \le 1$, no guesswork needed here! Now, we get 74.6% MC efficiency.

Importance sampling — example

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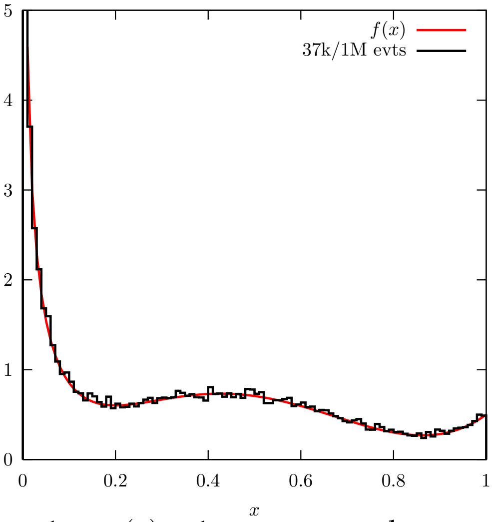
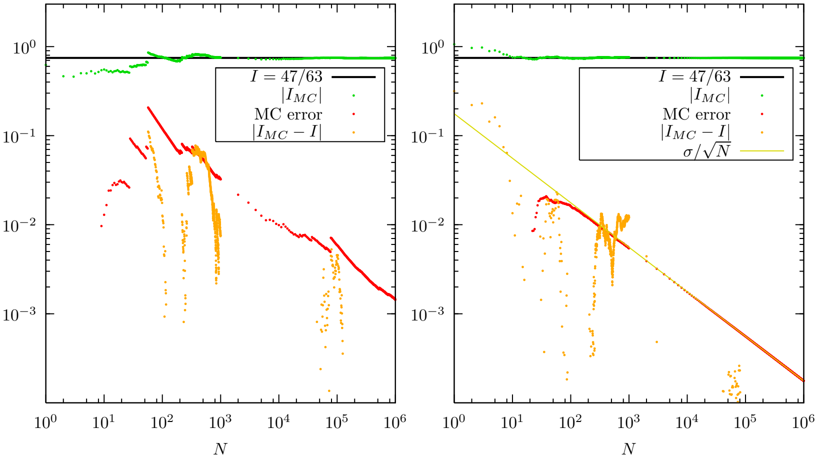


Events generated with $w_{\max} = 1$, as $p(x) \leq 1$, no guesswork needed here! Now, we get 74.6% MC efficiency.
...as opposed to 3.7%.

Importance sampling — example

Crude MC vs Importance sampling.

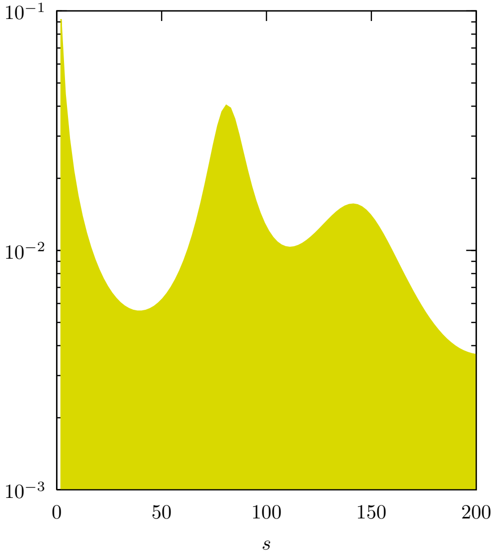


100× more events needed to reach same accuracy.

Multichannel MC

Typical problem:

- $f(s)$ has multiple peaks ($\times$ wiggles from ME).



Multichannel MC

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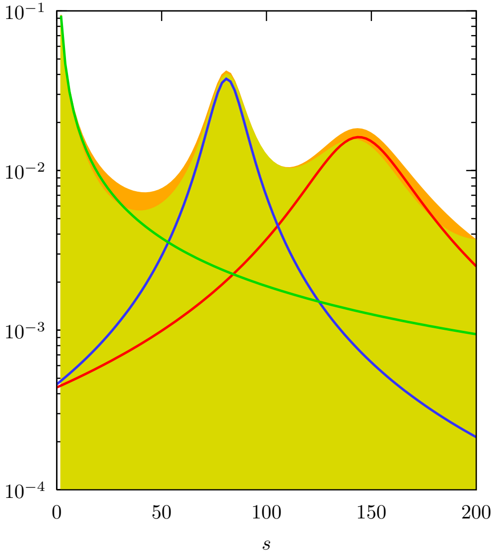
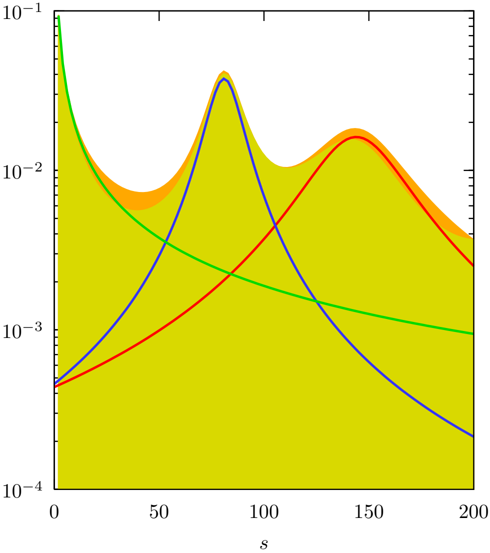


Multichannel MC

Typical problem:

- $f(s)$ has multiple peaks ($\times$ wiggles from ME).
- Usually have some idea of the peak structure.
- Encode this in sum of sample functions $g_i(s)$ with weights $\alpha_i, \sum_i \alpha_i = 1$.

$$g(s) = \sum_i \alpha_i g_i(s) \; .$$



Multichannel MC

Now rewrite

$$\begin{aligned}
\int_{s_0}^{s_1} f(s)\, ds &= \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g(s)\, ds \\
&= \int_{s_0}^{s_1} \frac{f(s)}{g(s)} \sum_i \alpha_i g_i(s)\, ds \\
&= \sum_i \alpha_i \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g_i(s)\, ds
\end{aligned}$$

Now $g_i(s)\, ds = d\rho_i$ (inverting the integral).

Multichannel MC

Now rewrite

$$\begin{aligned}\int_{s_0}^{s_1} f(s)\,ds &= \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g(s)\,ds \\ &= \int_{s_0}^{s_1} \frac{f(s)}{g(s)} \sum_i \alpha_i g_i(s)\,ds \\ &= \sum_i \alpha_i \int_{s_0}^{s_1} \frac{f(s)}{g(s)} g_i(s)\,ds\end{aligned}$$

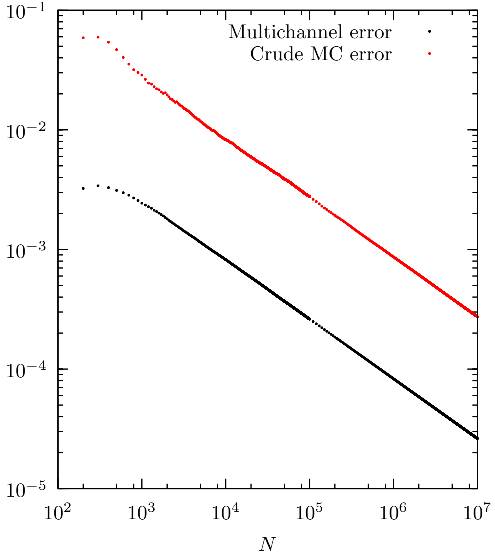
Now $g_i(s)\,ds = d\rho_i$ (inverting the integral).

Select the distribution $g_i(s)$ you'd like to sample next event from acc to weights α_i.

α_i can be optimized after a number of trials.

Multichannel MC

Works quite well:

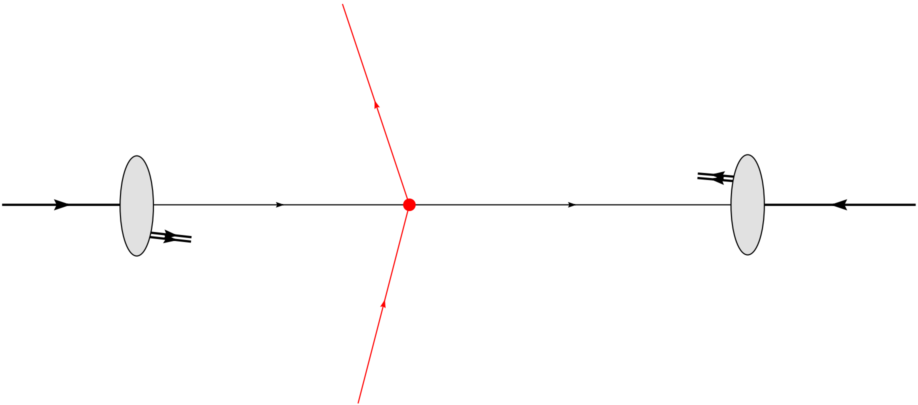


Some Remarks/Real Life MC

- Didn't discuss random number generators. Please make sure to use 'good' random numbers.
- Didn't discuss *stratified sampling* (VEGAS).
  Sample where variance is biggest.
  (not necessarily where PS is most populated).
- Only discussed one–dimensional case here. N–particle PS has $3N - 4$ dimensions...
- Didn't discuss tools geared towards this, like `RAMBO` (generates flat N particles PS).
- generalisation straightforward, particularly
  MCError $\sim \frac{1}{\sqrt{N}}$,
  compare eg Trapezium rule Error $\sim \frac{1}{N^{2/D}}$.
- Many important techniques covered here in detail! Should be good starting point.

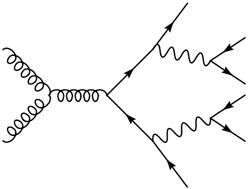
Hard Scattering

Hard scattering



Matrix elements

- Perturbation theory/Feynman diagrams give us (fairly accurate) final states for a few number of legs ($O(1)$).



- OK for very inclusive observables.

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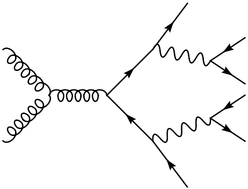

- OK for very inclusive observables.
- Starting point for further simulation.
- Want exclusive final state at the LHC ($O(100)$).

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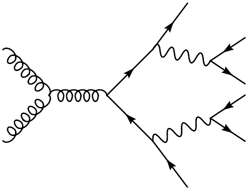

- OK for very inclusive observables.
- Starting point for further simulation.
- Want exclusive final state at the LHC ($\mathcal{O}(100)$).
- Want arbitrary cuts.
- $\rightarrow$ use Monte Carlo methods.

Matrix elements

Where do we get (LO) $|M|^2$ from?

- Most/important simple processes (SM) are 'built in'.
- Calculate yourself (≤ 3 particles in final state).
- Matrix element generators:
  - MadGraph/MadEvent.
  - Comix/AMEGIC (part of Sherpa).
  - HELAC/PHEGAS.
  - Whizard.
  - CalcHEP/CompHEP.

  generate code or event files that can be further processed.
- $\rightarrow$ FeynRules interface to ME generators.

Cross section formula

From Matrix element, we calculate

$$\sigma = \int f_i(x_1,\mu^2) f_j(x_2,\mu^2) \frac{1}{F} \overline{\sum} |M|^2 \qquad\qquad \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n \ ,$$

Cross section formula

From Matrix element, we calculate

$$\sigma = \int f_i(x_1, \mu^2) f_j(x_2, \mu^2) \frac{1}{F} \overline{\sum} |M|^2 \, \Theta(\text{cuts}) \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n \ ,$$

Cross section formula

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now,

$$\frac{1}{F} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n = J(\vec{x}) \prod_{i=1}^{3n-2} \mathrm{d}x_i \qquad \left(\mathrm{d}\Phi_n = (2\pi)^4 \delta^{(4)}(\ldots) \prod_{i=1}^{n} \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_i} \right)$$

such that

$$\sigma = \int g(\vec{x}) \, \mathrm{d}^{3n-2}\vec{x} \ , \qquad \left(g(\vec{x}) = J(\vec{x}) f_i f_j \overline{\sum} |M|^2 \Theta(\mathrm{cuts}) \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{g(\vec{x}_i)}{p(\vec{x}_i)} = \frac{1}{N} \sum_{i=1}^{N} w_i \ .$$

Cross section formula

From Matrix element, we calculate

$$\sigma = \int f_i(x_1, \mu^2) f_j(x_2, \mu^2) \frac{1}{F} \overline{\sum} |M|^2 \, \Theta(\text{cuts}) \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n \ ,$$

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such that

$$\begin{aligned} \sigma &= \int g(\vec{x}) \, \mathrm{d}^{3n-2}\vec{x} \ , \qquad \left(g(\vec{x}) = J(\vec{x}) f_i f_j \overline{\sum} |M|^2 \Theta(\text{cuts}) \right) \\ &= \frac{1}{N} \sum_{i=1}^{N} \frac{g(\vec{x}_i)}{p(\vec{x}_i)} = \frac{1}{N} \sum_{i=1}^{N} w_i \ . \end{aligned}$$

We generate events $\vec{x}_i$ with weights w_i.

Mini event generator

- We generate pairs $(\vec{x}_i, w_i)$.

Mini event generator

- We generate pairs $(\vec{x}_i, w_i)$.
- Use immediately to book weighted histogram of arbitrary observable (possibly with additional cuts!)

Mini event generator

- We generate pairs $(\vec{x}_i, w_i)$.
- Use immediately to book weighted histogram of arbitrary observable (possibly with additional cuts!)
- Keep event $\vec{x}_i$ with probability

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Generate events with same frequency as in nature!

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where $w_{\max}$ has to be chosen sensibly.
$\to$ reweighting, when $\max(w_i) = \bar{w}_{\max} > w_{\max}$, as

$$P_i = \frac{w_i}{\bar{w}_{\max}} = \frac{w_i}{w_{\max}} \cdot \frac{w_{\max}}{\bar{w}_{\max}} \ ,$$

i.e. reject events with probability $(w_{\max}/\bar{w}_{\max})$ afterwards.
(can be ignored when #(events with $w_i > \bar{w}_{\max}$) small.)

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Matrix elements

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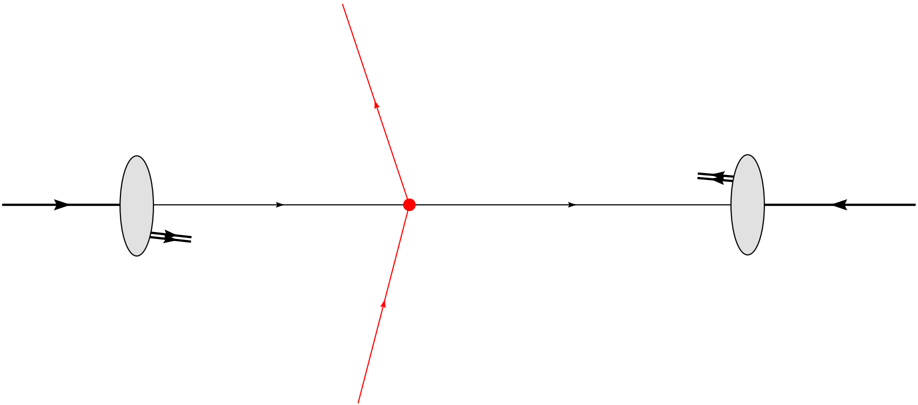
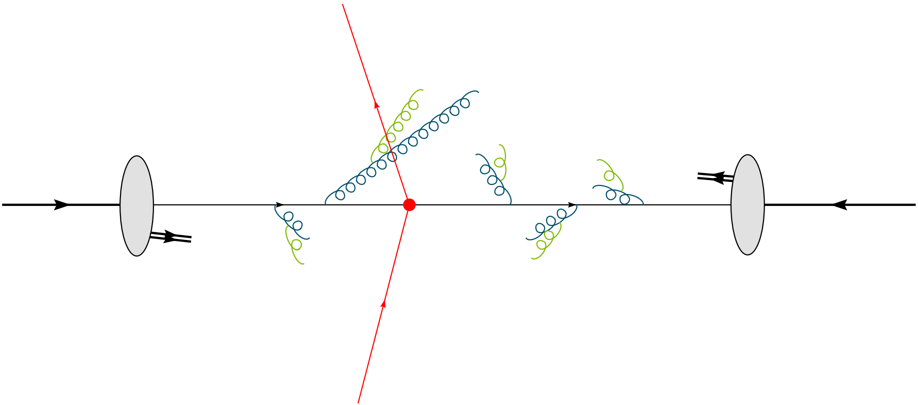
- Clear from above: efficient generation closely tied to knowledge of $f(\vec{x}_i)$, *i.e.* the matrix element's propagator structure.
  $\to$ build phase space generator already while generating ME's automatically.

Parton Showers

Hard matrix element



Hard matrix element → parton showers



Parton showers

Quarks and gluons in final state, pointlike.

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Generated from emissions *ordered* in Q.
Soft and/or collinear emissions.

Good starting point: $e^+e^- \to q\bar{q}g$:

Final state momenta in one plane (orientation usually averaged).
Write momenta in terms of

$$x_i = \frac{2p_i \cdot q}{Q^2} \quad (i = 1,2,3) \ ,$$

$$0 \le x_i \le 1 \ , x_1 + x_2 + x_3 = 2 \ ,$$

$$q = (Q,0,0,0) \ ,$$

$$Q \equiv E_{cm} \ .$$

$(x_1, x_2) = (x_q, x_{\bar{q}})$ –plane:

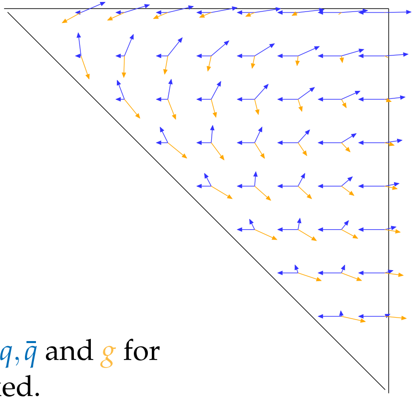
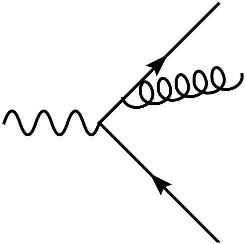


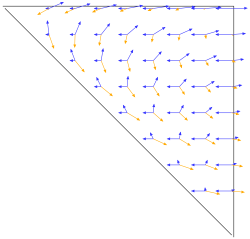
Fig: momentum configuration of $q, \bar{q}$ and g for given point (x_1, x_2), $\bar{q}$ direction fixed.

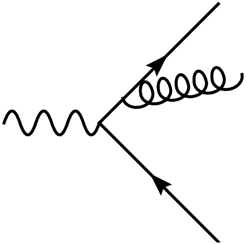


Differential cross section:

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \frac{x_1 + x_2}{(1 - x_1)(1 - x_2)}$$

Collinear singularities: $x_1 \to 1$ or $x_2 \to 1$.
Soft singularity: $x_1, x_2 \to 1$.





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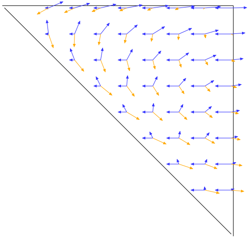

Rewrite in terms of x_3 and $\theta = \angle(q, g)$:

$$\frac{d\sigma}{d\cos\theta\, dx_3} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \left[\frac{2}{\sin^2\theta} \frac{1+(1-x_3)^2}{x_3} - x_3 \right]$$

Singular as $\theta \to 0$ and $x_3 \to 0$.

Can separate into two jets as

$$\frac{2\mathrm{d}\cos\theta}{\sin^2\theta} = \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\theta}{1+\cos\theta}$$
$$= \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\bar{\theta}}{1-\cos\bar{\theta}}$$
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$$\mathrm{d}\sigma = \sigma_0 \sum_{\text{jets}} \frac{\mathrm{d}\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} C_F \frac{1+(1-z)^2}{z^2} \mathrm{d}z$$

e^+e^- annihilation

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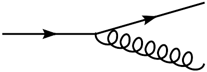
$$= \sigma_0 \sum_{\text{jets}} \frac{\mathrm{d}\theta^2}{\theta^2}\frac{\alpha_S}{2\pi} P(z)\mathrm{d}z$$

with DGLAP splitting function $P(z)$.

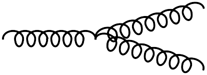
Collinear limit

Universal DGLAP splitting kernels for collinear limit:

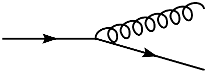
$$\mathrm{d}\sigma = \sigma_0 \sum_{\text{jets}} \frac{\mathrm{d}\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} P(z) \mathrm{d}z$$



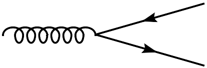
$$P_{q\to qg}(z) = C_F \frac{1+z^2}{1-z}$$



$$P_{g\to gg}(z) = C_A \frac{(1-z(1-z))^2}{z(1-z)}$$



$$P_{q\to gq}(z) = C_F \frac{1+(1-z)^2}{z}$$



$$P_{g\to qq}(z) = T_R(1-2z(1-z))$$

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Universal DGLAP splitting kernels for collinear limit:

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Note: Other variables may equally well characterize the collinear limit:

$$\frac{\mathrm{d}\theta^2}{\theta^2} \sim \frac{\mathrm{d}Q^2}{Q^2} \sim \frac{\mathrm{d}p_\perp^2}{p_\perp^2} \sim \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2} \sim \frac{\mathrm{d}t}{t}$$

whenever $Q^2, p_\perp^2, t \to 0$ means "collinear".

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- θ: HERWIG
- Q^2: PYTHIA ≤ 6.3, old SHERPA.
- $p_\perp$: PYTHIA ≥ 6.4, ARIADNE, Catani–Seymour showers in HERWIG++ and SHERPA.
- $\tilde{q}$: Herwig++.

Resolution

Need to introduce resolution t_0, e.g. a cutoff in $p_\perp$. Prevent us from the singularity at $\theta \to 0$.

Emissions below t_0 are unresolvable.

Finite result due to virtual corrections:

 = finite.

unresolvable + virtual emissions are included in Sudakov form factor via unitarity (see below!).

Towards multiple emissions

Starting point: factorisation in collinear limit, single emission.

$$\sigma_{2+1}(t_0) = \sigma_2(t_0) \int_{t_0}^{t} \frac{\mathrm{d}t'}{t'} \int_{z_-}^{z_+} \mathrm{d}z \, \frac{\alpha_S}{2\pi} \hat{P}(z) \; = \sigma_2(t_0) \int_{t_0}^{t} \mathrm{d}t \, W(t) \; .$$

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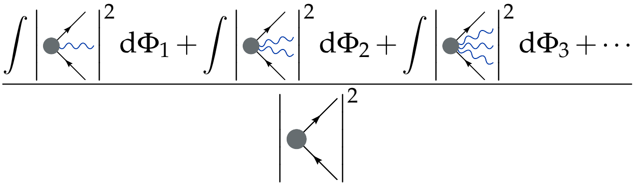
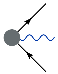
Simple example:
Multiple photon emissions, strongly ordered in t.
We want

$$W_{\rm sum} = \sum_{n=1} W_{2+n} =$$

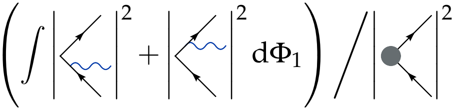


for any number of emissions.

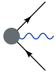
Towards multiple emissions

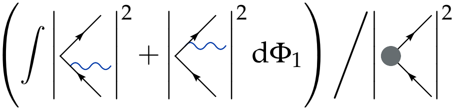
$(n = 1)$ 

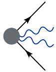
$$W_{2+1} =$$

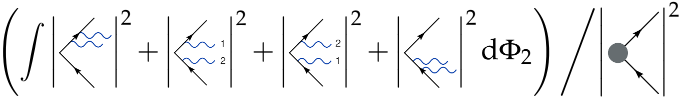


$$= \frac{2}{1!}\int_{t_0}^{t} \mathrm{d}t\, W(t)\ .$$

$(n = 1)$ 

$W_{2+1} =$  $= \dfrac{2}{1!} \displaystyle\int_{t_0}^{t} \mathrm{d}t\, W(t)$.

$(n = 2)$ 

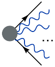
$W_{2+2} =$ 

$$= 2^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\, W(t') W(t'') = \frac{2^2}{2!} \left(\int_{t_0}^{t} \mathrm{d}t\, W(t) \right)^2 .$$

We used

$$\int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n \;\; W(t_1) \ldots W(t_n) = \frac{1}{n!} \left(\int_{t_0}^{t} \mathrm{d}t\, W(t) \right)^n .$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

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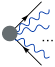
Easily generalized to n emissions  by induction. *i.e.*

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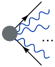
Easily generalized to n emissions  by induction. *i.e.*

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$$= \sigma_2(t_0) \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right)$$

Sudakov Form Factor

$$\Delta(t_0, t) = \exp\left[- \int_{t_0}^{t} \mathrm{d}t\, W(t) \right]$$

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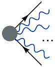
Easily generalized to n emissions  by induction. *i.e.*

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Sudakov Form Factor in QCD

$$\Delta(t_0,t) = \exp\left[-\int_{t_0}^{t}\mathrm{d}t\, W(t)\right] = \exp\left[-\int_{t_0}^{t}\frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\right]$$

Sudakov form factor

Note that

$$\sigma_{\text{all}} = \sigma_2 + \sigma_{>2} = \sigma_2 + \sigma_2 \left(\frac{1}{\Delta^2(t_0,t)} - 1 \right) ,$$
$$\Rightarrow \Delta^2(t_0,t) = \frac{\sigma_2}{\sigma_{\text{all}}} .$$

$$\text{Two jet rate} = \Delta^2 = P^2(\text{No emission in the range } t \to t_0) .$$

Sudakov form factor = No emission probability .

Often $\Delta(t_0,t) \equiv \Delta(t)$.

- Hard scale t, typically CM energy or $p_\perp$ of hard process.
- Resolution t_0, two partons are resolved as two entities if inv mass or relative $p_\perp$ above t_0.
- P^2 (not P), as we have two legs that evolve independently.