Basics of QCD perturbation theory

D. E. Soper

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Abstract

A prediction for experiment based on perturbative QCD combines

• a particular calculation of Feynman diagrams (easy at leading order, not so easy at next-to-leading order).

• use of general features of the theory that allow the Feynman diagrams to be related to experiment:

- renormalization group and the running coupling;
- existence of infrared safe observables;
- isolation of hadron structure in parton distribution functions.

I will discuss these structural features of the theory that allow a comparison of theory and experiment. Along the way we will discover something about certain important processes: e^+e^- annihilation, deeply inelastic scattering, and jet production in hadron-hadron collisions.

Disclaimer. We will not learn how to do significant calculations in QCD perturbation theory. Four hours is not enough for that.

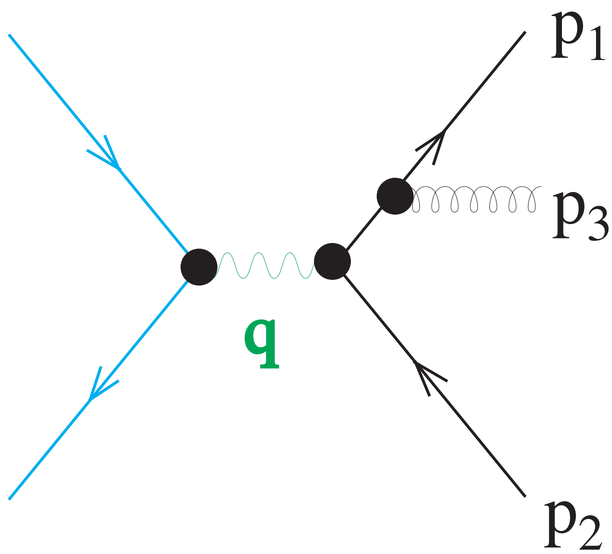
Electron-positron annihilation and jets

Exploring the QCD final state

See lectures of G. Cowan and W. Gary.

A) Kinematics of $e^+e^- \to 3\ partons$.
B) Structure of the cross section.
C) Null plane coordinates.
D) Space-time picture of the singularities.
E) The long time problem and infrared safe observables.
F) Examples of infrared safe observables.
G) C_F and C_A.

Kinematics of $e^+e^- \to 3$ partons



Energy fractions:

$$x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2p_i \cdot q}{s}, \qquad \Rightarrow \quad 0 < x_i.$$

Energy conservation:

$$\sum_i x_i = \frac{2(\sum p_i) \cdot q}{s} = 2.$$

Angles:

$$2p_1 \cdot p_3 = (p_1 + p_3)^2 = (q - p_2)^2 = s - 2q \cdot p_2.$$

$$2E_1 E_3 (1 - \cos\theta_{13}) = s(1 - x_2).$$

$$x_1 x_3 (1 - \cos\theta_{13}) = 2(1 - x_2).$$

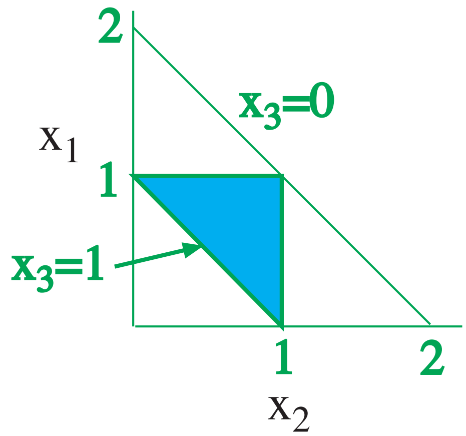
So

$$x_i < 1.$$

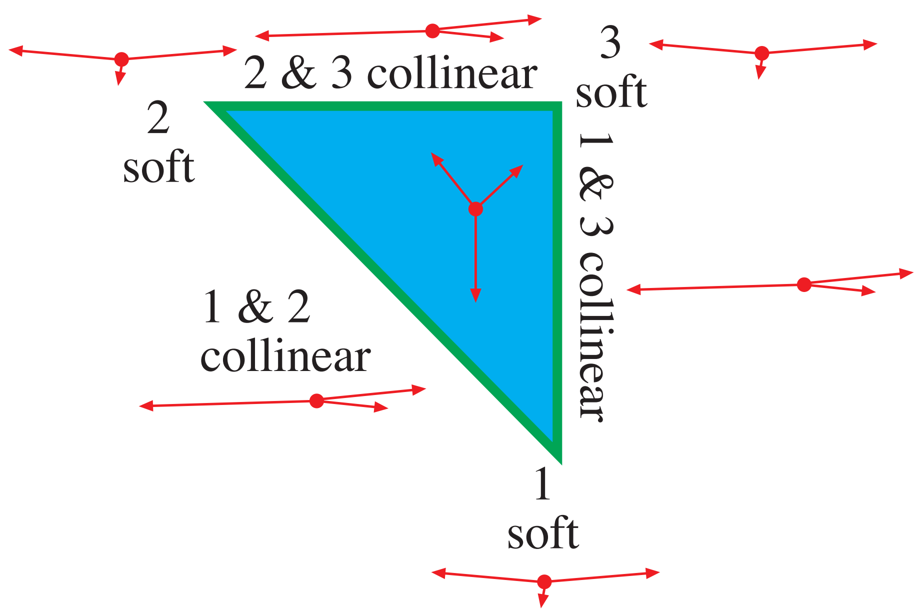
$$\theta_{13} \to 0 \quad \Leftrightarrow \quad x_2 \to 1.$$

Regions for the energy fractions

The energy fractions x_i lie within a triangle.

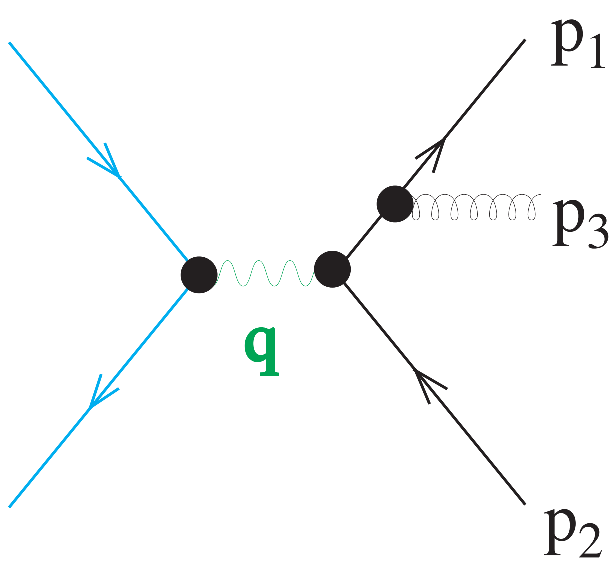


The edges of the allowed region correspond to two partons being collinear.



The corners correspond to one parton being soft.

Structure of the cross section



$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

where $C_F = 4/3$ and $\sigma_0 = (4\pi\alpha^2/s)\sum e_q^2$ is the total cross section for $e^+e^- \to hadrons$ at order α_s^0.

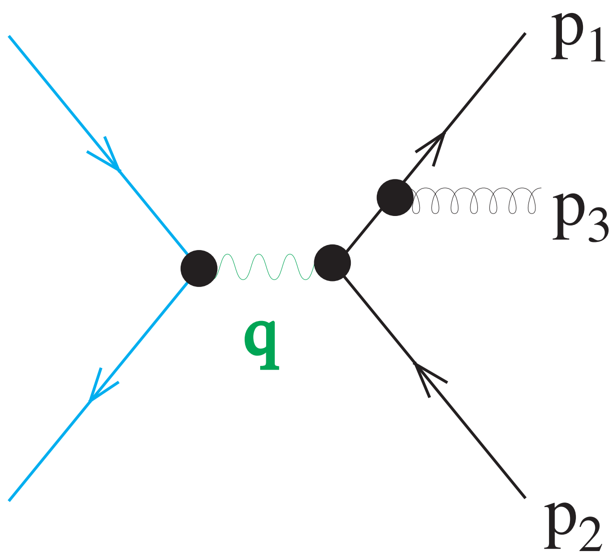
Collinear singularities: $(1-x_1) \to 0$ (2&3 collinear).
$(1-x_2) \to 0$ (1&3 collinear).

Soft singularity (3 soft):

$$(1-x_1) \to 0, \quad (1-x_2) \to 0, \quad \frac{(1-x_1)}{(1-x_2)} \sim \text{const.}$$

- *Cf.* lecture of S. Kuhlmann: $e^+e^- \to q\bar{q}\gamma$.

$e^+e^- \to 3$ partons in terms of energy and angle



$$\frac{1}{\sigma_0}\frac{d\sigma}{dE_3\, d\cos\theta_{13}} = \frac{\alpha_s}{2\pi}\, C_F\, \frac{f(E_3,\theta_{13})}{E_3(1-\cos\theta_{13})}$$

where $f(E_3,\theta_{13})$ is finite for $E_3 \to 0$ and for $\theta_{13} \to 0$.

Collinear singularity, $\theta_{13} \to 0$:

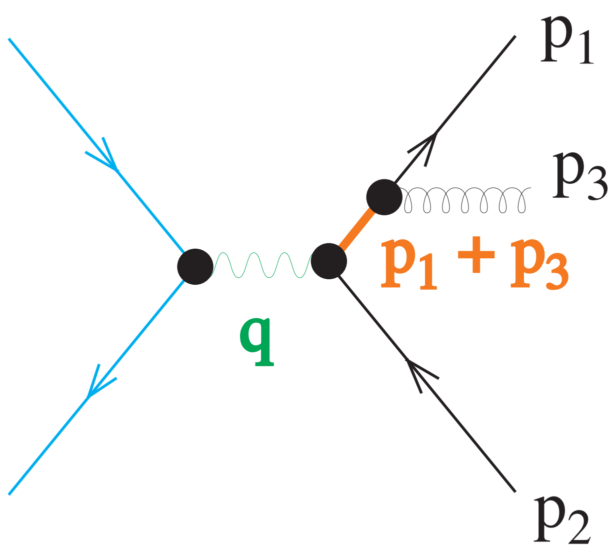
$$\int_a^1 d\cos\theta_{13}\ \frac{d\sigma}{dE_3\, d\cos\theta_{13}} = \log(\infty).$$

Soft singularity, $E_3 \to 0$:

$$\int_0^a dE_3\ \frac{d\sigma}{dE_3\, d\cos\theta_{13}} = \log(\infty).$$

That's great, but is there a general reason for it?

Why is $e^+e^- \to 3$ partons singular?



$\mathcal{M}$ contains a factor $1/(p_1 + p_3)^2$ where

$$(p_1 + p_3)^2 = 2p_1 \cdot p_3 = 2E_1 E_3 (1 - \cos\theta_{13}).$$

Also, a numerator factor $\propto \theta_{13}$ in the collinear limit. So

$$|\mathcal{M}|^2 \propto \left[\frac{\theta_{13}}{E_3 \theta_{13}^2}\right]^2$$

for $E_3 \to 0$ or $\theta_{13} \to 0$.
Integration:

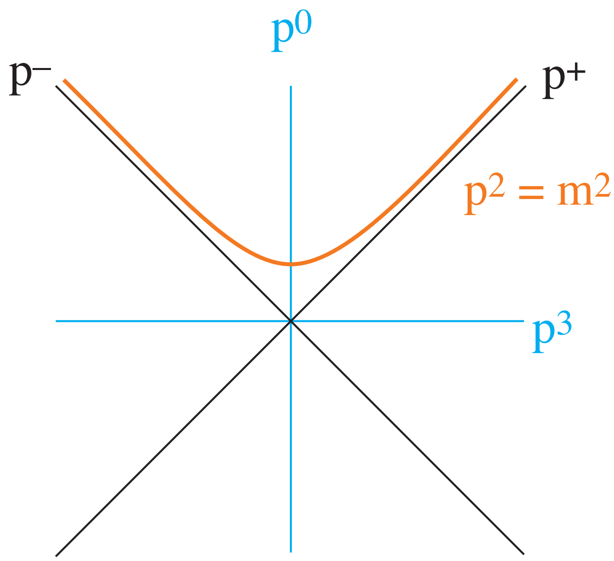
$$\int \frac{E_3^2 dE_3 d\cos\theta_{13} d\phi}{E_3} \sim \int E_3 dE_3 d\theta_{13}^2 d\phi.$$

Together:

$$d\sigma \sim \int E_3 dE_3 d\theta_{13}^2 d\phi \left[\frac{\theta_{13}}{E_3 \theta_{13}^2}\right]^2 \sim \int \frac{dE_3}{E_3} \, \frac{d\theta_{13}^2}{\theta_{13}^2} \, d\phi.$$

Note the universal nature of these factors.

Interlude: Null plane coordinates



Use $p^\mu = (p^+, p^-, p^1, p^2)$ where

$$p^\pm = (p^0 \pm p^3)/\sqrt{2}.$$

• For a particle with large momentum in the $+z$ direction and limited transverse momentum, p^+ is large and p^- is small.

• Often one *chooses* the + axis so that a particle or group of particles of interest have large p^+ and small p^- and p_T.

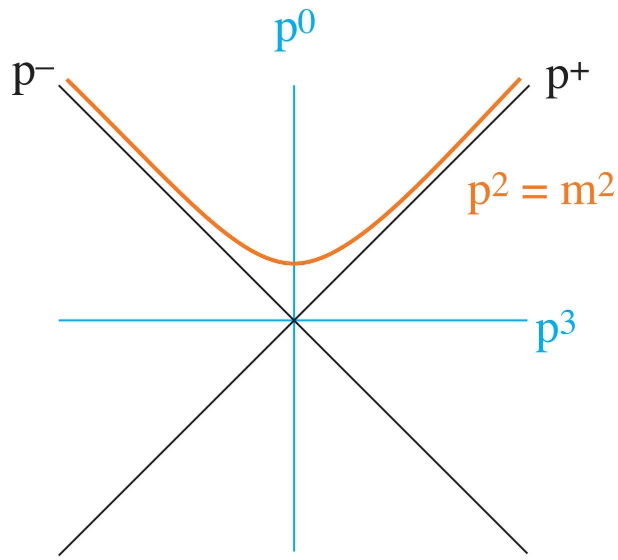
• Covariant square:

$$p^2 = 2p^+p^- - \mathbf{p}_T^2.$$

• p^- for a particle on its mass shell:

$$p^- = \frac{\mathbf{p}_T^2 + m^2}{2p^+}.$$

Kinematics of null plane coordinates, continued.



- For a particle on its mass shell,

$$p^+ > 0\,, \qquad p^- > 0\,.$$

- Integration over the mass shell:

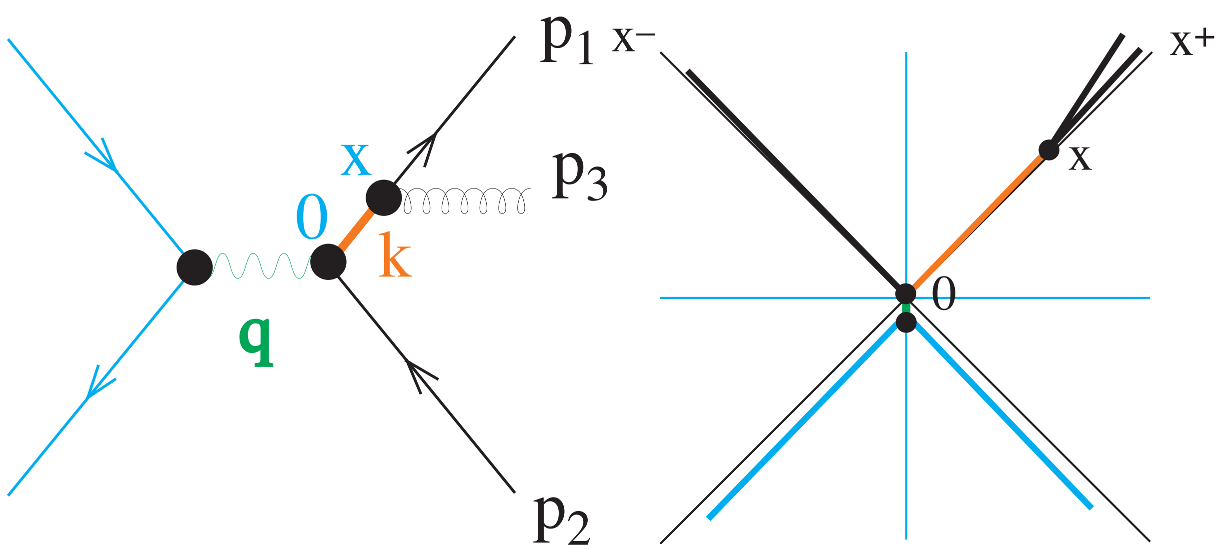
$$(2\pi)^{-3}\int \frac{d^3\vec{p}}{2\sqrt{\vec{p}^2+m^2}}\cdots = (2\pi)^{-3}\int d^2\mathbf{p}_T \int_0^\infty \frac{dp^+}{2p^+}\cdots.$$

- Fourier transform:

$$p\cdot x = p^+x^- + p^-x^+ - \mathbf{p}_T\cdot\mathbf{x}_T.$$

So x^- is conjugate to p^+ and x^+ is conjugate to p^-.
(Sorry.)

Space-time picture of the singularities



Define $p_1^\mu + p_3^\mu = k^\mu$.
Choose null-plane coordinates with k^+ large and $\mathbf{k}_1^T = \mathbf{0}$.
Then $k^2 = 2k^+k^-$ becomes small when

$$k^- = \frac{\mathbf{p}_3^2}{2p_1^+} + \frac{\mathbf{p}_3^2}{2p_3^+}$$

becomes small. (Collinear or soft singularity.)

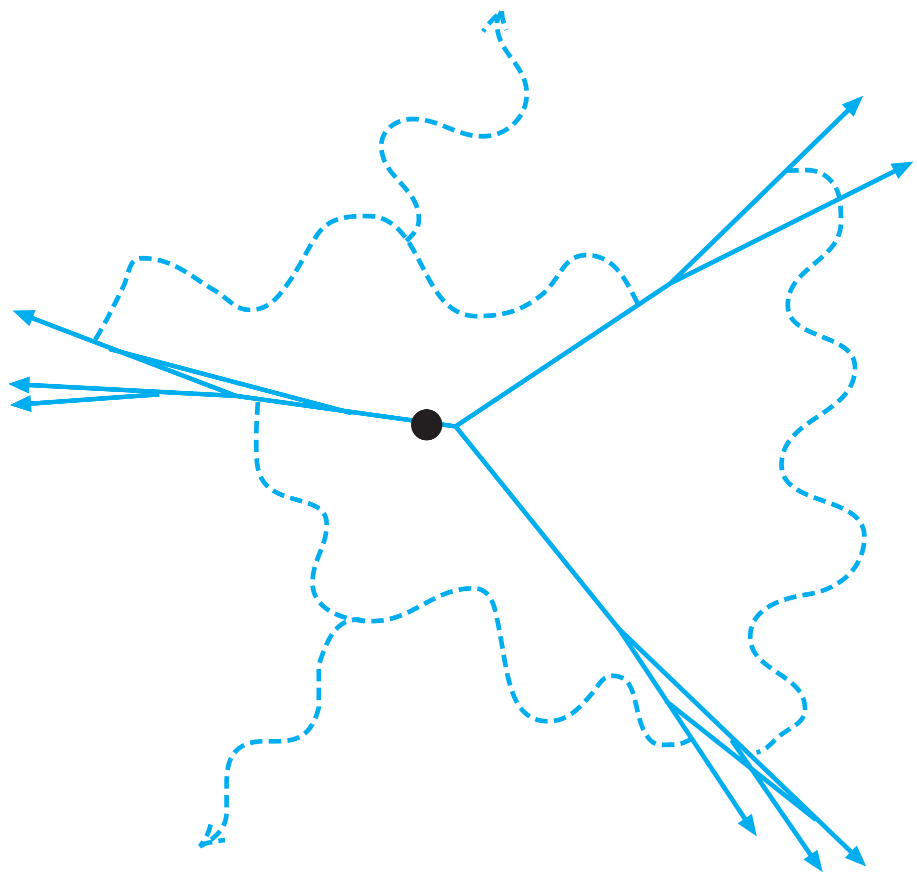
Consider the fourier transform.

$$S_F(k) = \int dx^+ dx^- d\mathbf{x}\, \exp(i[k^+ x^- + k^- x^+ - \mathbf{k}\cdot\mathbf{x}])\; S_F(x).$$

Contributing values of x have small x^- large x^+.

Long time picture

Perturbation theory suggests the generic structure of long time physics:



Thus QCD suggests a jet structure of final state hadrons.

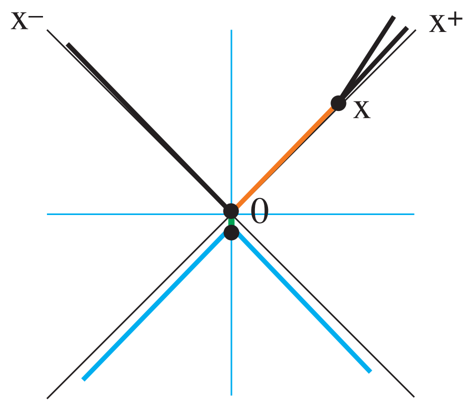
• This structure is modelled in Monte Carlo event generators. See lectures of S. Mrenna.

"Summed" perturbation theory suggests this is OK.
But beware of "nonperturbative" effects!

That's a qualitative success. But can you predict reliable numbers?

The long time problem

Perturbation theory not effective for long time physics. But the detector is a long distance away!

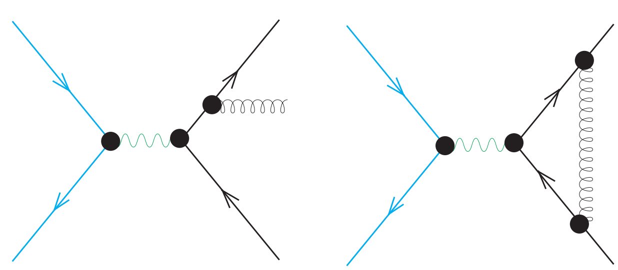


Answer

Use measurements that are not sensitive to long time physics.
Example: the e^+e^- annihilation total cross section.
Effects from $\Delta x \gg 1/\sqrt{s}$ cancel because of unitarity:

$$\langle 0|J(y')U(y',\infty)U(\infty,y)J(y)|0\rangle$$
$$= \langle 0|J(y')U(y',y)J(y)|0\rangle$$

At order α_s, this works by a cancellation between real gluon emission graphs and virtual gluon graphs.



If the total cross section is all you can look at, QCD physics will be a little boring!

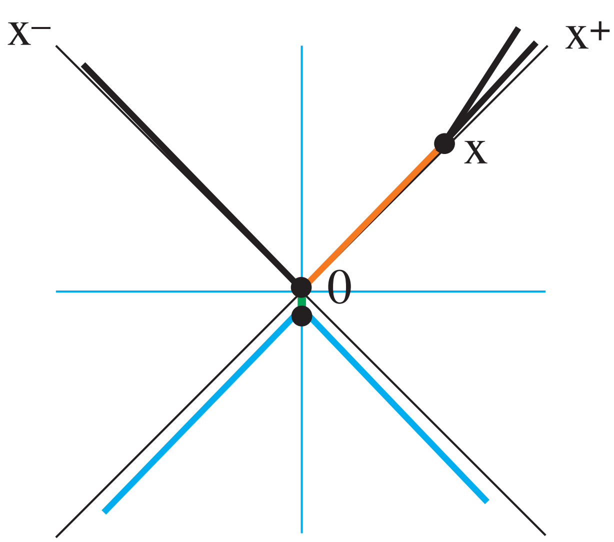
Infrared safe quantities

Some quantities are not sensitive to infrared effects.

$$
\begin{aligned}
\mathcal{I} = {} & \frac{1}{2!}\int d\Omega_2\ \frac{d\sigma[2]}{d\Omega_2}\ \mathcal{S}_2(p_1^\mu, p_2^\mu) \\
& + \frac{1}{3!}\int d\Omega_2 dE_3 d\Omega_3\ \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3}\ \mathcal{S}_3(p_1^\mu, p_2^\mu, p_3^\mu) \\
& + \frac{1}{4!}\int d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4 \\
& \qquad\qquad \times \frac{d\sigma[4]}{d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4}\ \mathcal{S}_4(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) \\
& + \cdots.
\end{aligned}
$$

Need (for $\lambda = 0$ or $0 < \lambda < 1$)

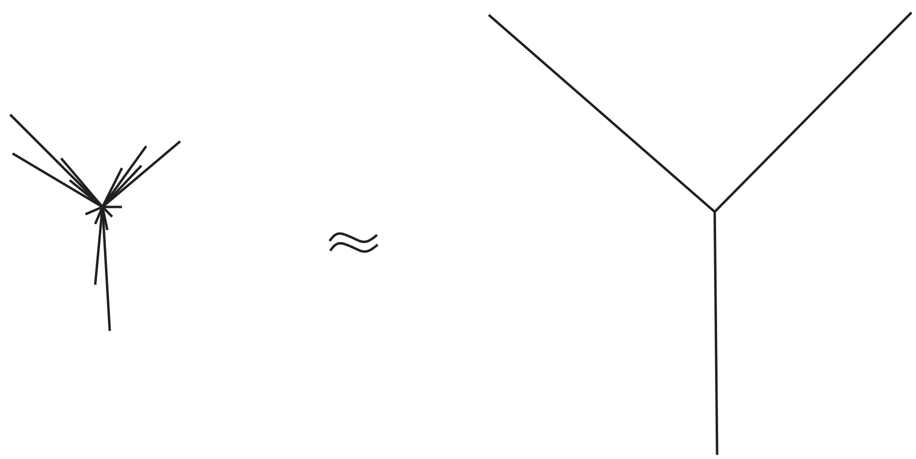
$$
\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu).
$$



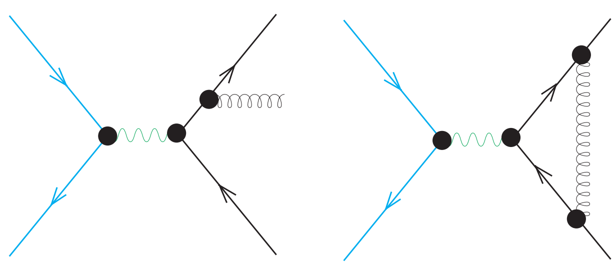
What does infrared safety mean?

$$\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu).$$

The physical meaning is that for an IR-safe quantity a physical event with hadron jets should give approximately the same measurement as a parton event:



The calculational meaning is that infinities cancel.



Examples of infrared safe observables

$$
\begin{aligned}
\mathcal{I} = {} & \frac{1}{2!} \int d\Omega_2 \; \frac{d\sigma[2]}{d\Omega_2} \; \mathcal{S}_2(p_1^\mu, p_2^\mu) \\
& + \frac{1}{3!} \int d\Omega_2 dE_3 d\Omega_3 \; \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} \; \mathcal{S}_3(p_1^\mu, p_2^\mu, p_3^\mu) \\
& + \frac{1}{4!} \int d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4 \\
& \qquad\qquad \times \frac{d\sigma[4]}{d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4} \; \mathcal{S}_4(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) \\
& + \cdots ,
\end{aligned}
$$

with

$$
\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda) p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu).
$$

The simplest example is the total cross section σ_T:

$$
\mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu) = 1.
$$

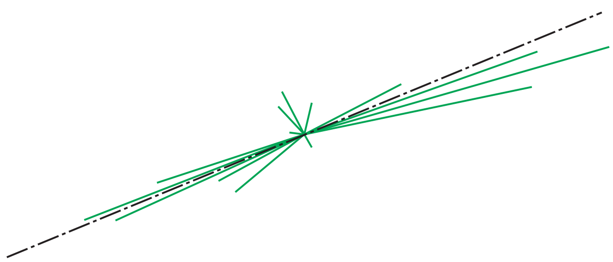
Thrust Distribution

$$\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu).$$

An example is the thrust distribution $d\sigma/dT$:

$$\mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu) = \delta\left(T - \mathcal{T}_n(p_1^\mu, \ldots, p_n^\mu)\right) \ .$$

$$\mathcal{T}_n(p_1^\mu, \ldots, p_n^\mu) = \max_{\vec{u}} \frac{\sum_{i=1}^n |\vec{p}_i \cdot \vec{u}|}{\sum_{i=1}^n |\vec{p}_i|} \ .$$



- Contribution from a particle with $\vec{p} \to 0$ drops out.
- Replacing one particle by two collinear particles doesn't change the thrust:

$$|(1-\lambda)\,\vec{p}_n \cdot \vec{u}| + |\lambda\,\vec{p}_n \cdot \vec{u}| = |\vec{p}_n \cdot \vec{u}|,$$

and

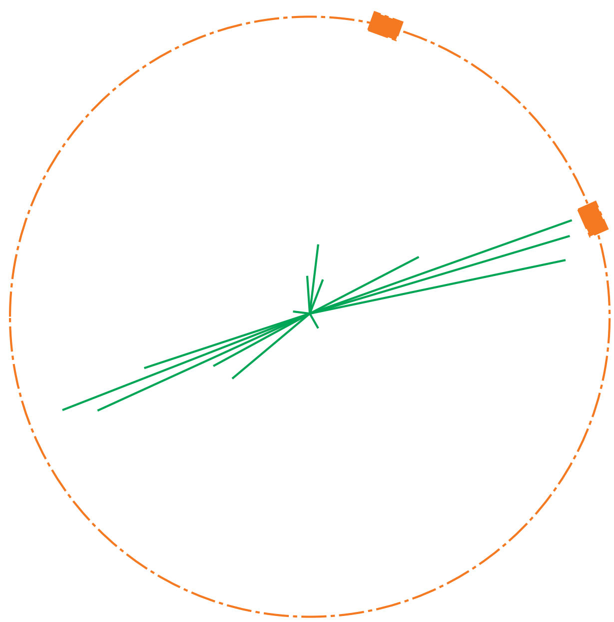
$$|(1-\lambda)\,\vec{p}_n| + |\lambda\,\vec{p}_n| = |\vec{p}_n|.$$

Energy-energy correlation function

$$\mathcal{S}_{n+1}(p_1^\mu, \ldots, (1-\lambda)p_n^\mu, \lambda p_n^\mu) = \mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu).$$

Another example is the energy-energy correlation function $d\Sigma/d\cos(\theta)$:

$$\mathcal{S}_n(p_1^\mu, \ldots, p_n^\mu) = \sum_{ij} \frac{E_i E_j}{s}\, \delta\left(\cos(\theta_{ij}) - \cos(\theta)\right) \ .$$



- Contribution from a particle with $E_i \to 0$ drops out.
- Replacing one particle by two collinear particles doesn't change the thrust:

$$(1-\lambda)\, E_n\, E_j + \lambda\, E_n\, E_j = E_n\, E_j,$$

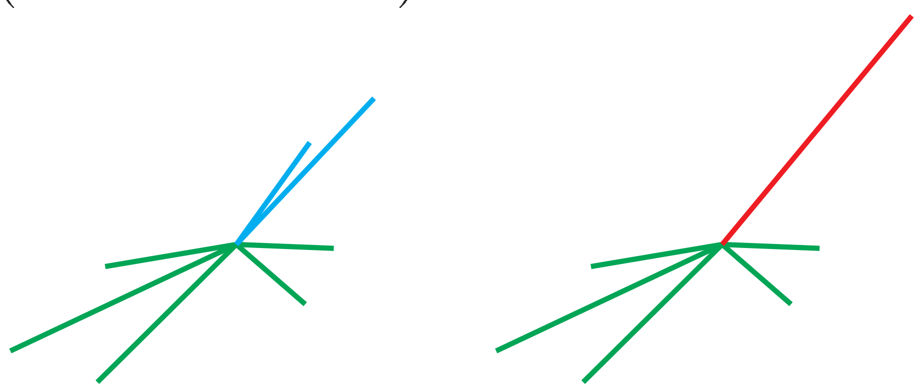
This works for the autocorrelation term too:

$$(1-\lambda)^2\, E_n^2 + 2\lambda(1-\lambda)\, E_n^2 + \lambda^2\, E_n^2 = E_n^2.$$

Jet cross sections

Another example is the n jet cross section σ_n.

There are several algorithms to choose from. Here is the simplest (but not the best) one.



• Start with a list of momenta $p_1^\mu, p_2^\mu, \ldots, p_N^\mu$. At the start, these represent the momenta of particles. (In a perturbative calculation, they are the momenta of partons.)
• Choose a parameter $y_{\rm cut}$.

1. Find the pair (i,j) such that $(p_i + p_j)^2$ is the smallest.
2. If $(p_i + p_j)^2 > y_{\rm cut}\, s$, exit. Else continue.
3. Replace the two momenta p_i and p_j in the list by their sum $p_k^\mu = p_i^\mu + p_j^\mu$.
4. Go to 1.

This produces a list of momenta p_i of jets. σ_n is the cross section to have n jets.

Variations on this theme: change the resolution condition or the combination prescription.

Infrared safety of σ_n is self-evident.

C_F and C_A

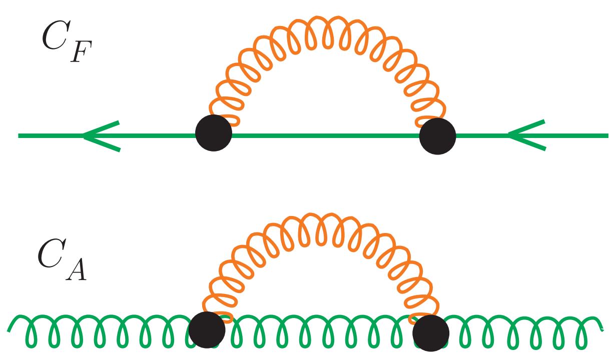
Recall that form of the tree level cross section for $e^+e^- \to$ 3 *partons* has the form

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

where $C_F = 4/3$. The factor C_F comes from the color structure,

$$\sum_{a=1}^{8} t_a^{(3)}\, t_a^{(3)} = C_F \times 1.$$

Here the $t_a^{(3)}$ are the generators of the **3** representation of $SU(3)$.



If we use the **8** representation, we get

$$\sum_{a=1}^{8} t_a^{(8)}\, t_a^{(8)} = C_A \times 1.$$

The value is $C_A = 3$.

Effectively, gluons have a bigger color charge than quarks.

- See lecture of W. Gary.

Deeply inelastic scattering I

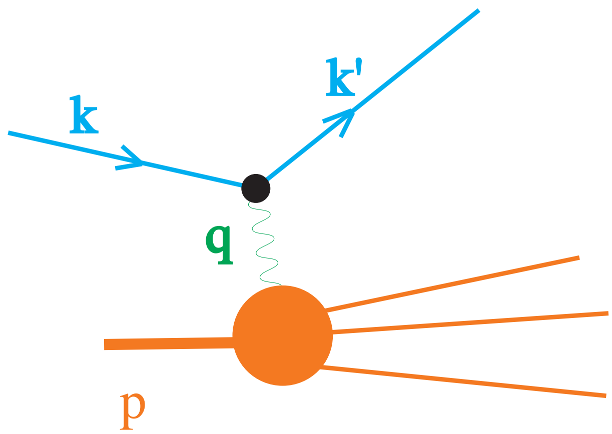
The effect of partons

See lectures of A. Doyle

A) Kinematics of deeply inelastic scattering.
B) Structure functions for DIS.
C) Space-time structure of DIS.
D) Factored cross section.
E) The hard scattering cross section.
F) Factorization for the structure functions.

Kinematics of deeply inelastic lepton scattering

$$\ell(k) + h(p) \to \ell'(k') + X.$$
$$q^\mu = k^\mu - k'^\mu$$



$$Q^2 = -q^2$$
$$x_{\rm bj} = \frac{Q^2}{2p\cdot q} \quad \text{or} \quad A\,\frac{Q^2}{2p\cdot q}$$

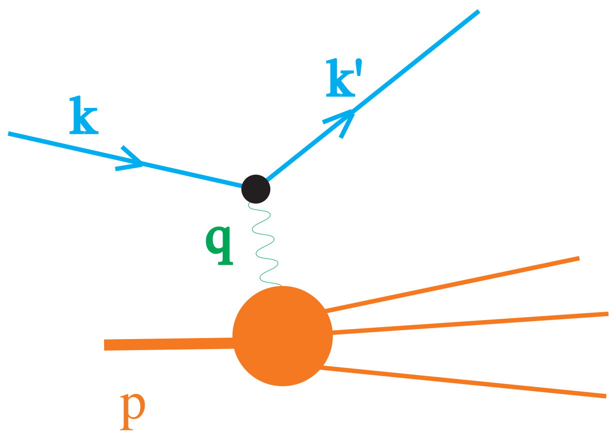
"Deeply inelastic" $\Rightarrow Q^2 \to \infty$, x fixed.
Then also

$$W^2 = (p+q)^2 = m_h^2 + \frac{1-x}{x}Q^2 \to \infty.$$

Lepton variables related to hadron variables by

$$y = \frac{p\cdot q}{p\cdot k}.$$

Structure functions for DIS



Included here: γ or W exchange. For HERA need also Z exchange.

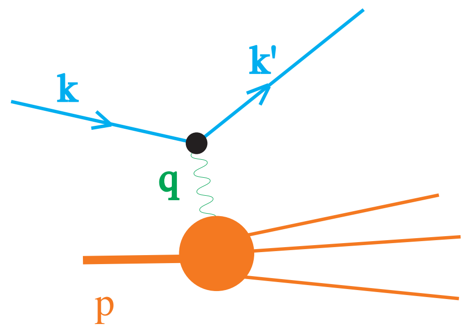
Analysis does not require QCD, just electroweak theory:

$$d\sigma = \frac{4\alpha^2}{s} \frac{d^3\mathbf{k}'}{2|\mathbf{k}'|} \frac{1}{(q^2 - M^2)^2} L^{\mu\nu}(k,q)\, W_{\mu\nu}(p,q).$$

$$L^{\mu\nu} = \frac{1}{2} \mathrm{Tr}\left(k \cdot \gamma\, \Gamma^\mu k' \cdot \gamma\, \Gamma^\nu\right).$$

$$\begin{aligned} W_{\mu\nu} = &- \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_1(x, Q^2) \\ &+ \left(p_\mu - q_\mu \frac{p \cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p \cdot q}{q^2}\right) \frac{1}{p \cdot q}\, F_2(x, Q^2) \\ &- i\epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma \frac{1}{p \cdot q}\, F_3(x, Q^2). \end{aligned}$$

Cross section in terms of structure functions



Result (neglecting m_h^2/Q^2):

$$\frac{d\sigma}{dx\,dy} = \tilde{N}(Q^2)\left[y F_1 + \frac{1-y}{xy} F_2 + \delta_V \,(1 - \frac{y}{2}) F_3 \right].$$

Here

$$\tilde{N} = \frac{4\pi\alpha^2}{2Q^2}, \qquad \delta_V = 0, \quad e^- + h \to e^- X,$$

$$\tilde{N} = \frac{\pi\alpha^2 Q^2}{4\sin^2(\theta_W)\ (Q^2+M)^2}, \qquad \delta_V = 1, \quad \nu + h \to e^- X,$$

$$\tilde{N} = \frac{\pi\alpha^2 Q^2}{4\sin^2(\theta_W)\ (Q^2+M)^2}, \qquad \delta_V = -1, \quad \bar{\nu} + h \to e^+ X.$$

Use y dependence to determine F_1, F_2, F_3.

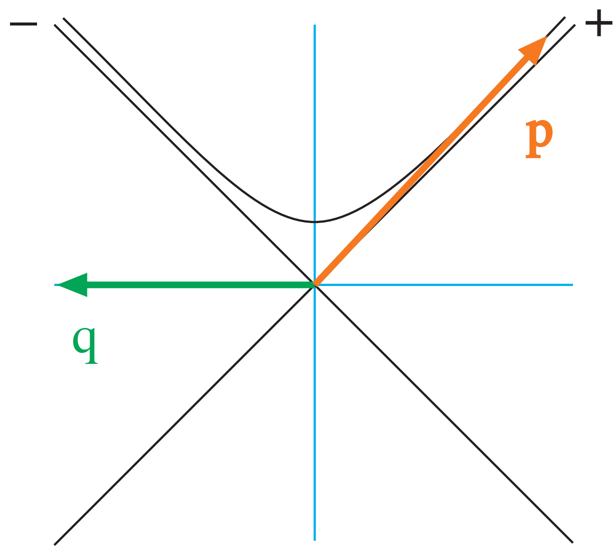
Space-time structure of DIS

A convenient reference frame

A convenient frame is [components $(v^+, v^-, \mathbf{v}_T)$]:

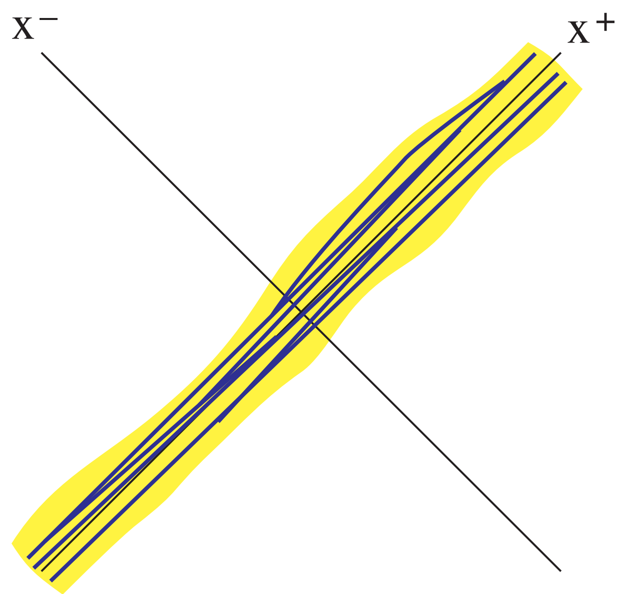
$$(q^+, q^-, \mathbf{q}) = \frac{1}{\sqrt{2}}\,(-Q, Q, \mathbf{0})$$

$$(p^+, p^-, \mathbf{p}) \approx \frac{1}{\sqrt{2}}\,(\frac{Q}{x}, \frac{xm_h^2}{Q}, \mathbf{0})$$



- Hadron momentum is big.
- Momentum transfer is big.

Interactions within a fast moving hadron



Lorentz transformation spreads out interactions. Hadron at rest has separation between interactions

$$\Delta x^+ \sim \Delta x^- \sim \frac{1}{m}.$$

Moving hadron has

$$\Delta x^+ \sim \frac{1}{m} \times \frac{Q}{m} = \frac{Q}{m^2}, \qquad \Delta x^- \sim \frac{1}{m} \times \frac{m}{Q} = \frac{1}{Q}.$$

The virtual photon meets the fast moving hadron

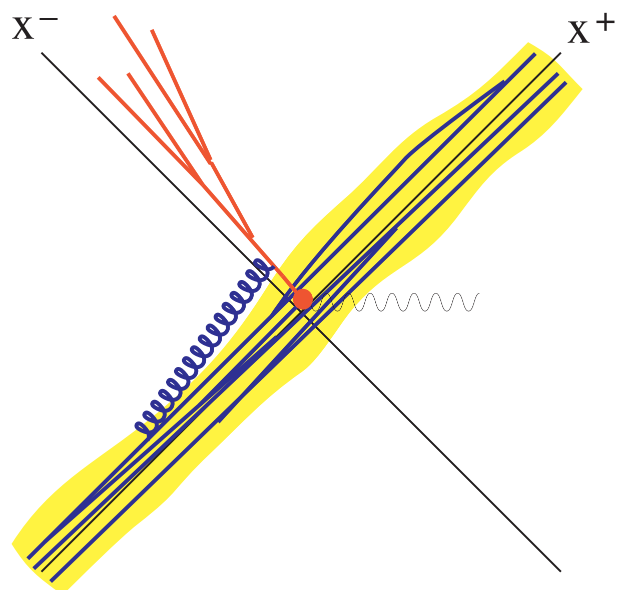
Moving hadron has

$$\Delta x^+ \sim Q/m^2.$$

Interaction with photon with $q^- \sim Q$ is localized to within

$$\Delta x^+ \sim 1/Q.$$

Thus quarks and gluons = "**partons**" are effectively free.



At a given x^+, find partons with an amplitude

$$\psi(p_1^+, \mathbf{p}_1; p_2^+, \mathbf{p}_2; \cdots), \qquad 0 < p_i^+.$$

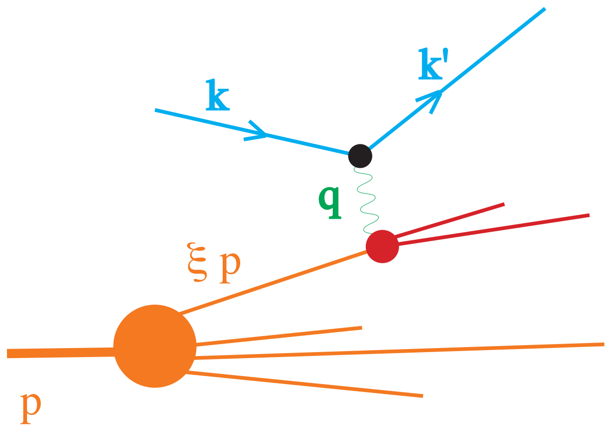
The $\mathbf{p}_i$ are negligible. For p_i^+, use momentum fractions

$$\xi_i = p_i^+/p^+, \qquad 0 < \xi_i < 1.$$

$\Rightarrow$ Hadron is like a collection of free massless partons with $v = 1$, parallel momenta.

Factored cross section

Treat hadron as a collection of free massless partons with parallel momenta.



$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q).$$

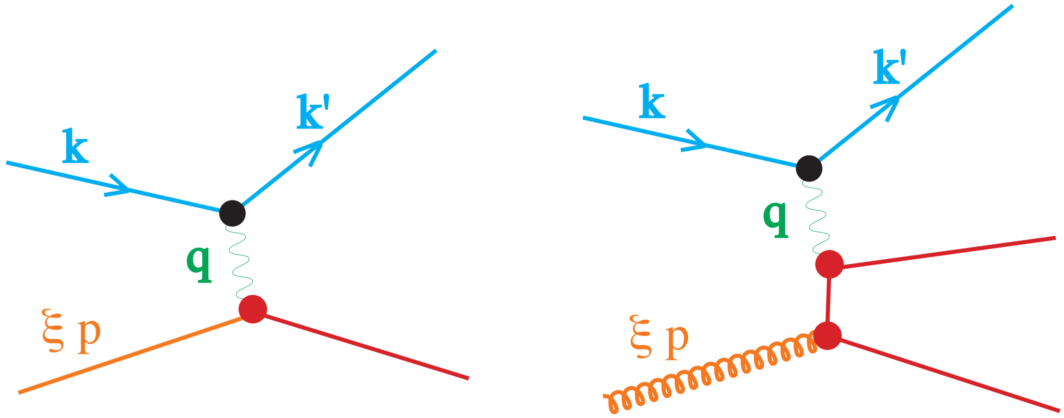
$f_{a/h}(\xi,\mu)\; d\xi$ = probability to find a parton
with flavor $a = g, u, \bar{u}, d, \ldots$,
in hadron h,
carrying momentum fraction $\xi = p_i^+/p^+$.

$d\hat{\sigma}_a/dE'\,d\omega'$ = cross section for scattering that parton.

We delay discussion of the μ dependence.

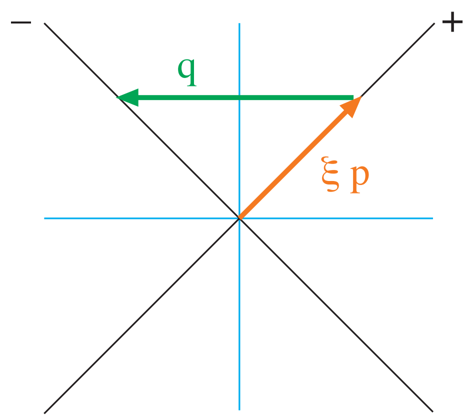
The hard scattering cross section

To calculate $d\hat{\sigma}_a(\mu)/dE'\,d\omega'$ use diagrams like



Lowest order. Higher order.

Kinematics of lowest order diagram:



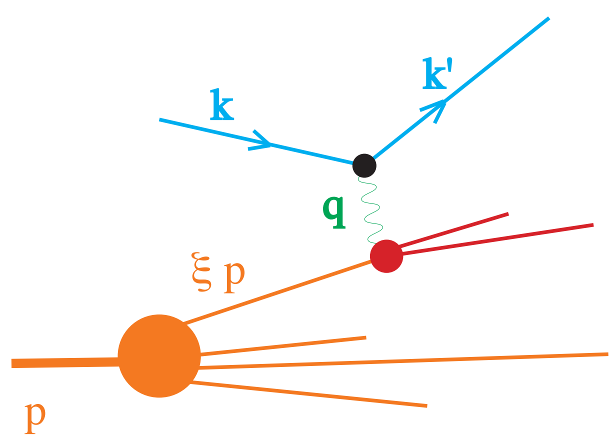
$\xi p^+ + q^+ = 0.$
$p^+ = Q/(x\sqrt{2}),\ q^+ = -Q/\sqrt{2}.$
So

$$\xi = x.$$

Factorization for the structure functions

We will look at DIS in a little detail since it is so important. Our object is to derive a formula relating the measured structure functions to structure functions calculated at the parton level. Then we will look at the parton level at lowest order. Start with

$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi)\,\frac{d\hat{\sigma}_a}{dE'\,d\omega'} + \mathcal{O}(m/Q).$$



Write this in terms of x and y variables:

$$y = \frac{p\cdot q}{p\cdot k} = \frac{\xi p\cdot q}{\xi p\cdot k}.$$

$$x = \frac{Q^2}{2p\cdot q} = \xi\,\frac{Q^2}{2\xi p\cdot q} = \xi\hat{x}.$$

$$\frac{d\sigma}{dx\,dy} \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi)\,\frac{1}{\xi}\left[\frac{d\hat{\sigma}_a}{d\hat{x}\,dy}\right]_{\hat{x}=x/\xi} + \mathcal{O}(m/Q).$$

Relate cross sections to structure functions

$$\frac{d\sigma}{dx\,dy} \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi)\ \frac{1}{\xi}\left[\frac{d\hat\sigma_a}{d\hat x\,dy}\right]_{\hat x = x/\xi} + \mathcal{O}(m/Q).$$

For γ exchange,

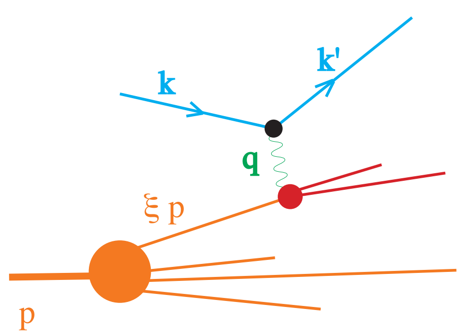
$$\frac{d\sigma}{dx\,dy} = \tilde N(Q^2)\left[yF_1(x,Q^2) + \frac{1-y}{xy}F_2(x,Q^2)\right] + \mathcal{O}(m/Q).$$

$$\frac{d\hat\sigma_a}{d\hat x\,dy} = \tilde N(Q^2)\left[y\hat F_1^a(x/\xi,Q^2) + \frac{1-y}{(x/\xi)y}\hat F_2^a(x/\xi,Q^2)\right].$$

So the structure functions can be factored as

$$F_1(x,Q^2) \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi)\ \frac{1}{\xi}\hat F_1^a(x/\xi,Q^2) + \mathcal{O}(m/Q).$$

$$F_2(x,Q^2) \sim \int_0^1 d\xi \sum_a f_{a/h}(\xi)\ \hat F_2^a(x/\xi,Q^2) + \mathcal{O}(m/Q).$$



Structure functions at lowest order

A simple calculation gives

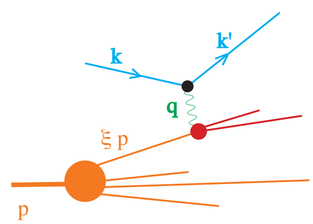
$$\hat{F}_1^a(x/\xi, Q^2) = \frac{1}{2} Q_f^2 \; \delta(x/\xi - 1) + \mathcal{O}(\alpha_s),$$

while we recall

$$F_1(x, Q^2) \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi) \; \frac{1}{\xi} \hat{F}_1^a(x/\xi, Q^2) + \mathcal{O}(m/Q).$$

So

$$F_1(x, Q^2) \sim \frac{1}{2} \sum_a \; Q_a^2 \; f_{a/h}(x) + \mathcal{O}(\alpha_s) + \mathcal{O}(m/Q).$$



Similarly, a simple calculation gives

$$\hat{F}_2^a(x/\xi, Q^2) = Q_a^2 \; \delta(x/\xi - 1) + \mathcal{O}(\alpha_s),$$

while we recall

$$F_2(x, Q^2) \sim \int_0^1 d\xi \sum_a \; f_{f/h}(\xi) \; \hat{F}_2^a(x/\xi, Q^2) + \mathcal{O}(m/Q).$$

So

$$F_2(x, Q^2) \sim \sum_a \; Q_a^2 \; x \, f_{a/h}(x) + \mathcal{O}(\alpha_s) + \mathcal{O}(m/Q).$$

Factor 1/2 between F_1 and F_2: quarks have spin 1/2.

Summary so far

• Singularities for collinear parton splitting and joining.

• Singularities for soft gluon emission and absorption.

• This suggests a jet structure of final states.

• The singularities reflect long time physics.

• For short time physics, use infrared safe observables.

• In DIS, collinear splitting and joining in initial hadron

• This long distance physics $\to$ parton distributions.

• At lowest order, the hard scattering part of DIS is trivial, so measured structure functions $\approx$ parton distributions.

Renormalization and factorization scales

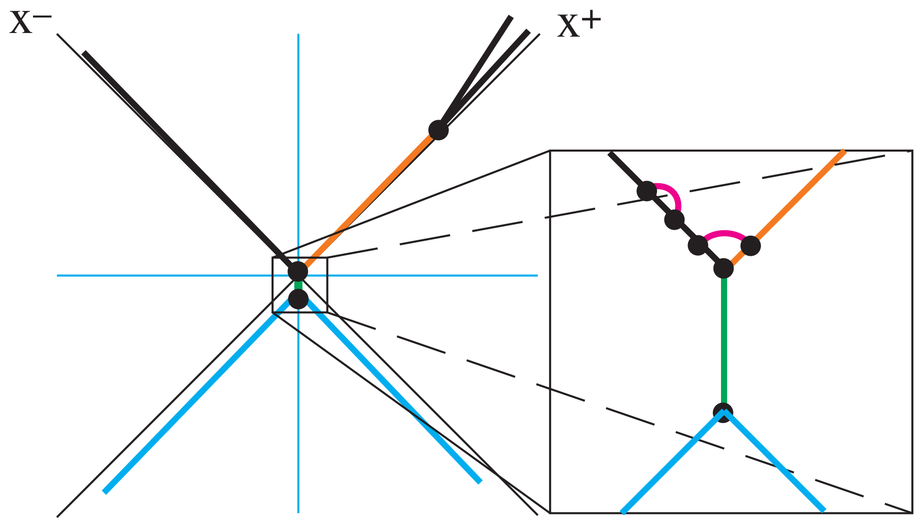
What QCD looks like depends on the time scale at which you look.

A) What renormalization does.
B) The running coupling.
C) The choice of renormalization scale.
D) The scale dependent parton distributions.
E) The choice of factorization scale.

What renormalization does

Use $\overline{\rm MS}$ renormalization with renormalization scale μ:

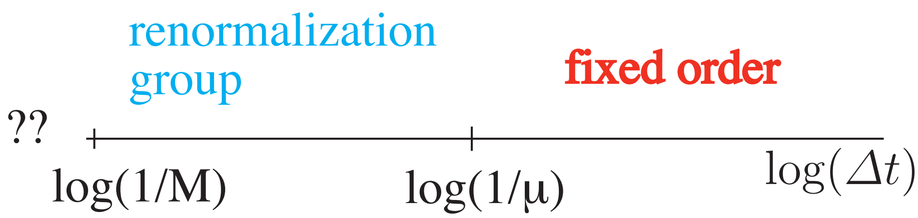
• Physics of time scales $|t| \ll 1/\mu$ removed from perturbative calculation.
• Effect of small time physics accounted for by adjusting value of the coupling*: $\alpha_s = \alpha_s(\mu)$.



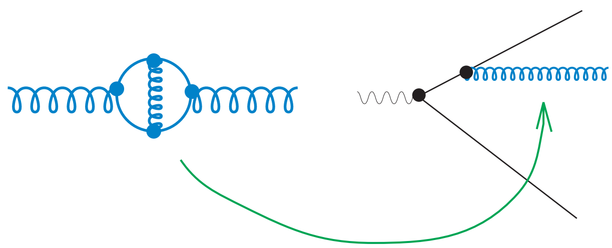
*This is not exactly the truth. There are also running masses $m(\mu)$ and there are μ dependent adjustments to the normalizations of the field operators. In addition, renormalization by dimensional regularization and minimal subtraction is not exactly the same as imposing a cutoff $|\Delta x| > 1/\mu$.

The running coupling

We account for time scales much smaller than $1/\mu$ (but bigger than a cutoff M at the "GUT scale") by using the running coupling.



This sums the effects of short time fluctuations of the fields.



Result of one-loop renormalization group equation:

$$
\begin{aligned}
\alpha_s(\mu) &\sim \alpha_s(M) - (\beta_0/4\pi)\log(\mu^2/M^2)\ \alpha_s^2(M) \\
&\quad + (\beta_0/4\pi)^2\log^2(\mu^2/M^2)\ \alpha_s^3(M) + \cdots \\
&= \frac{\alpha_s(M)}{1 + (\beta_0/4\pi)\alpha_s(M)\log(\mu^2/M^2)} \\
&= \frac{4\pi}{\beta_0 \log(\mu^2/\Lambda^2)}.
\end{aligned}
$$

- $\alpha_s(\mu)$ decreases as μ increases.

But what should the scale μ be?

Summation of perturbative effects

There are many places where calculation of a single Feynman graph won't do.

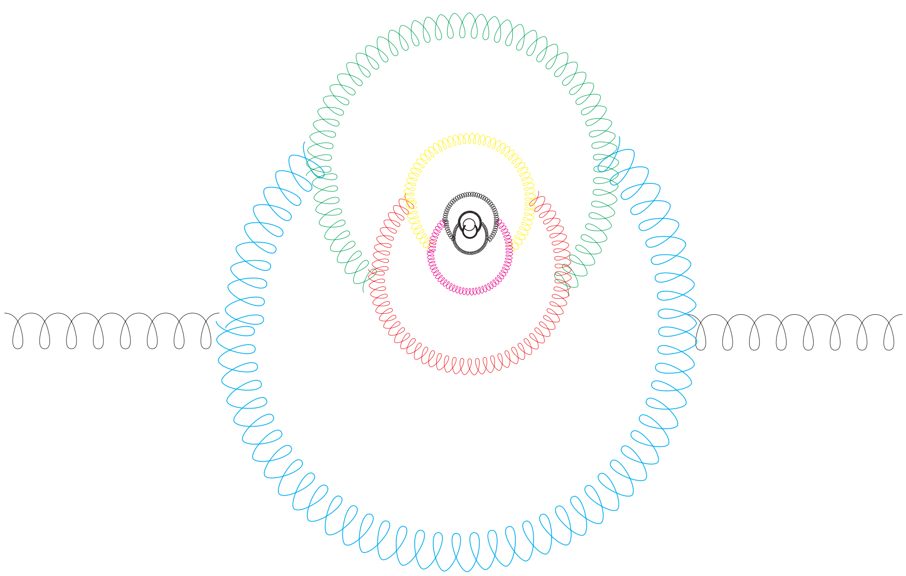
One needs to sum the most important part of each of an infinite number of graphs.

The result

$$\begin{aligned}\alpha_s(\mu) &\sim \alpha_s(M) - (\beta_0/4\pi)\log(\mu^2/M^2)\ \alpha_s^2(M) \\ &\quad + (\beta_0/4\pi)^2 \log^2(\mu^2/M^2)\ \alpha_s^3(M) + \cdots \\ &= \frac{\alpha_s(M)}{1 + (\beta_0/4\pi)\alpha_s(M)\log(\mu^2/M^2)}\end{aligned}$$

is our first example of such a summation.

The physical effect that we account for here is fluctuations within fluctuations within . . . .



- See lectures of C. Davies on lattice QCD.

The choice of scale

Example: Cross section for $e^+e^- \to$ hadrons via virtual photon:

$$\sigma_{\rm tot} = \frac{12\pi\alpha^2}{s}\left(\sum_f Q_f^2\right)[1+\Delta]$$

$$\begin{aligned}\Delta(\mu) = {} & \frac{\alpha_s(\mu)}{\pi} + \left[1.4092 + 1.9167\ \log\left(\mu^2/s\right)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \\ & + \left[-12.805 + 7.8179\ \log\left(\mu^2/s\right) + 3.674\ \log^2\left(\mu^2/s\right)\right] \\ & \quad \times \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \\ & + \cdots\end{aligned}$$

Clearly, $\log\left(\mu^2/s\right)$ should not be big.

- α_s depends on μ.
- Coefficients depend on μ.
- Physical cross section does not depend on μ.
- The harder we work, the less the calculated cross section depends on μ:

$$\frac{d}{d\log\mu}\sum_{n=1}^{N} c_n(\mu)\ \alpha_s(\mu)^n \sim \mathcal{O}(\alpha_s(\mu)^{N+1})$$

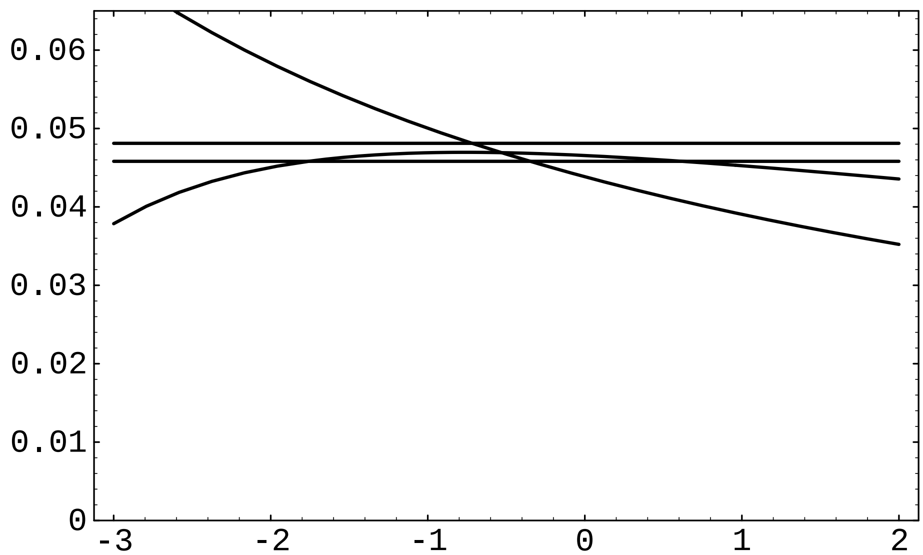
Choosing μ

Recall $\Delta(\mu)$ defined by

$$\sigma_{\mathrm{tot}} = \left(12\pi\alpha^2/s\right)\left(\sum Q_f^2\right)[1+\Delta].$$

Take $\alpha_s(M_Z) \approx 0.117$, $\sqrt{s} = 34$ GeV, 5 flavors. I plot $\Delta(\mu)$ versus p defined by

$$\mu = 2^p\sqrt{s}.$$



First curve: $\Delta_1(\mu) = \alpha_s(\mu)/\pi$.
Second curve (note improvement):

$\Delta_2(\mu)$ including α_s^2 term.

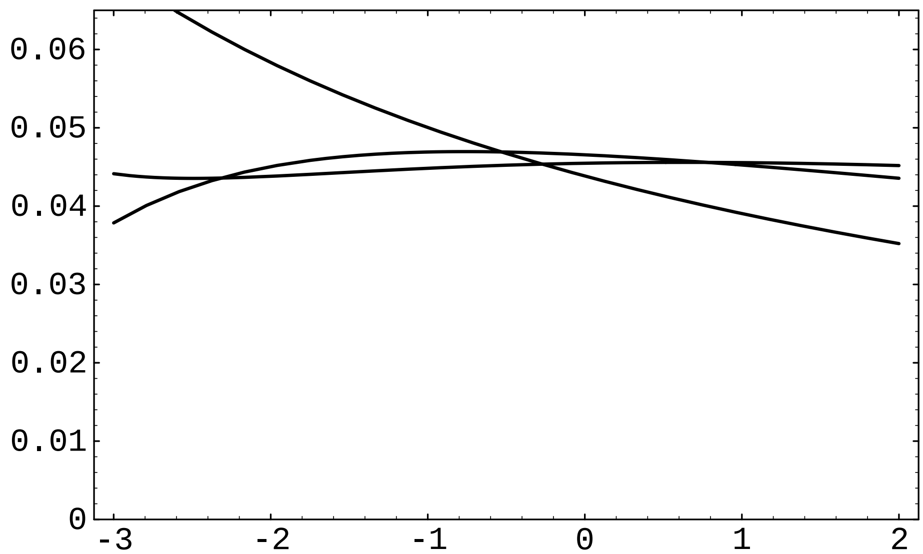
- Possible choice:

$$\Delta_{PMS} = \Delta(\hat{\mu})\,, \qquad \left[\frac{d\Delta(\mu)}{d\log\mu}\right]_{\mu=\hat{\mu}} = 0.$$

Error band: estimated using $\mu = 2\hat{\mu}$ or $\mu = (1/2)\hat{\mu}$.

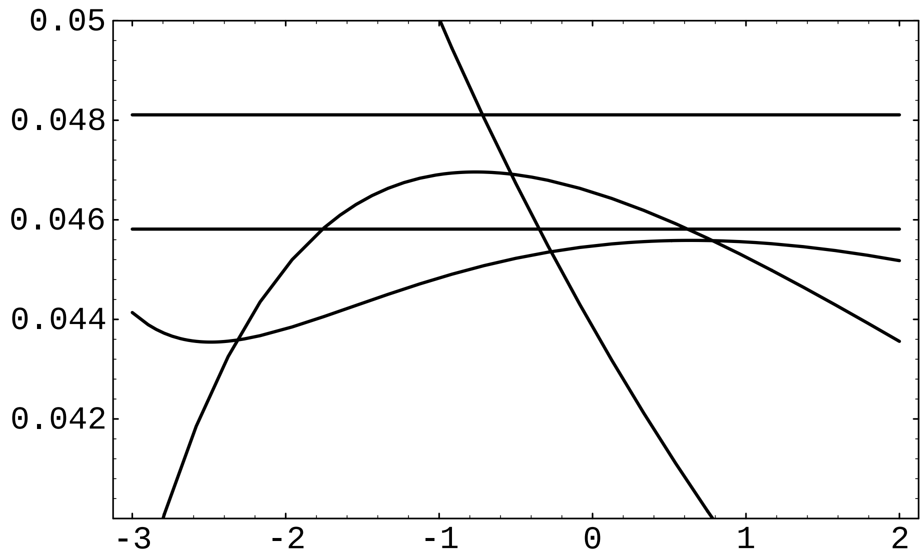
One more order

I plot again $\Delta(\mu)$ versus p ($\mu = 2^p \sqrt{s}$).



Three curves: $\Delta_1(\mu)$, $\Delta_2(\mu)$, $\Delta_3(\mu)$.

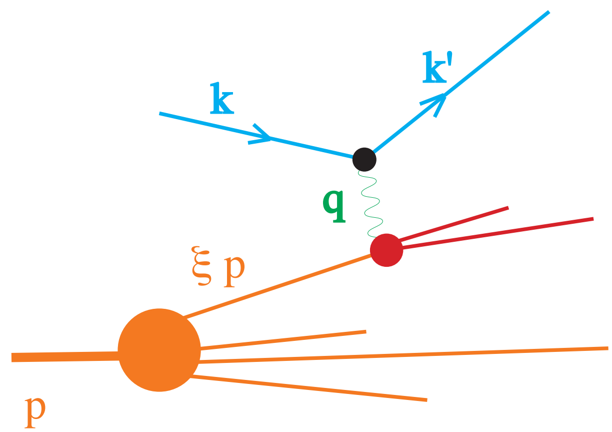
Magnified view (including our $\Delta_2(\mu)$ error band):



Was the error estimate valid?

Scales in the factored cross section

Recall the expression for the factored cross section in DIS:



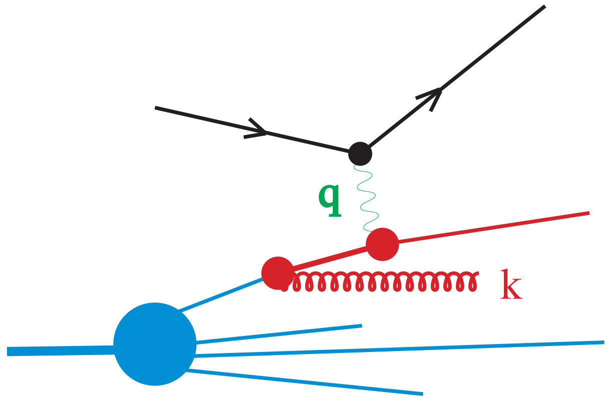
$$\frac{d\sigma}{dE'\,d\omega'} \sim \int_0^1 d\xi \sum_a \; f_{a/h}(\xi,\mu)\; \frac{d\hat{\sigma}_a(\mu)}{dE'\,d\omega'} + \mathcal{O}(m/Q).$$

$f_{a/h}(\xi,\mu)\; d\xi =$ probability to find a parton
with flavor $a = g, u, \bar{u}, d, \ldots,$
in hadron h,
carrying momentum fraction $\xi = p_i^+/p^+$.

$d\hat{\sigma}_a/dE'\,d\omega' =$ cross section for scattering that parton.

What about the μ dependence?

μ_F dependence

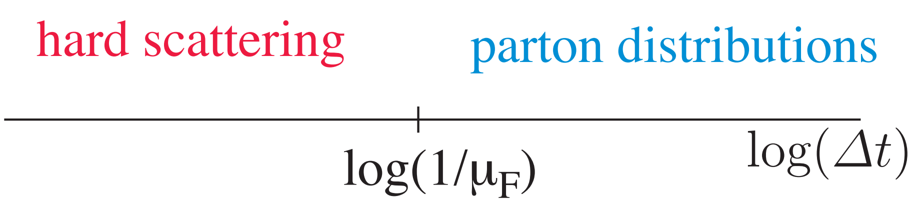


Δx^+'s cover a range from Q/m^2 to $1/Q$ to $\ll 1/Q$.

A gluon emission with $\mathbf{k}^2 \sim m^2$ is part of $f(\xi)$.

A gluon emission with $\mathbf{k}^2 \sim Q^2$ is part of $d\hat{\sigma}$.

When calculating $\hat{\sigma}$, we (roughly speaking) count it as part of $\phi(\xi)$ for $\mathbf{k}^2 < \mu_F^2$ and as part of $d\hat{\sigma}$ for $\mu_F^2 < \mathbf{k}^2$.



$\Rightarrow d\hat{\sigma}_a(\mu_F)/dE'\,d\omega'$ and $f_{a/h}(\xi, \mu_F)$ depend on μ_F.

- μ_F in $f_{f/h}(\xi, \mu_F)$ = "factorization scale."

- μ in $\alpha_s(\mu)$ = "renormalization scale."

As with μ, the higher order calculation you use, the less dependence on μ_F there is.

- Often one sets $\mu_F = \mu$.

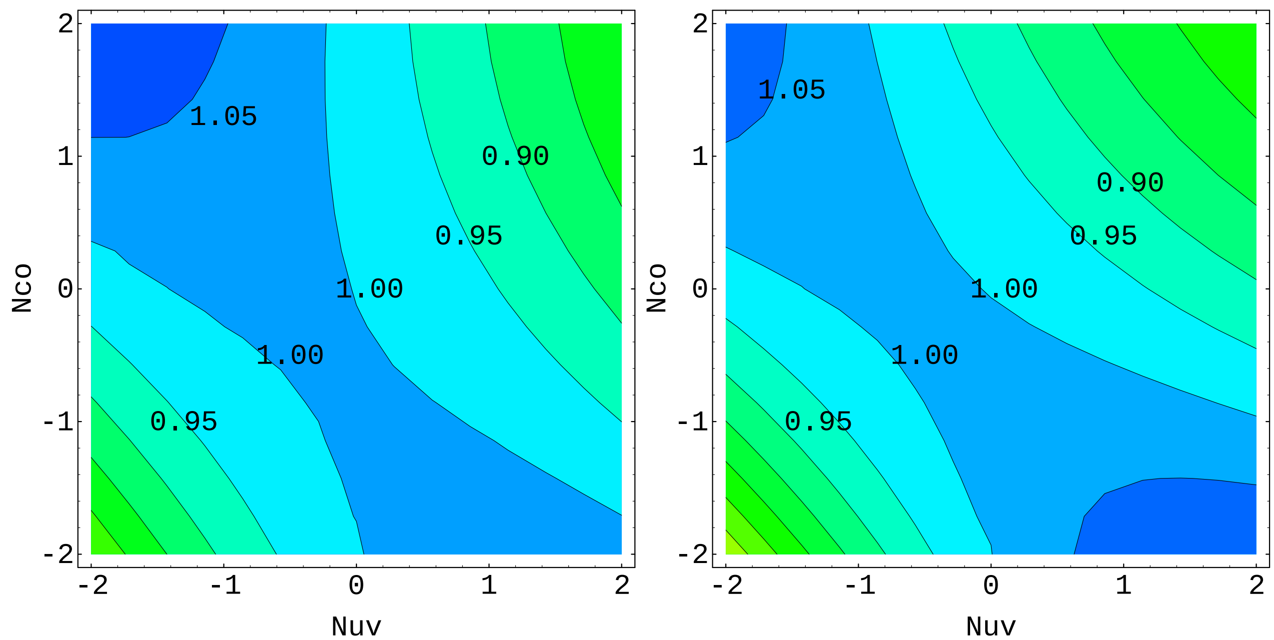
Contour graphs of scale dependence

As an example, look at the one jet inclusive cross section in proton antiproton collisions at $\sqrt{s} = 1800$ GeV.

How does it depend on μ in $\alpha_s(\mu)$ and μ_F in $f_{a/p}(x, \mu_F)$?

$\mu = (E_T/2) \times 2^{N_{\rm UV}}$, $\mu_F = (E_T/2) \times 2^{N_{\rm CO}}$

$d\sigma/dE_T\, d\eta$ at rapidity $\eta = 0$ with arbitrary normalization, 5% contour lines.



$E_T = 100$ GeV $\qquad$ $E_T = 500$ GeV

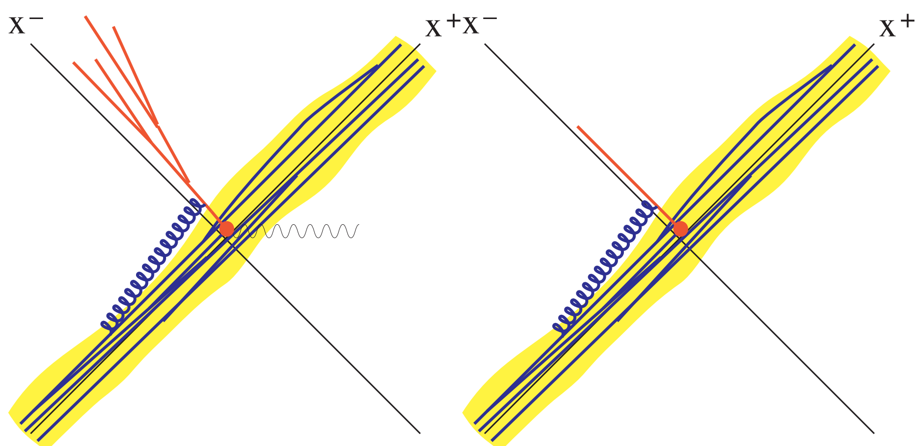
• Variation with scale is roughly $\pm 10\%$ both for medium and large E_T.

$\overline{\mathrm{MS}}$ definition of parton distribution functions

Quarks:

$$f_{i/h}(\xi,\mu) = \frac{1}{2}\int \frac{dy^-}{2\pi}\; e^{i\xi p^+ y^-} \langle p|\bar\psi_i(0,y^-,\mathbf{0})\gamma^+ F\psi_i(0)|p\rangle.$$

$$F = \mathcal{P}\exp\left(-ig\int_0^{y^-} dz^-\, A_a^+(0,z^-,\mathbf{0})\, t_a\right).$$



DIS | Parton distribution

This is renormalized ($\overline{\mathrm{MS}}$) with scale μ:

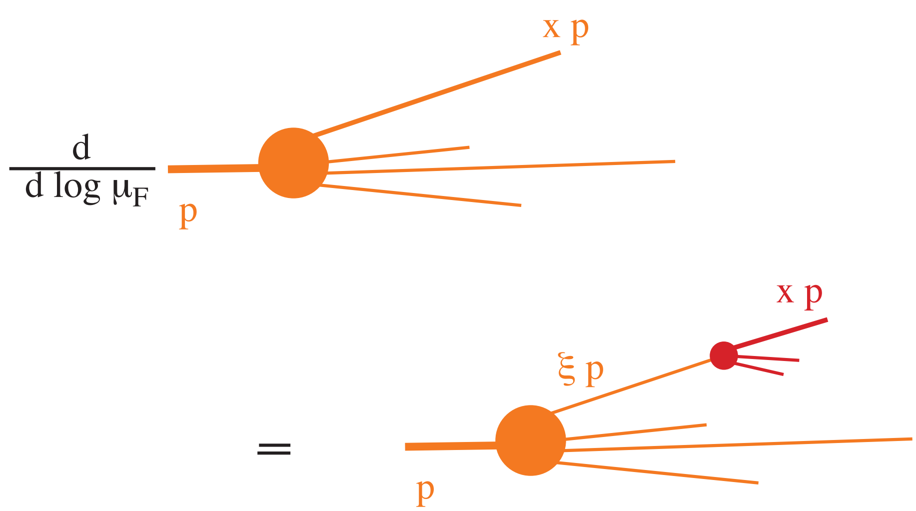
$$\mathbf{k}^2 < \mu^2 \text{ included in } \phi.$$

Gluons: similar definition using gluon field.

Evolution of the parton distributions

There is a renormalization group equation that gives the μ_F dependence:

$$
\frac{d}{d\log\mu_F} f_{a/h}(x,\mu_F) =
\sum_b \int_x^1 \frac{d\xi}{\xi}\, P_{ab}(x/\xi,\alpha_s(\mu_F))\; f_{b/h}(\xi,\mu_F).
$$



$$
\begin{aligned}
P_{ab}(x/\xi,\alpha_s(\mu_F)) &= P^{(1)}_{ab}(x/\xi)\;\frac{\alpha_s(\mu_F)}{\pi} \\
&\quad + P^{(2)}_{ab}(x/\xi)\;\left(\frac{\alpha_s(\mu_F)}{\pi}\right)^2 \\
&\quad + \cdots .
\end{aligned}
$$

• This gets modified for the distribution of partons in a photon. See lecture of J. Owens.

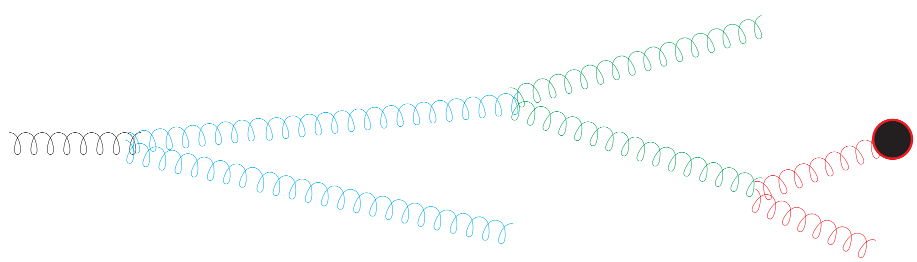
Summation of perturbative effects, again

One often needs to sum the most important part of each of an infinite number of graphs. The differential equation

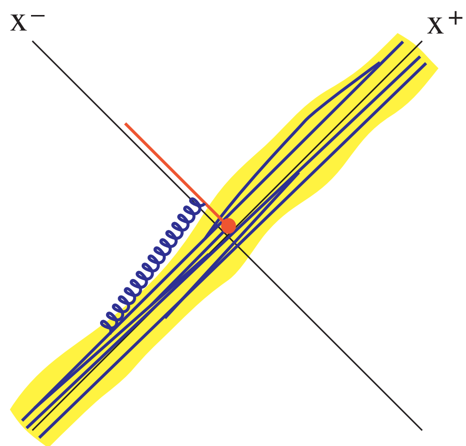
$$\frac{d}{d\log\mu_F} f_{a/h}(x,\mu_F) = \sum_b \int_x^1 \frac{d\xi}{\xi}\, P_{ab}(x/\xi,\alpha_s(\mu_F))\; f_{b/h}(\xi,\mu_F).$$

accomplishes such a summation.

The physical effect that we account for here is fluctuations within fluctuations within ....



Comments on the parton distributions



• The $\overline{\rm MS}$ definition in terms of operators is process independent.

• Sum rules are automatic. Eg.

$$\sum_a \int_0^1 d\xi \; \xi \; f_{a/h}(\xi, \mu) = 1.$$

• We don't calculate f, but the definition adopted determines how $d\hat{\sigma}$ is calculated.

• The parton distributions appear in the QCD formula for any process with one or two hadrons in the initial state.

• Comparison of theory with experiment allows one to fit the parton distributions.

• The evolution is predicted, so one has only to fit the parton distributions at a starting scale μ_0.

• There are lots of experiments, so this program won't work unless QCD is right.

• See lecture of W. Giele

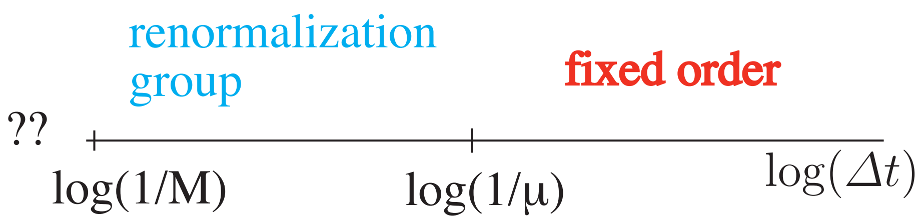
Physics at the shortest time scales

How quantum field theory hides the truth.

A) Unknown short-time physics affects the values of the parameters of the Standard Model.
B) Unknown short-time physics leads to new terms in the effective lagrangian.

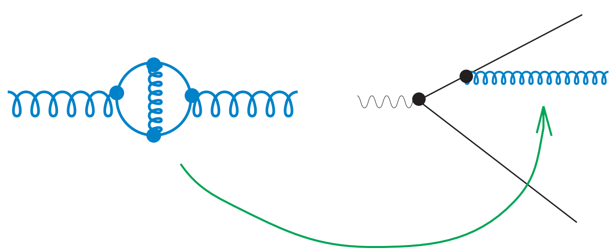
Beyond the Standard Model

At some scale M, we run into the unknown!

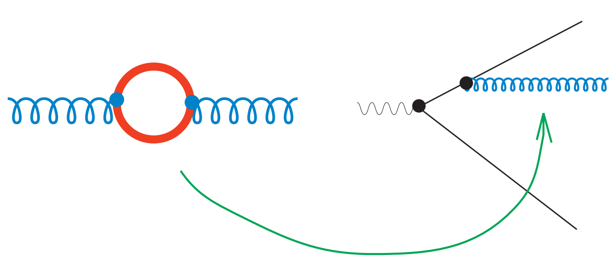


How can we see the unknown in current experiments?

• Recall that known physics affects the effective value of the strong coupling:



• Unknown physics also affects the effective value of the strong coupling:

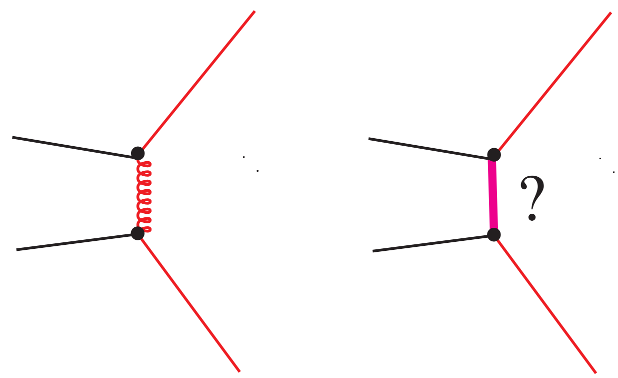


• The unknown physics affects α_s, α_{em}, $\sin^2(\theta_W)$.
• The unknown physics affects masses of $u, d, \ldots, e, \mu, \ldots$.

Thus, the unknown physics affects the parameters of the Standard Model.

New physics and the effective lagrangian

The unknown physics can also introduce new terms into the lagrangian that we investigate experimentally.



$$\Delta\mathcal{L} = \frac{\tilde{g}^2}{M^2}\ \bar{\psi}\gamma^\mu\psi\ \bar{\psi}\gamma_\mu\psi.$$

The new terms are hard to see because they have $1/M^2$.

Looking for new terms in the effective lagrangian

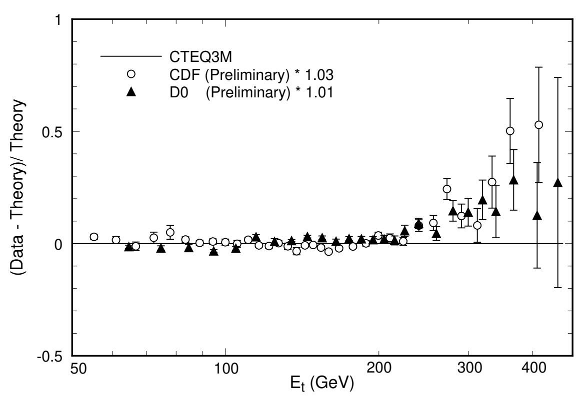
Look for terms like

$$\Delta\mathcal{L} = \frac{\tilde{g}^2}{M^2}\ \bar\psi\gamma^\mu\psi\ \bar\psi\gamma_\mu\psi.$$

These terms are small. Therefore need one of
- high precision;
- an effect forbidden in the Standard Model;
- moderate precision and high energies.

Example: Look at $p + \bar{p} \to jet + X$ as a function of the transverse energy ($\sim P_T$) of the jet. Look for

$$\frac{\text{Data} - \text{Theory}}{\text{Theory}} \propto \tilde{g}^2\,\frac{E_T^2}{M^2}.$$



- This is the kind of signal one looks for.
- But possibly the excess is due to our poor knowledge of the gluon distribution in the proton.
- See lectures of A. Doyle for how this works in *ep*.

QCD in hadron-hadron collisions

Initial state, hard scattering, final state

A) Kinematics: rapidity.
B) Production of γ^*, W, Z.
C) Heavy quark production.
D) Jet production and jet definitions.

Kinematics: rapidity

Rapidity y (or η) is useful for hadron-hadron collisions.
Choose c.m. frame with z-axis along the beam direction.

Massive particle (e.g. Z-boson production):

- Momentum $q^\mu = (q^+, q^-, \mathbf{q})$.

$$y = \frac{1}{2}\log\left(\frac{q^+}{q^-}\right).$$

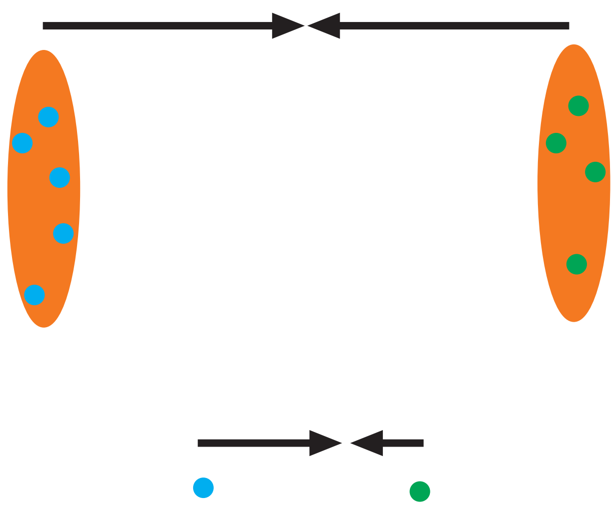
$$q^\mu = (e^y\sqrt{(\mathbf{q}^2 + M^2)/2},\ e^{-y}\sqrt{(\mathbf{q}^2 + M^2)/2},\ \mathbf{q}).$$

- Transformation property under a boost along z-axis:

$$q^+ \to e^\omega q^+, \quad q^- \to e^{-\omega} q^-, \quad \mathbf{q} \to \mathbf{q}.$$

$$y \to y + \omega.$$

- Good because the c.m. frame isn't so special.

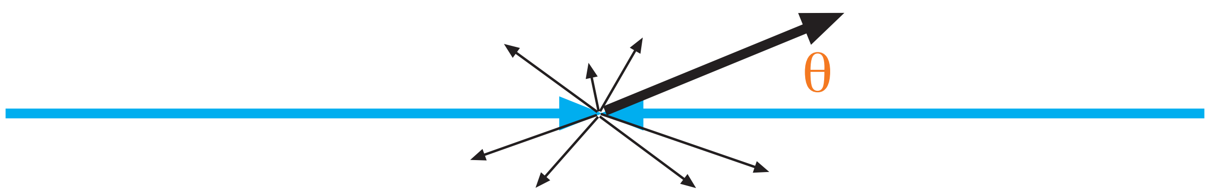


Pseudorapidity

Recall the definition of rapidity:

$$y = \frac{1}{2} \log\left(\frac{q^+}{q^-}\right) .$$

For a massless particle this is



$$y = -\log\left(\tan(\Theta/2)\right)$$

If the particle isn't quite massless, $-\log\left(\tan(\Theta/2)\right)$ is the "pseudorapidity."

γ^*, W, Z production in hadron-hadron collisions

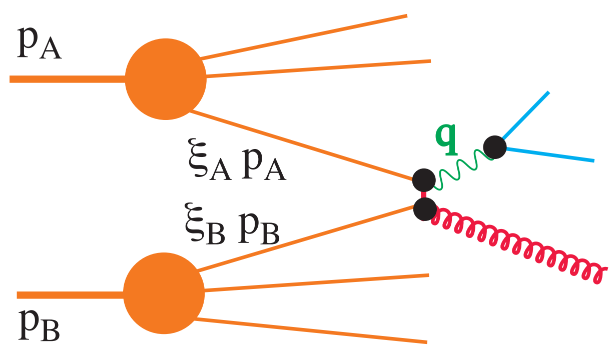
- See lectures of R. Brock.

Consider the process ("Drell-Yan")

$$A + B \to Z + X.$$

Let

$$x_A = e^{y}\sqrt{M^2/s}, \qquad x_B = e^{-y}\sqrt{M^2/s}.$$



Factored form of cross section:

$$\frac{d\sigma}{dy} \approx \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B\ f_{a/A}(\xi_A,\mu)\ f_{b/B}(\xi_B,\mu)\ \frac{d\hat{\sigma}_{ab}(\mu)}{dy}.$$

This has corrections of order m/M.

When $d\hat{\sigma}_{ab}/dy$ is calculated to order α_s^N then there are corrections of order α_s^{N+1}.

We integrate over $\mathbf{q}$; Z's are mostly at $\mathbf{q} \ll M$.

Historical importance of vector boson production

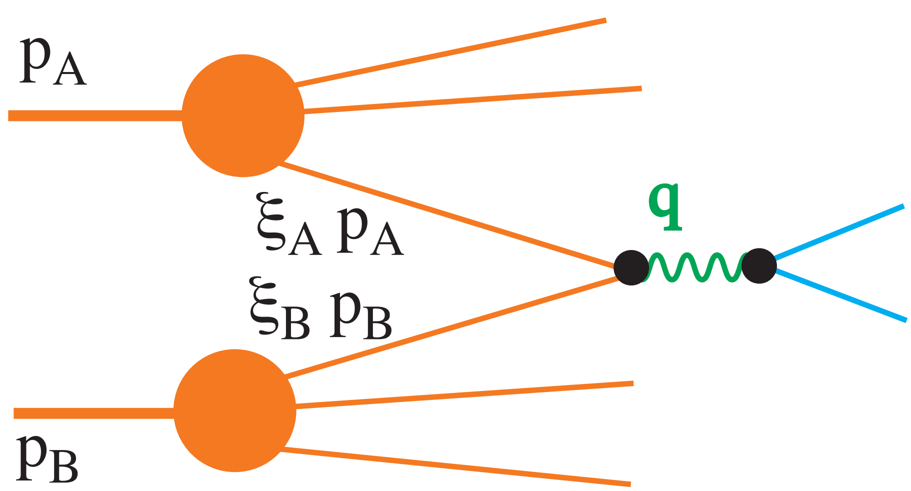
For $A + B \to \mu^+ + \mu^- + X$ one has the formula.

$$\frac{d\sigma}{dQ^2 dy} \approx$$

$$\sum_{a,b} \int_{x_A}^1 d\xi_A \int_{x_B}^1 d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}_{ab}(\mu)}{dQ^2 dy}$$

where Q^2 is the squared mass of the muon pair.

- Before QCD, one had partons and QED. Partons and QED did a good job explaining deeply inelastic scattering.
- But there were other ways to explain DIS.
- Drell and Yan proposed to explain the Lederman *et al.* experiment using the lowest order version of this formula.



- It worked!

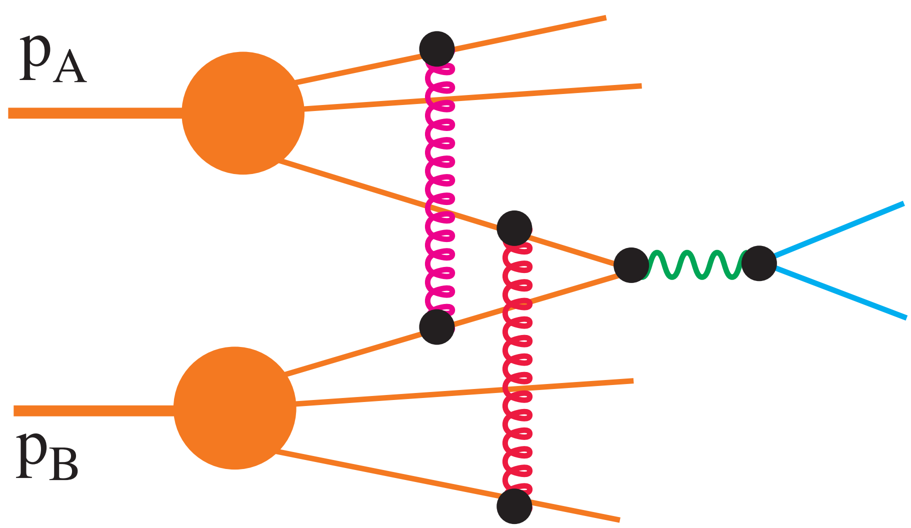
Factorization is not so obvious

For $A + B \to \mu^+ + \mu^- + X$ one has the formula

$$\frac{d\sigma}{dQ^2 dy} \approx$$

$$\sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B \; f_{a/A}(\xi_A, \mu) \; f_{b/B}(\xi_B, \mu) \; \frac{d\hat{\sigma}_{ab}(\mu)}{dQ^2 dy}$$

up to m^2/Q^2 corrections.



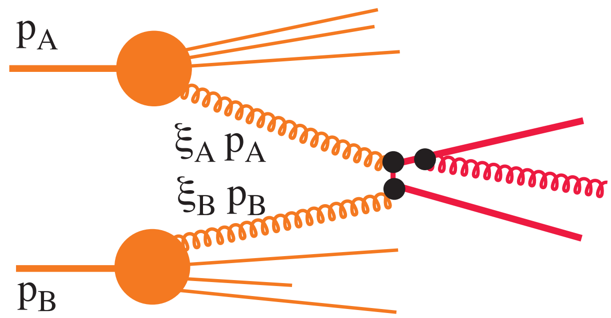
This result is not so obvious, and in fact does not hold graph by graph.

- Need unitarity.
- Need causality.
- Need gauge invariance.

Heavy quark production

- See lectures of R. K. Ellis.

$$A + B \to Q + \bar{Q} + X.$$



Here the big momentum scale (like Q in DIS) is M_Q.

$$\sigma_T \approx \sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B\ f_{a/A}(\xi_A, \mu)\ f_{b/B}(\xi_B, \mu)\ \hat{\sigma}_T{}^{ab}(\mu).$$

Is this really perturbative?

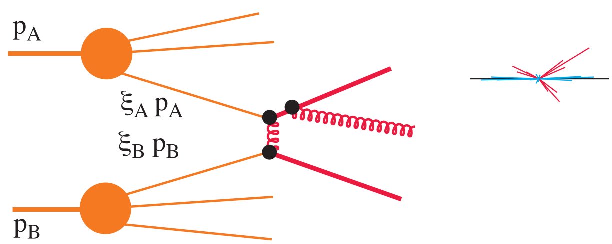
The crucial point is that even if the final state heavy quarks are on-shell, the "exchanged" heavy quark has virtuality at least as big as M_Q^2.

Jet production

- See lectures of N. Glover and N. Varelas

One can also measure cross sections to make jets,

$$A + B \to jet + X.$$



The idea is that the partons in the final state turn into collimated sprays of physical particles ("jets"). The cross section has the factored form

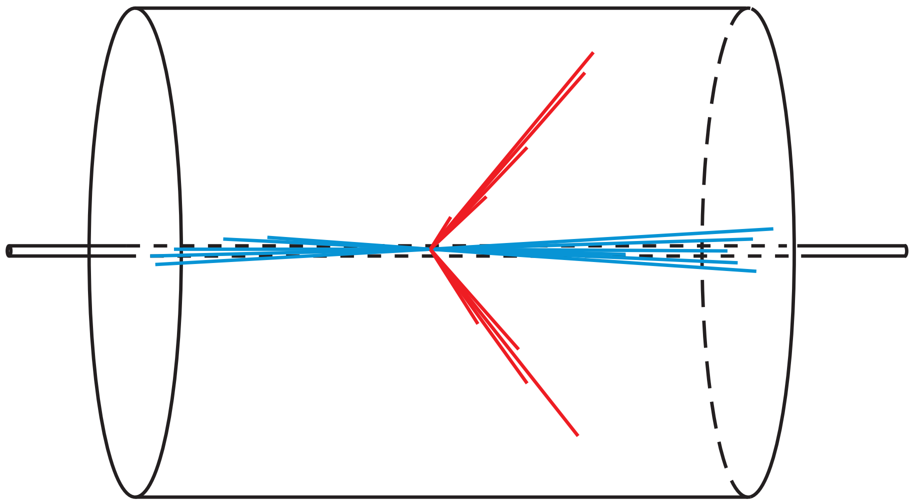
$$\frac{d\sigma}{dE_T\, d\eta} \approx$$
$$\sum_{a,b} \int_{x_A}^{1} d\xi_A \int_{x_B}^{1} d\xi_B\; f_{a/A}(\xi_A, \mu)\; f_{b/B}(\xi_B, \mu)\; \frac{d\hat{\sigma}^{ab}(\mu)}{dE_T\, d\eta}.$$

But what do you mean by a jet?

What does one mean by a jet?

Consider

$$\frac{d\,\sigma}{d\,E_T\;d\,\eta}\;,$$

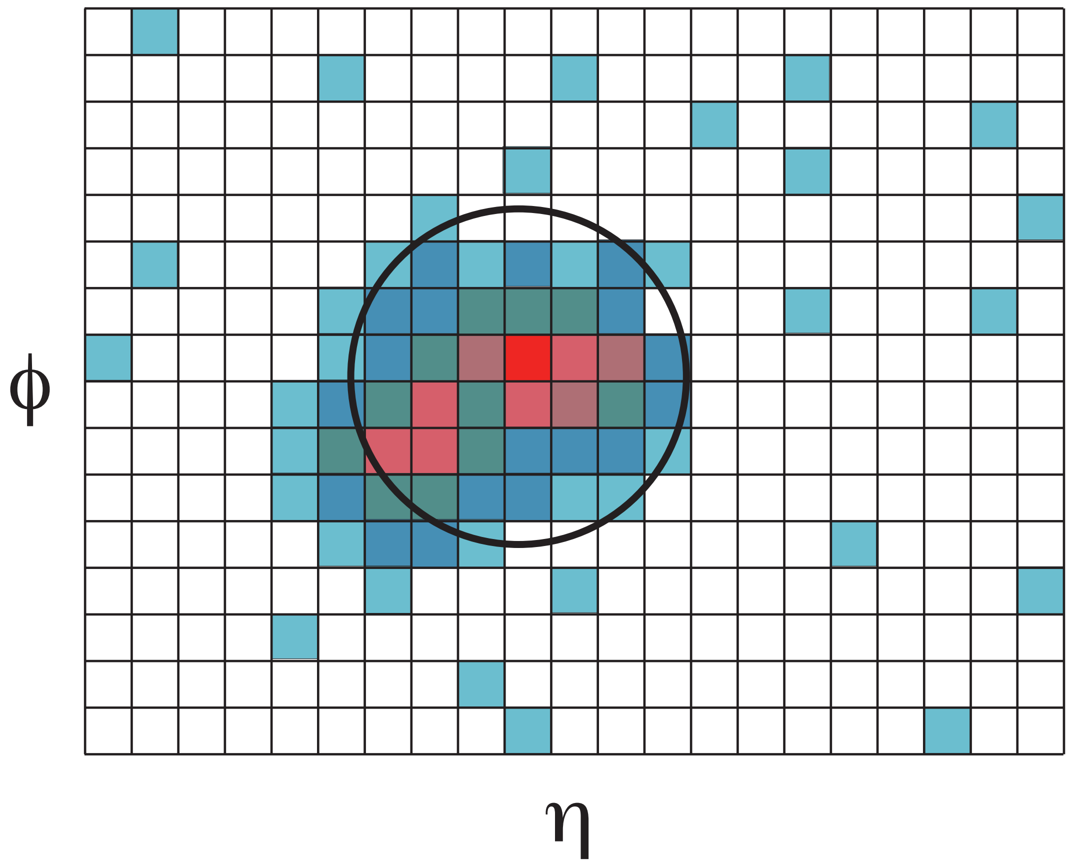


E_T = transverse energy [$\sim$ transverse momentum] of jet.
η = rapidity [$\sim -\log(\tan(\theta/2)$] of jet .

- Substantial E_T at large angles $\Rightarrow$ care with the definition.
- In particular, the definition should be infrared safe.
- There are several possibilities. The one most used in hadron-hadron collisions is based on cones.

“Snowmass Accord” definition

Define jet cone of radius R in η-ϕ space.



$$E_{T,J} = \sum_{i \in \text{cone}} E_{T,i}$$

Jet axis:

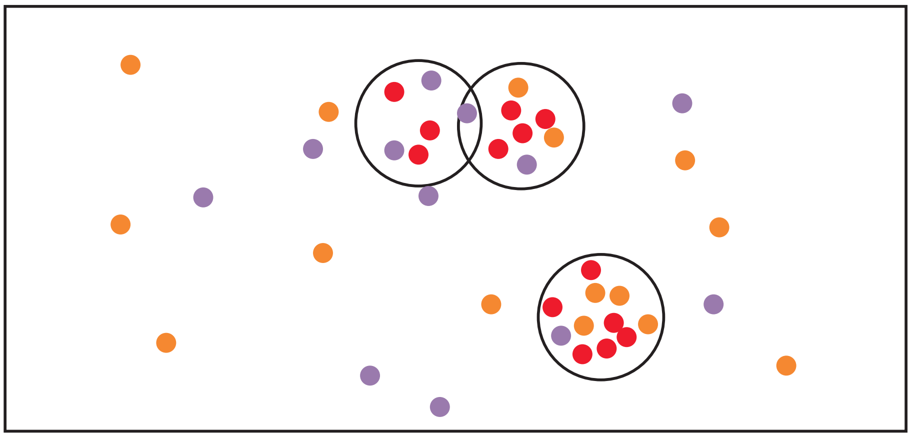
$$\phi_J = \frac{1}{E_{T,J}} \sum_{i \in \text{cone}} E_{T,i}\ \phi_i$$

$$\eta_J = \frac{1}{E_{T,J}} \sum_{i \in \text{cone}} E_{T,i}\ \eta_i$$

- The cone axis must agree with the jet axis.

Beyond Snowmass: the fine print

- Sometimes the cones overlap.
  - $\Rightarrow$If there is lots E_T in the overlap region, merge them.
  - $\Rightarrow$If not much E_T in the overlap region, split them.



- Subtract $\rho\,\pi R^2$ from E_T.

  $\rho =$ average E_T per unit $d\eta d\phi$ in minimum bias events.

  $\rho\,\pi(0.7)^2 \approx 1.1$ GeV

  Do *not* do this in order α_s^3 theoretical calculation.

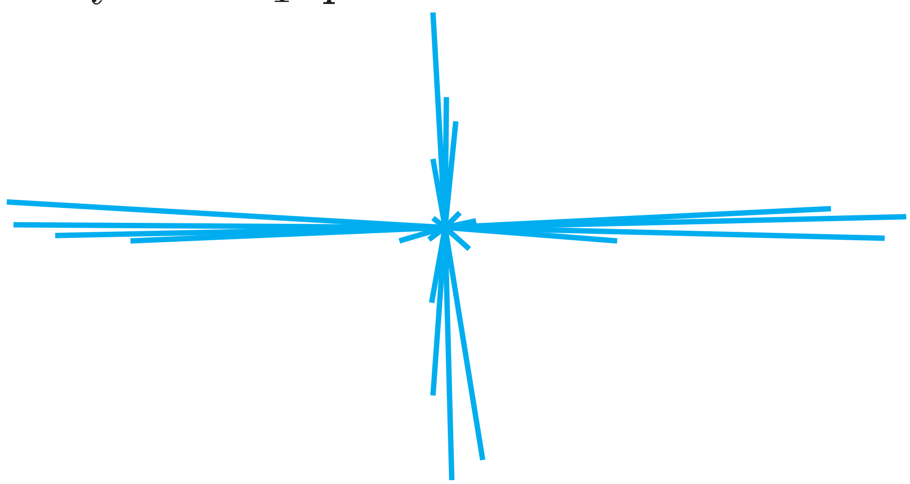
- Include $R_{\rm sep}$ in the theoretical calculation: don't merge two partons if their angular separation is greater than $R_{\rm sep}$.

k_T algorithm

We have seen that in application the Snowmass definition is not as simple as it first appears.

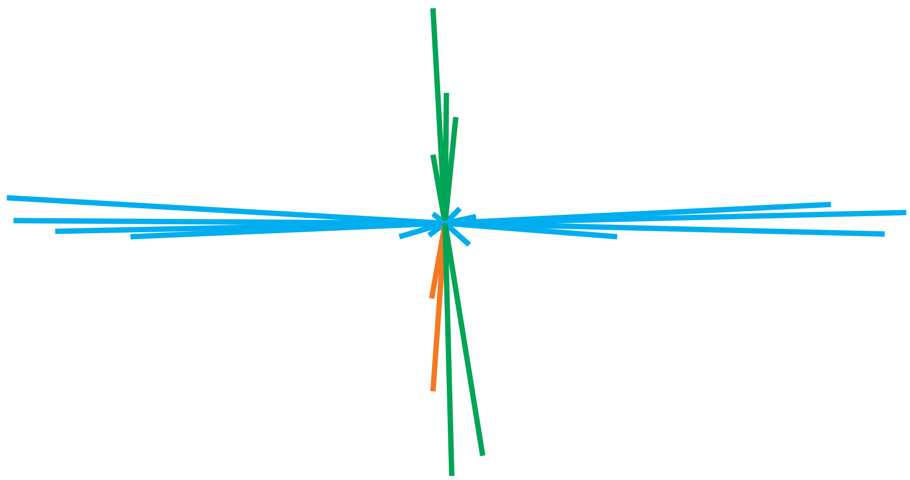
It is possible to use an iterative successive combination algorithm, as in e^+e^- annihilation.

The main idea is to use E_T, η and ϕ as variables, and to take the many low E_T particles into account.



- Choose a merging parameter R.
- Start with a list of "protojets" with momenta $p_1^\mu, \ldots, p_N^\mu$.
- We also start with an empty list of finished jets.

- Result is a list of momenta p_k of jets, ordered in E_T.
- Many will have small E_T and are really minijets, or just part of low E_T debris.
- For an exclusive n jet cross section, use an $E_{T,\min}$



1. For each pair of protojets define

$$d_{ij} = \min(E_{T,i}^2, E_{T,j}^2)\,[(\eta_i - \eta_j)^2 + (\phi_i - \phi_j)^2]/R^2$$

For each protojet define

$$d_i = E_{T,i}^2$$

2. Find the smallest of all the d_{ij} and the d_i. Call it $d_{\min}$
3. If $d_{\min}$ is a d_{ij}, merge protojets i and j into a new protojet k with

$$\begin{aligned} E_{T,k} &= E_{T,i} + E_{T,j} \\ \eta_k &= [E_{T,i}\,\eta_i + E_{T,j}\,\eta_j]/E_{T,k} \\ \phi_k &= [E_{T,i}\,\phi_i + E_{T,j}\,\phi_j]/E_{T,k} \end{aligned}$$

4. If If $d_{\min}$ is a d_i, then protojet i is "not mergable." Remove it from the list of protojets and add it to the list of jets.
5. If protojets remain, go to 1.

Summary of today's topics

• Renormalization removes from the theory effects from $\Delta t \ll 1/\mu$.

• The coupling α_s *etc.* depend on how much you removed.

• Choose $\mu \sim p$ to avoid $\ln(\mu^2/p^2)$.

• Another scale, μ_F, divides Δt included in parton distributions and Δt included in hard scattering.

• Parton distributions depend on μ_F.

• Choose $\mu \sim p$ to avoid $\ln(\mu_F^2/p^2)$.

• Unknown physics at very short times $\Delta t = 1/M$ is hidden in the values of the parameters of the Standard Model and in terms in the lagrangian suppressed by $1/M$.

• Hard processes in hadron-hadron collisions factor into

$$parton\ \ dist. \times parton\ \ dist. \times hard\ \ scattering$$

• Examples: γ^*, W, Z; heavy quarks; jets.