# A NLO Calculation of pQCD
# $P\overline{P} \to W^+ + X$

**C.–P. Yuan**

Michigan State University

CTEQ Summer School, June 2001

## Outline

1. **Parton Model**

$\Rightarrow$ **Born Cross Section**

2. **Factorization Theorem**

$\Rightarrow$ **How to organize a NLO calculation of pQCD**

3. **Feynman rules and Feynman diagrams**

$\Rightarrow$ **"Cut diagram" notation**

4. **Immediate Problems (Singularities)**

$\Rightarrow$ **Dimensional Regularization**

5. **Virtual Corrections**

6. **Real Emission Contribuation**

7. **Perturbative Parton Distribution Functions**

8. **Summary of NLO [$O(\alpha_s)$] Corrections**

**Appendices:**

$A.$ **$\gamma$-matrices in $n$ dimensions**

$B.$ **Some integrals and "special functions"**

$C.$ **Angular integrals in $n$ dimensions**

$D.$ **Two-particle phase space in $n$ dimensions**

$E.$ **Explicit Calculations**

*(Typesetting: prepared by Qing-Hong Cao at MSU.)*

**A few references can be found in**

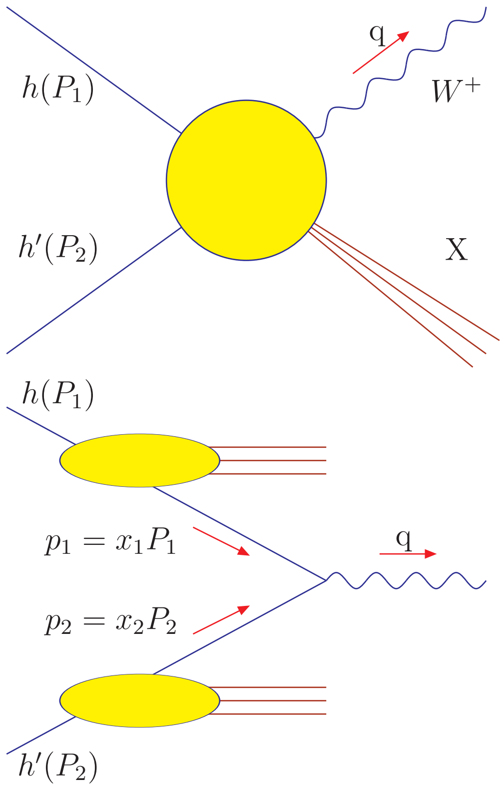
**"Handbook of pQCD"**

**on CTEQ website**

**http://www.phys.psu.edu/$\sim$ cteq/**

# Parton Model

$$\sigma_{hh' \to W^+ X} =$$




$$=$$

$$\sigma_{hh' \to W^+ X} = \sum_{f,f'=q,\bar{q}} \int_0^1 dx_1 dx_2 \left\{ \phi_{f/h}(x_1) \, \hat{\sigma}_{ff'} \phi_{\bar{f}'/h'}(x_2) + (x_1 \leftrightarrow x_2) \right\}$$

Partonic "Born"
Cross Section of $f\overline{f}' \to W^+$

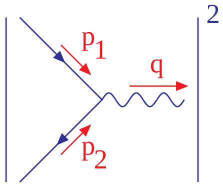
The probablility of finding a "parton" $f$ with fraction $x_1$ of the hadron $h$ momentum

# Born Cross Section

$$\hat{\sigma}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^3q}{(2\pi)^3\, 2q_0} (2\pi)^4 \,\delta^4\left(p_1 + p_2 - q\right) \cdot \overline{|\mathcal{M}|^2}$$

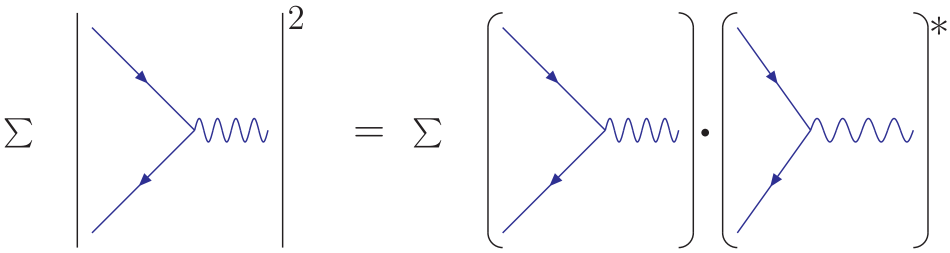
where

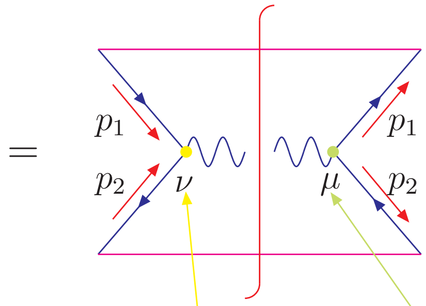
$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{3}\cdot\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)}_{\text{average color and spin}} \sum_{\substack{spin\\color}}$$




$$\left[\ \text{Or,}\quad -i\mathcal{M} = \bar{v}\left(p_2\right) \frac{ig_w}{\sqrt{2}} \gamma_\mu \frac{1}{2}\left(1 - \gamma_5\right) u\left(p_1\right)\ \right]$$

"Cut-diagram" notation




$\frac{ig_w}{\sqrt{2}}\gamma_\nu P_L$ $\qquad$ $-\frac{ig_w}{\sqrt{2}}\gamma_\mu P_L$ $\qquad$ $P_L \equiv \frac{1}{2}(1-\gamma_5)$

$$= \left(\frac{g_W}{\sqrt{2}}\right)^2 Tr\left[\not{p}_1\gamma_\mu P_L\ \not{p}_2\gamma_\nu P_L\right] \cdot \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) \cdot Tr\, I_{3\times 3}$$

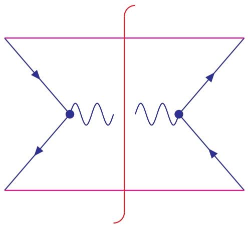
Doesn't contribute for $m_q = 0$,
due to Ward identity

Color

$$
\begin{aligned}
& Tr\left[\not{p}_1\gamma_\mu P_L \not{p}_2\gamma^\mu P_L\right](-1) \\
&= Tr\left[\not{p}_1\gamma_\mu \not{p}_2\gamma^\mu P_L\right](-1) \\
&= (-2)\, Tr\left[\not{p}_1 \not{p}_2 P_L\right](-1) \\
&= (-2)\cdot\frac{1}{2}\cdot 4\,(p_1\cdot p_2)\,(-1) \\
&= +2\hat{s}
\end{aligned}
$$

$$
\begin{aligned}
& P_L P_L = P_L = \frac{1}{2}(1-\gamma_5) \\
& \gamma_\mu \not{p}_2\gamma^\mu = -2\not{p}_2 \\
& Tr\left(\not{p}_1 \not{p}_2\right) = 4\,(p_1\cdot p_2) \\
& Tr\left(\not{p}_1 \not{p}_2\gamma_5\right) = 0
\end{aligned}
$$

$Tr\left[I_{3\times 3}\right] = 3$ $\qquad$ $\left(\hat{s}\equiv (p_1+p_2)^2 = q^2 \text{ and } p_1^2 = p_2^2 = 0\right)$

$\Rightarrow$  $= \left(\frac{g_w}{\sqrt{2}}\right)^2\cdot(+2\hat{s})\,(3) = 3\,g_w^2\,\hat{s}$

$$
\begin{aligned}
\int\frac{d^3q}{2q_0}\delta^4\left(p_1+p_2-q\right) &= \int d^4q\,\delta^4\left(p_1+p_2-q\right)\delta^+\left(q^2-M^2\right) \\
&= \delta\left(q^2-M^2\right)
\end{aligned}
$$

where $M$ is the mass of $W$-boson.

Thus,

$$
\begin{aligned}
\hat{\sigma}_{q\bar{q}'} &= \frac{1}{2\hat{s}}(2\pi)\cdot\delta\left(\hat{s}-M^2\right)\cdot\left(\frac{1}{3}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\cdot g_w^2\hat{s} \\
&= \frac{\pi}{12}g_w^2\delta\left(\hat{s}-M^2\right) \\
&= \frac{\pi}{12\hat{s}}g_w^2\delta\left(1-\hat{\tau}\right)
\end{aligned}
$$

$$
\left(\begin{array}{c} \hat{\tau} = M^2/\hat{s}\,,\ \hat{s} = x_1x_2S \ \text{ for} \\ S = (P_1+P_2)^2 \ \text{ and } P_1^2 = P_2^2 = 0 \end{array}\right)
$$

# Factorization Theorem

$$\sigma_{hh'} = \Sigma_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/h}\left(x, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/h'}\left(x_2, Q^2\right)$$

Nonperturbative,
but universal,
hence, measurable

IRS, Calculable
in pQCD

**Procedure:**

(1) Compute $\sigma_{kl}$ in pQCD with $k, l$ partons

(not $h, h'$ hadron)

$$\sigma_{kl} = \sum_{i,j} \int_0^1 dx_1 dx_2 \, \phi_{i/k}\left(x, Q^2\right) H_{ij}\left(\frac{Q^2}{x_1 x_2 S}\right) \phi_{j/l}\left(x_2, Q^2\right)$$

(2) Compute $\phi_{i/k}, \phi_{j/l}$ in pQCD

(3) Extract $H_{ij}$ in pQCD

$H_{ij}$ IRS $\Rightarrow$ $H_{ij}$ indepent of $k, l$

$\Rightarrow$ same $H_{ij}$ with $(k \to h, l \to h')$

(4) Use $H_{ij}$ in the above equation with $\phi_{i/h}, \phi_{j/h'}$

# Extracting $H_{ij}$ in pQCD

- Expansions in $\alpha_s$:

$$\sigma_{kl} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sigma_{kl}^{(n)} \qquad\qquad \alpha_s = \frac{g^2}{4\pi}$$

$$H_{ij} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n H_{ij}^{(n)}$$

$$\phi_{i/k}(x) = \underbrace{\delta_{ik}\delta(1-x)}_{\phi_{i/k}^{(0)}} + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \phi_{i/k}^{(n)}$$

$\phi_{i/k}^{(0)}$ ($\alpha_s = 0 \Rightarrow$ Parton k " stays itself " )

- Consequences:

$$H_{ij}^{(0)} = \sigma_{ij}^{(0)} = \text{"Born"}$$

suppress "^" from now on

$$H_{ij}^{(1)} = \sigma_{ij}^{(1)} - \left[\ \sigma_{il}^{(0)} \phi_{l/j}^{(1)} + \phi_{k/i}^{(1)} \sigma_{kj}^{(0)}\ \right]$$

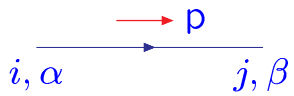
Computed from Feynman diagrams ( process dependent )

Computed from the definition of perturbative parton distribution function ( process independent, scheme dependent)

# Feynman Rules

- Quark Propagator

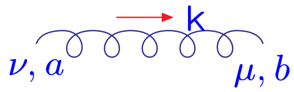
Take m=0 in
our calculation




$\frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}\delta_{ij}$

(i,j=1,2,3)

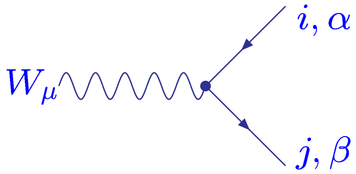
- Gluon Propagator




$\frac{i(-g_{\mu\nu})}{k^2+i\epsilon}\delta_{ab}$

(a,b=1,2...,8)

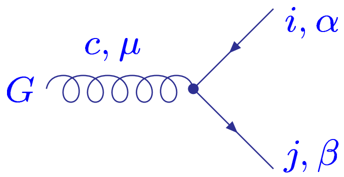
- Quark-W Vertex




$i\frac{g_W}{\sqrt{2}}(\gamma_\mu)_{\beta\alpha}\frac{(1-\gamma_5)}{2}\delta_{ij}$

$g_w = \frac{e}{\sin\theta_w}$, weak coupling

- Quark-Gluon Vertex




$-ig\,(t_c)_{ji}\,(\gamma_\mu)_{\beta\alpha}$

$t_c$ is the $SU(N)_{N\times N}$ generator

- Quark Color Generators

$$[t_a, t_b] = i f_{abc} t_c$$

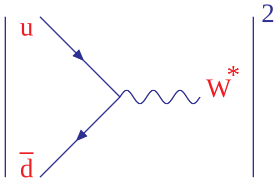
$$\sum_c t_c^2 = C_F\, I_{N\times N}$$

$$C_F = \frac{N^2-1}{2N} = \frac{4}{3}\ , \quad (N = 3)$$

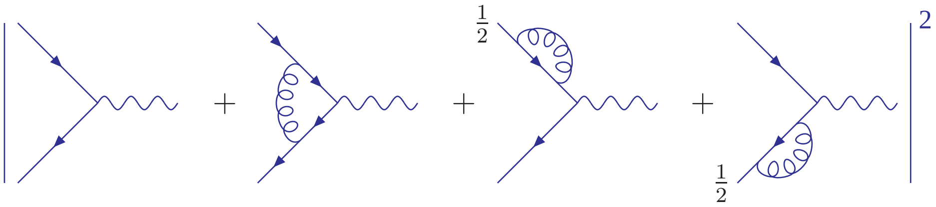
$$Tr(\sum_c t_c^2) = N\, C_F$$

# Feynman Diagrams

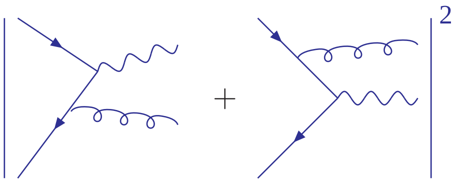
- Born level $\alpha_s^{(0)}$ $(q\overline{q'})_{Born}$



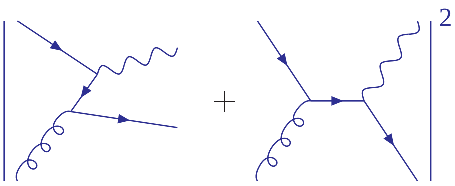

- NLO: $(\alpha_s^{(1)})$ virtual corrections $(q\overline{q'})_{virt}$



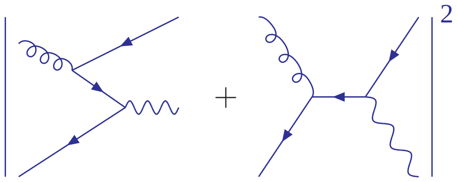

- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(q\overline{q'})_{real}$




- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(qG)_{real}$

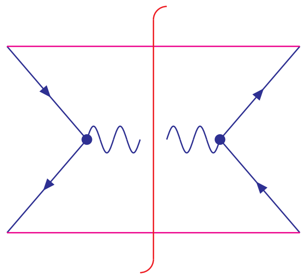



- NLO: $(\alpha_s^{(1)})$ real emission diagrams $(G\overline{q'})_{real}$

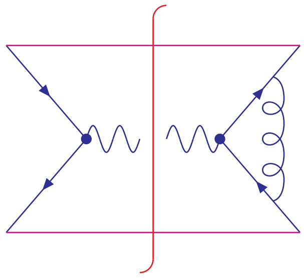
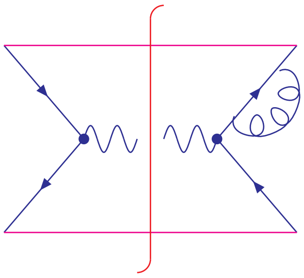
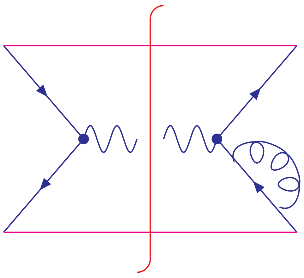


# In "Cut-diagram" notation

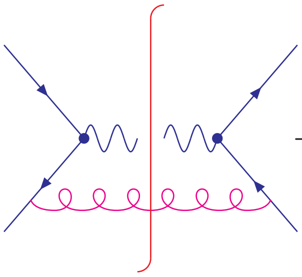
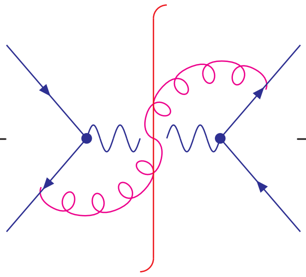
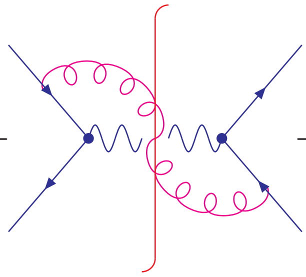
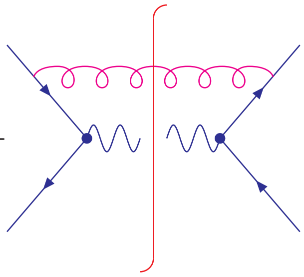
- $(q\overline{q'})_{Born}$



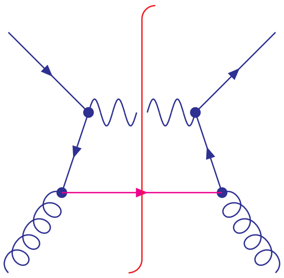
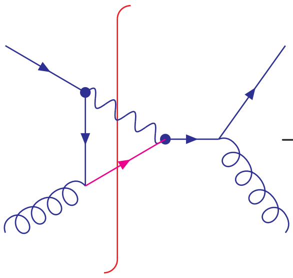
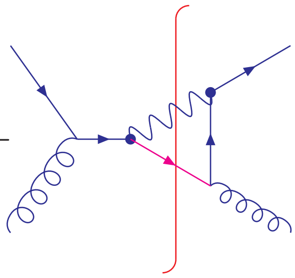
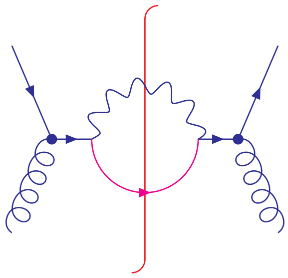
- $(q\overline{q'})_{virt}$

$2 \times Re \Bigg[$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $\Bigg]$

- $(q\overline{q'})_{real}$

 $+$  $+$  $+$ 

- $(qG)_{real}$

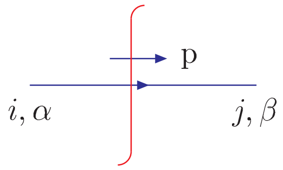
 $+$  $+$  $+$ 

- $(G\overline{q'})_{real}$

Same as $(qG)_{real}$ after replacing $q$ by $\overline{q'}$.

# Feynman rules for cut-diagrams

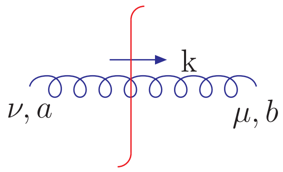
- quark line




$$(2\pi)\delta^+(p^2 - m^2)(\not{p} + m)_{\beta\alpha}\delta_{ij}$$

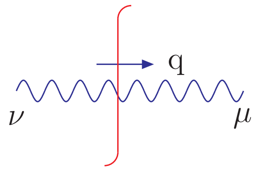
$\delta(p^2 - m^2)\theta(p_0)$

- gluon line




$$(2\pi)\delta^+(k^2)(-g_{\mu\nu})\delta_{ab}$$

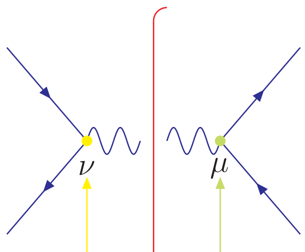
- $W$-boson line




$$(2\pi)\delta^+(q^2 - M^2)(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M^2})$$

Doesn't contribute for $m_f = 0$ because of Ward identity

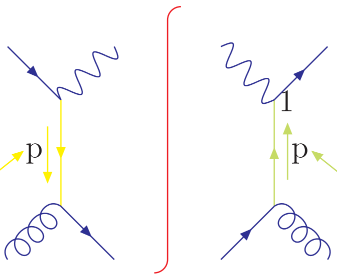
-




$i\frac{g_W}{\sqrt{2}}\gamma_\nu\frac{1}{2}(1-\gamma_5)$

$-i\frac{g_W}{\sqrt{2}}\gamma_\mu\frac{1}{2}(1-\gamma_5)$

-

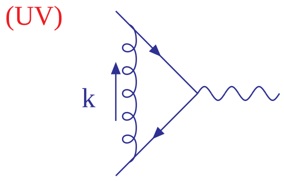



$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$

$\frac{-i(\not{p}+m)}{p^2-m^2-i\epsilon}$

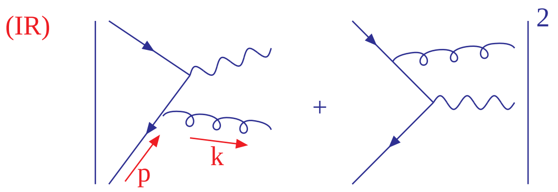
# Immediate problems (Singularities)

- Ultraviolet singularity




$$\sim \quad \int^{\infty} d^4k \frac{\not{k}\ \not{k}}{(k^2)\,(k^2)\,(k^2)} \quad \to \quad \infty$$

- Infrared singularities




$$\to \quad \infty$$

as $k^\mu \to 0$ (soft divergence)

or $k^\mu \parallel p^\mu$ (collinear divergence)

(Similar singularities also exist in virtual diagrams.)

- Solutions

Compute $H_{ij}$ in pQCD in $n = 4 - 2\varepsilon$ dimensions

(dimensional regularization)

(1) $n \neq 4 \Rightarrow$ UV & IR divergences appear as $\frac{1}{\varepsilon}$ poles

in $\sigma_{ij}^{(1)}$ (Feynman diagram calculation)

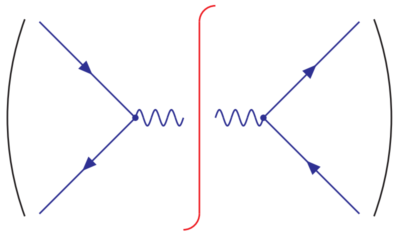
(2) $H_{ij}$ is IR safe $\Rightarrow$ no $\frac{1}{\varepsilon}$ in $H_{ij}$

($H_{ij}$ is UV safe after "renormalization".)

# Dimensional Regularization
## (Revisit the Born Cross Section in $n$ dimensions)

- $$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{1}{2\hat{s}} \int \frac{d^{n-1}q}{(2\pi)^{n-1}\, 2q_0} (2\pi)^n \cdot \delta^n (p_1 + p_2 - q) \cdot \overline{|m|^2}$$

- $$\overline{|m|^2} = \left(\frac{1}{3} \cdot \frac{1}{3}\right) \left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot$$



In n-dim, the polarization degree of freedom is (2) for a quark, and (n-2) for a gluon.

- Using the Naive-$\gamma^5$ prescription:

$$\begin{aligned}
& Tr\,[\not{p}_1 \gamma_\mu P_L \; \not{p}_2 \gamma^\mu P_L]\,(-1) \\
&= Tr\,[\not{p}_1 \gamma_\mu \; \not{p}_2 \gamma^\mu P_L]\,(-1) \\
&= (-2)\,(1-\varepsilon)\,Tr\,[\not{p}_1 \; \not{p}_2 P_L]\,(-1) \\
&= (-2)\,(1-\varepsilon) \cdot \frac{1}{2} \cdot 4\,(p_1 \cdot p_2)\,(-1) \\
&= 2\,(1-\varepsilon)\,\hat{s}
\end{aligned}$$

$$\gamma_\mu \; \not{p}_2 \gamma^\mu = -2\,(1-\varepsilon) \; \not{p}_2$$

- In $n$ dimensions

$$\hat{\sigma}^{(0)}_{q\bar{q}'} = \frac{\pi}{12\hat{s}} g_w^2 \cdot (1-\varepsilon) \cdot \delta\,(1-\hat{\tau})$$

## Strong Coupling $g$ in $n$ dimensions

- In $n$ dimensions

$$\int d^n x \mathcal{L}$$
$$\longrightarrow \int d^n x \left\{ \bar{\psi} i \not{\partial} \psi - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + g t^a \bar{\psi} \gamma^\mu G_\mu \psi + \cdots \right\}$$

The dimension in mass unit ($\mu$)

$$[\psi] \sim \mu^{\frac{n-1}{2}}$$
$$[G] \sim \mu^{\frac{n-2}{2}}$$
$$\left[\bar{\psi} G \psi\right] \sim \mu^{\frac{n-1}{2} \times 2 + \frac{n-2}{2}} = \mu^{\frac{3n}{2} - 2}$$

Since $\left[g \bar{\psi} G \psi\right] \sim \mu^n$ , so

$$[g] \sim \mu^{\frac{-n}{2} + 2} \qquad n = 4 - 2\varepsilon$$
$$= \mu^{\varepsilon}$$

$\Rightarrow$ $g$ has a dimension in mass when $\varepsilon \neq 0$

$\Rightarrow$ Feynman rules should read $g \to g\mu^{\varepsilon}$

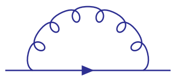
## Calculations

- Tools needed for a NLO calculation are collected in Appendices A-D

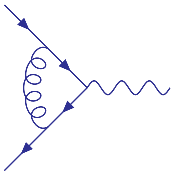
- The detailed calculation for each subprocess can be found in Appendices E

- In the following, I shall summarize the result for each subprocess

# Virtual Corrections $(q\bar{q}')_{virt}$
## (in Feynman Gauge )

-  $=$ 0

  $\frac{1}{\varepsilon_{IR}}$ and $\frac{1}{\varepsilon_{UV}}$ poles cancel when $\varepsilon_{UV} = -\varepsilon_{IR} \equiv \varepsilon$

- 

  $\frac{1}{\varepsilon_{UV}}$ cancel $\Rightarrow$ Electroweak coupling is not renormalized by QCD interactions at one-loop order
  (Ward identity,
  a renormalizable theory)

  $\frac{1}{\varepsilon_{IR}}$ poles remain

- $\sigma_{virt}^{(1)}$ is free of ultraviolent singularity.

$$\sigma_{virt}^{(1)} = \sigma^{(0)} \frac{\alpha_s}{2\pi} \delta\left(1-\hat{\tau}\right) \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon} \frac{\Gamma\left(1-\varepsilon\right)}{\Gamma\left(1-2\varepsilon\right)} \cdot \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} \right\} \cdot \left(C_F\right)$$

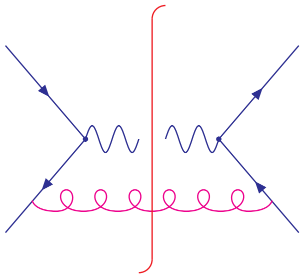
$-\frac{2}{\varepsilon^2}$: soft and collinear singularities

$-\frac{3}{\varepsilon}$: soft or collinear singularities

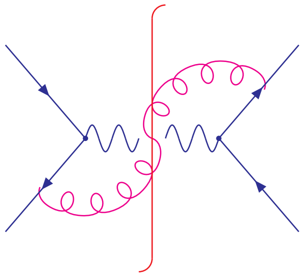
$C_F$: color factor

$\sigma^{(0)} \equiv \frac{\pi}{12\hat{s}} g_w^2$

# Real Emission Contribution $(q\bar{q}')_{real}$

- 

$\sim \frac{1}{\varepsilon}$ Collinear



$\sim \frac{1}{\varepsilon^2}$ Soft and Collinear

- 

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right) = \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta(1-\hat{\tau}) - \frac{2}{\varepsilon}\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau}\right\}$$

Note: $[\cdots]_+$ is a distribution,

$$\int_0^1 dz\, f(z)\left[\frac{1}{1-z}\right]_+ = \int_0^1 dz\frac{f(z)-f(1)}{1-z},$$ which is finite.

- In the soft limit, $\hat{\tau}\to 1$ $\left(\hat{\tau}=\frac{M^2}{\hat{s}}\right)$,

$$\sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right) \longrightarrow \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)}\cdot C_F$$
$$\cdot\left\{\frac{2}{\varepsilon^2}\delta(1-\hat{\tau}) - \frac{4}{\varepsilon}\frac{1}{(1-\hat{\tau})_+} + 8\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right\}$$

# $\left(q\bar{q}'\right)_{virt} + \left(q\bar{q}'\right)_{\text{real}}$ at NLO

- $$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(1)}_{virt}\left(q\bar{q}'\right) + \sigma^{(1)}_{\text{real}}\left(q\bar{q}'\right)$$
$$= \sigma^{(0)}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\cdot C_F$$
$$\cdot\left\{\frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+ - 2\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\ln\hat{\tau} + 4\left(1+\hat{\tau}^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+\right.$$
$$\left. + \left(\frac{2\pi^2}{3} - 8\right)\delta(1-\hat{\tau})\right\}$$

Where we have used
$$\frac{-2}{\varepsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right] = \frac{-2}{\varepsilon}\left(\frac{1+\hat{\tau}^2}{1-\hat{\tau}}\right)_+$$

- All the soft singularities $\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)$ cancel in $\sigma^{(1)}_{q\bar{q}'}$

$\Rightarrow$ $KLN$ theorem

(Kinoshita-Lee-Navenberg)

- $$\sigma^{(1)}_{q\bar{q}'} \sim \underset{\substack{\uparrow \\ \boxed{\text{Collinear Singularity}}}}{\frac{1}{\varepsilon}(\text{term})} + \text{finite (terms)}$$

# Factorization Theorem

- Perturbative PDF

$\phi^{(0)}_{i/k} = \delta_{ik}\delta(1-x)$

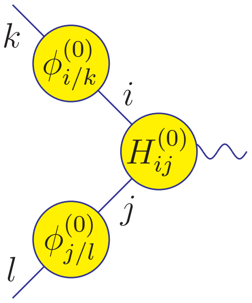
$\frac{\alpha_s}{\pi}\phi^{(1)}_{i/k}$ can be calculated from the definition of PDF.

(Process independent,but factorization scheme dependent)

-

(1)

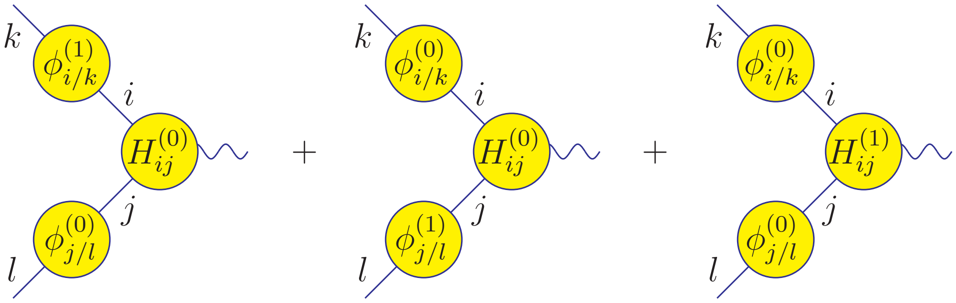
$$\sigma^{(0)}_{kl} =$$

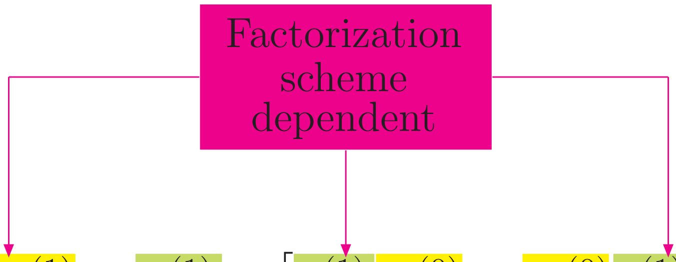



$$\Rightarrow H^{(0)}_{kl} = \sigma^{(0)}_{kl}$$

(2)

$$\sigma^{(1)}_{kl} =$$



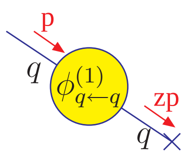

$$\Rightarrow H^{(1)}_{kl} = \sigma^{(1)}_{kl} - \left[\phi^{(1)}_{i/k} H^{(0)}_{il} + H^{(0)}_{kj}\phi^{(1)}_{j/l}\right]$$

Finite Divergent

# Perturbative PDF

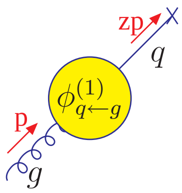
- In $\overline{MS}$-scheme (modified minimal subtraction)

$$\phi^{(1)}_{q/q}(z) = \phi^{(1)}_{\bar{q}/\bar{q}}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow q}(z)$$
$$\phi^{(1)}_{q/g}(z) = \phi^{(1)}_{\bar{q}/g}(z) = \frac{-1}{\varepsilon}\frac{1}{2}\left(4\pi e^{-\gamma_E}\right)^{\varepsilon} P^{(1)}_{q\leftarrow g}(z)$$

where the splitting kernel for 

is

$$P^{(1)}_{q\leftarrow q}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$
$$= C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)$$

and for 

:

$$P^{(1)}_{q\leftarrow g}(z) = \frac{1}{2}\left(z^2 + (1-z)^2\right)$$

$$\left(\begin{array}{l}\text{Note: The Pole part in the } \overline{MS} \text{ scheme is} \\ \qquad \frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon}(4\pi e^{-\gamma_E})^{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E \\ \text{In the MS scheme, the pole part is just } \frac{1}{\varepsilon}\end{array}\right)$$

## **Find $H^{(1)}_{q\bar{q}'}$ (in the $\overline{MS}$ scheme)**

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma^{(1)}_{q\bar{q}'} = \sigma^{(0)} \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \left[ \ln\left(\frac{M^2}{\mu^2}\right) - \frac{1}{\varepsilon} + \gamma_E - \ln 4\pi \right] + C_F \left[ -\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln\hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\}$$

- 

$$\begin{aligned} H^{(1)}_{q\bar{q}'}(\hat{\tau}) &= \sigma^{(1)}_{q\bar{q}'} - \left[2\phi^{(1)}_{q\leftarrow q}\sigma^{(0)}_{q\bar{q}'}\right] \\ &= \sigma^{(0)} \cdot \left\{ P^{(1)}_{q\leftarrow q}(\hat{\tau}) \ln\left(\frac{M^2}{\mu^2}\right) \right. \\ &\quad \left. + C_F \left[ -\frac{1+\hat{\tau}^2}{1-\hat{\tau}} \ln\hat{\tau} + 2\left(1+\tau^2\right)\left(\frac{\ln(1-\hat{\tau})}{1-\hat{\tau}}\right)_+ + \left(\frac{\pi^2}{3} - 4\right)\delta(1-\hat{\tau}) \right] \right\} \end{aligned}$$

where

$$\hat{\tau} = \frac{M^2}{\hat{s}} = \frac{M^2}{x_1 x_2 S},$$

$$\sigma^{(0)} = \frac{\pi}{12\hat{s}} g_w^2 = \frac{\pi g_w^2}{12S}\frac{1}{x_1 x_2}$$

- pQCD prediction

$$\begin{aligned} \sigma_{hh'} &= \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\sigma^{(0)}\delta(1-\hat{\tau})\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right) \\ &+ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{f\bar{f}}(\hat{\tau})\right] \phi_{\bar{f}/h'}\left(x_2,\mu^2\right) \\ &+ \left\{ \sum_{f=q,\bar{q}'} \int dx_1 dx_2 \phi_{f/h}\left(x_1,\mu^2\right) \left[\frac{\alpha_s\left(\mu^2\right)}{\pi} H^{(1)}_{fG}(\hat{\tau})\right] \phi_{G/h'}\left(x_2,\mu^2\right) + (x_1 \leftrightarrow x_2) \right\} \end{aligned}$$

# Find $H_{qG}^{(1)}$ (in the $\overline{MS}$ scheme)

- Take off the factor $\left(\frac{\alpha_s}{\pi}\right)$

$$\sigma_{qG}^{(1)} = \sigma^{(0)} \cdot \frac{1}{4} \cdot \left\{ 2P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[ \frac{-1}{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}} \right] \right.$$
$$\left. + \frac{3}{2} + \hat{\tau} - \frac{3}{2}\hat{\tau}^2 \right\}$$

- 

$$H_{qG}^{(1)}(\hat{\tau}) = \sigma_{qG}^{(1)} - \left[ \sigma_{q\bar{q}'}^{(0)} \phi_{\bar{q}'\leftarrow G}^{(1)} \right]$$
$$= \frac{\sigma^{(0)}}{2} \cdot \left\{ P_{q\leftarrow g}^{(1)}(\hat{\tau}) \left[ \ln\left(\frac{M^2}{\mu^2}\right) + \ln\left(\frac{(1-\hat{\tau})^2}{\hat{\tau}}\right) \right] \right.$$
$$\left. + \frac{3}{4} + \frac{\hat{\tau}}{2} - \frac{3}{4}\hat{\tau}^2 \right\}$$

- Similarly,

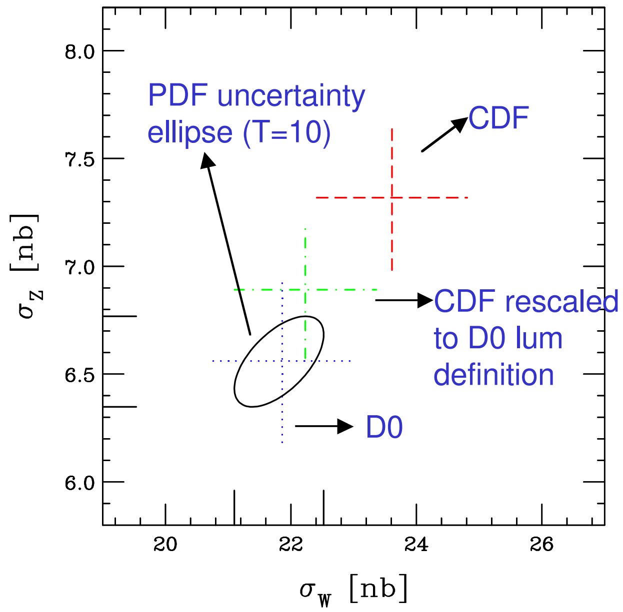
$$H_{G\bar{q}'}^{(1)} = \sigma_{G\bar{q}'}^{(1)} - \left[ \phi_{q\leftarrow G}^{(1)} \sigma_{q\bar{q}'}^{(0)} \right]$$
$$= H_{qG}^{(1)}$$

( Note: If we choose the renormalization scale $\mu^2 = M^2$, then $\ln\left(\frac{M^2}{\mu^2}\right) = 0$ )

# W and Z production

- CDF and D0 would like to use their W and Z cross sections for luminosity determination
- D0 cross sections close to center of PDF prediction ellipse; not the case with CDF

MSU study




J. Pumplin, D. Stump, R. Brock, D. Casey, J. Huston, J. Kalk, H.L. Lai, W.K. Tung: hep-ph/0101051

# Summary

- $\phi_{f/h}(x,\mu^2)$ depends on scheme ($\overline{MS}$,DIS,...)
  $\Rightarrow H_{ij}$ **scheme dependent**

- Evolution equations allow us to perdict
  $q^2-$**dependent of** $\phi(x,q^2)$

- Essentially identical procedure for
  $hh' \to jets$, inclusive $Q\overline{Q}$,...
  But, when the Born level process involves
  strong interaction (eg. $q\overline{q} \to t\overline{t}$),
  it is also necessary to renormalize the
  strong coupling $\alpha_s$, etc, to eliminate
  ultraviolate singularities

# Appendix A
# $\gamma$-matrices in $n$ dimensions

- Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\}_+ \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$
$$\mu, \nu = 1, 2, \cdots, n \qquad g^{\mu\nu} = diag\,(1, -1, \cdots, -1)$$
$$g^{\mu\nu} g_{\mu\nu} = n$$
$$\left\{\gamma^\mu, \gamma^5\right\}_+ = 0 \qquad \left(\text{Naive-}\gamma^5\text{prescription}\right)$$

This works in calculating the inclusive rate of $W$-boson , but fails in the differential distributions of the leptons from the $W$-boson decay.

- Matrix identities

$n = 4 - 2\varepsilon$

$$\gamma_\mu \not{a} \gamma^\mu = -2\,(1 - \varepsilon)\not{a}$$
$$\gamma_\mu \not{a} \not{b} \gamma^\mu = 4a \cdot b - 2\varepsilon \not{a} \not{b}$$
$$\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2 \not{c} \not{b} \not{a} + 2\varepsilon \not{a} \not{b} \not{c}$$

- Traces

$$Tr\,[\not{a}\not{b}] = 4\,(a \cdot b)$$
$$Tr\,[\not{a}\not{b}\not{c}\not{d}] = 4\,\{(a \cdot b)\,(c \cdot d) - (a \cdot c)\,(b \cdot d) + (a \cdot d)\,(b \cdot c)\}$$
$$Tr\,[\gamma_5 \not{a}\not{b}] = 0$$

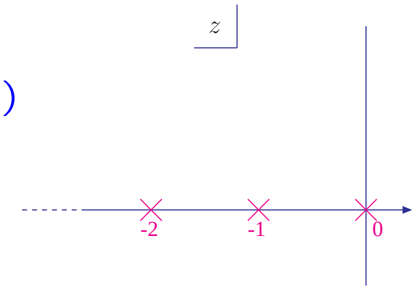
# Appendix B
# Some integrals and "special functions"

- The "Gamma function"

$$\Gamma(z) = \int_0^\infty dx x^{z-1} e^{-x} \qquad (\mathrm{Re}(z) > 0)$$

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} \quad \text{(for all z)}$$

$\Rightarrow$ Poles in $\Gamma(z)$




$$\Gamma(n) = (n-1)! \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + \frac{\varepsilon}{2}\left(\gamma_E^2 - \frac{\pi^2}{6}\right) + \cdots$$

$(\gamma_E = 0.5772\cdots, \text{ Euler constant})$

$$\Gamma(1-\varepsilon) = -\varepsilon\Gamma(\varepsilon) = 1 + \varepsilon\gamma_E + \frac{1}{2}\varepsilon^2\left(\frac{\pi^2}{6} + \gamma_E^2\right) + O\left(\varepsilon^3\right)$$

$$\Gamma(1-\varepsilon)\Gamma(1+\varepsilon) = 1 + \varepsilon^2\frac{\pi^2}{6} + O\left(\varepsilon^4\right)$$

$$z^\varepsilon = e^{\ln z^\varepsilon} = e^{\varepsilon \ln z} = 1 + \varepsilon \ln z + \cdots$$

- The "Beta function"

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$B(\alpha,\beta) = \int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1} = \int_0^\infty dy\, y^{\alpha-1}(1+y)^{-\alpha-\beta}$$

$$= 2\int_0^{\frac{\pi}{2}} d\theta\, (\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}$$

- Feynman trick

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{\left[ax + b\left(1 - x\right)\right]^2}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma\left(\alpha + \beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \int_0^1 dx \frac{x^{\alpha-1}\left(1 - x\right)^{\beta-1}}{\left[ax + b\left(1 - x\right)\right]^{\alpha+\beta}}$$

- n-dimension integrals

$$\int d^n l \frac{l_\mu}{\left(l^2 - M^2\right)^\alpha} = 0$$

$$\int d^n l \frac{l_\mu l_\nu}{\left(l^2 - M^2\right)^\alpha} = \int d^n l \frac{\left(\frac{l^2 g_{\mu\nu}}{n}\right)}{\left(l^2 - M^2\right)^\alpha}$$

$$\int \frac{d^n l}{\left(2\pi\right)^n} \frac{1}{\left(l^2 - M^2\right)^\alpha} = \frac{\left(-1\right)^\alpha}{\left(4\pi\right)^{n/2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma\left(\alpha\right)} \left(\frac{1}{M^2}\right)^{\alpha - \frac{n}{2}}$$

$$\int d^n l \frac{l^2}{\left(l^2 - M^2\right)^\alpha} = \int d^n l \frac{\left(l^2 - M^2\right) + M^2}{\left(l^2 - M^2\right)^\alpha}$$

- $$\mathrm{Re}\left[\left(-1\right)^\varepsilon\right] = 1 - \varepsilon^2 \frac{\pi^2}{2} + O\left(\varepsilon^4\right)$$

- "plus distribution"— to isolate $\frac{1}{\varepsilon}$ poles

Consider $\dfrac{1}{(1-z)^{1+2\varepsilon}}$

$$= \frac{1}{(1-z)^{1+2\varepsilon}} - \left[\delta(1-z)\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} + \frac{1}{2\varepsilon}\delta(1-z)\right]$$

$\searrow \quad \swarrow$

cancel

because $\displaystyle\int_0^1 \frac{dz'}{(1-z')^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$ for $\varepsilon \to 0^-$

$$\equiv \left[\frac{1}{(1-z)^{1+2\varepsilon}}\right]_+ - \frac{1}{2\varepsilon}\delta(1-z)$$

$$= \frac{1}{(1-z)_+} - 2\varepsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + O\left(\varepsilon^2\right) - \frac{1}{2\varepsilon}\delta(1-z)$$

because $\displaystyle\frac{1}{(1-z)^{2\varepsilon}} = (1-z)^{-2\varepsilon}$

$$= 1 - 2\varepsilon\ln(1-z) + \cdots$$

- $[\cdots]_+$ is a distribution

$$\int_0^1 dz\, f(z)\left[\frac{1}{1-z}\right]_+$$

$$\equiv \int_0^1 dz\, \frac{f(z)}{1-z} - \int_0^1 dz\, f(z)\,\delta(1-z)\int_0^1 \frac{dz'}{(1-z')}$$

$$= \int_0^1 dz \frac{f(z)-f(1)}{1-z},$$ which is finite.

# Appendix C
# Angular integrals in $n$ dimensions

- In $n$ dimensions

$$\int d^n x = \int r^{n-1} d\Omega_{n-1}$$

-

$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int_0^\pi d\theta_{n-2} \sin^{n-2}\theta_{n-2} \cdots \int_0^\pi d\theta_1 \sin\theta_1 \int_0^{2\pi} d\phi$$

$$\Rightarrow \int d\Omega_1 = \int_0^{2\pi} d\phi \qquad \longrightarrow \Omega_1 = 2\pi$$

$$\int d\Omega_2 = \int_0^\pi d\theta_1 \sin\theta_1 \int d\Omega_1 \qquad \longrightarrow \Omega_2 = 4\pi$$

$$\vdots$$

$$\int d\Omega_n = \int_0^\pi d\theta_{n-1} \sin^{n-1}\theta_{n-1} \int d\Omega_{n-1}$$

$$\Rightarrow \quad \Omega_n = \frac{2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}$$ from repeated use of $B(\alpha, \beta)$

$$= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$ because $\Gamma(n) = \dfrac{2^{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$

# Appendix D
## Two-particle phase space in $n$ dimensions

- $$\int_{PS_2(p)} dk\, dq = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}\, 2k_0} \frac{d^{n-1}\vec{q}}{(2\pi)^{n-1}\, 2q_0} \cdot (2\pi)^n\, \delta^n\, (p-q-k)$$

  with $k^\mu = \left(k_0, \vec{k}\right)$ , etc.

  Use $\dfrac{d^{n-1}\vec{q}}{2q_0} = \int d^n q \delta^+ \left(q^2 - Q^2\right)$ , we get

- $$\int_{PS_2(p)} dk\, dq = \frac{1}{(2\pi)^{n-2}} \int \frac{d^{n-1}\vec{k}}{2k_0} \delta^+ \left((p-k)^2 - Q^2\right)$$
  $$= \frac{1}{(2\pi)^{n-2}} \int \frac{dk\, k^{n-3}}{2} \int d\Omega_{n-2} \delta \left(\hat{s} - 2k\sqrt{\hat{s}} - Q^2\right)$$
  $$\left(p^2 \equiv \hat{s}, k^2 = 0, k = \left|\vec{k}\right|\right)$$

  Use $n = 4 - 2\varepsilon$ , then in the c.m. frame $\left(p^\mu = \left(\sqrt{\hat{s}}, \vec{0}\right)\right)$ ,

- $$\int_{PS_2(p)} dk\, dq = \frac{\Omega_{n-3}}{(2\pi)^{2(1-\varepsilon)}} \int \frac{dk\, k^{1-2\varepsilon}}{4\sqrt{\hat{s}}} \int_0^\pi d\theta\, (\sin\theta)^{1-2\varepsilon} \cdot \delta \left(k - \frac{\hat{s} - Q^2}{2\sqrt{\hat{s}}}\right)$$

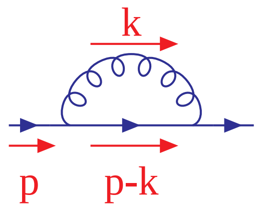
  Use new variables:

  $$z = \frac{Q^2}{\hat{s}}, y = \frac{1}{2}(1 + \cos\theta) \Rightarrow k = \frac{\sqrt{\hat{s}}}{2}(1-z),$$

- $$\int_{PS_2(p)} dk\, dq = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\right)^\varepsilon \frac{z^\varepsilon (1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy\, [y(1-y)]^{-\varepsilon}$$

# Appendix E
# Explicit Calculations

- Consider




$$\int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\mu \left( \not{p} - \not{k} \right) \gamma^\mu}{(k^2 + i\epsilon)\left((p-k)^2 + i\epsilon\right)}$$

$$\to \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left(\not{p} - \not{k}\right)}{\left[k^2 - 2k \cdot xp\right]^2} \qquad (l \equiv k - xp)$$

$$= \int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \frac{(2-n)\left[(1-x)\ \not{p} - \not{l}\right]}{\left[l^2 + i\epsilon\right]^2}$$

$$= \left[\left(1 - \frac{n}{2}\right)\ \not{p}\right] \cdot \underbrace{\int \frac{d^n l}{(2\pi)^n} \frac{1}{\left[l^2 + i\epsilon\right]^2}}$$

$$\downarrow$$

$$= 0 \quad \left(\begin{array}{c}\text{Because there is}\\ \text{no mass scale}\end{array}\right)$$

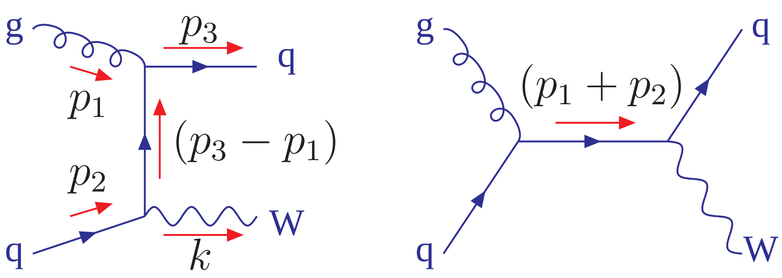
$$\uparrow$$

$$\left(\begin{array}{c}\text{Due to cancellation}\\ \text{of } \frac{1}{\varepsilon_{UV}} \text{ and } \frac{1}{\varepsilon_{IR}}\\ \text{Trick:} A = A - B + B\end{array}\right)$$

$$= \int \frac{d^n l}{(2\pi)^n} \left\{ \underbrace{\left[\frac{1}{(l^2)^2} - \frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\text{IR div.}} + \underbrace{\left[\frac{1}{(l^2 - \Lambda^2)^2}\right]}_{\text{UV div.}} \right\}$$

$$= \frac{i}{16\pi^2}\left(\frac{1}{\varepsilon_{IR}}\right) + \frac{i}{16\pi}\left(\frac{1}{\varepsilon_{UV}}\right), \qquad \left(\begin{array}{c} n - 4 = 2\varepsilon_{IR} \\ 4 - n = 2\varepsilon_{UV}\end{array}\right)$$

- consider the real emission process




Define the Mandelstam variables

$$\hat{s} = (p_1 + p_2)^2 = 2p_1 \cdot p_2$$
$$\hat{t} = (p_1 - p_3)^2 = -2p_1 \cdot p_3$$
$$\hat{u} = (p_2 - p_3)^2 = -2p_2 \cdot p_3$$

After averaging over colors and spins

$$\overline{|\mathcal{M}|^2} = \underbrace{\left(\frac{1}{2(1-\varepsilon)}\frac{1}{2}\right)}_{\text{Spin}} \cdot \underbrace{\left(\frac{1}{3}\cdot\frac{1}{8}\right) \cdot Tr\,(t^a t^a)}_{\text{Color}} \cdot (g\mu^\varepsilon)^2$$

$$\cdot\, g_w^2 \cdot 2\,(1-\varepsilon)$$

$$\cdot \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}}\right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon\right]$$

Note: The d.o.f. of gluon polarization is $2(1-\varepsilon)$, and that of quark polarization is 2.

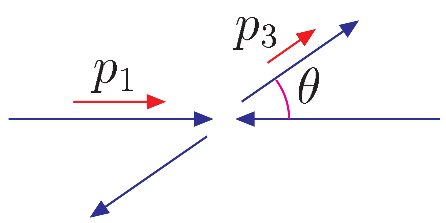
- In the parton c.m. frame, the constituent cross section

$$\hat{\sigma} = \frac{1}{2\hat{s}}\overline{|\mathcal{M}|^2} \cdot (PS_2)$$

$$= \frac{1}{2\hat{s}} \cdot \left\{\frac{1}{4} \cdot \frac{1}{6} \cdot 2g_s^2 \mu^{2\varepsilon} g_w^2 (1-\varepsilon) \cdot \left[(1-\varepsilon)\left(-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}}\right) - \frac{2\hat{u}M^2}{\hat{t}\hat{s}} + 2\varepsilon\right]\right\}$$

$$\cdot \left\{\frac{1}{8\pi}\left(\frac{4\pi}{M^2}\right)^{\varepsilon} \frac{1}{\Gamma(1-\varepsilon)} \hat{\tau}^{\varepsilon} (1-\hat{\tau})^{1-2\varepsilon} \int_0^1 dy \, [y(1-y)]^{-\varepsilon}\right\}$$

where $y \equiv \frac{1}{2}(1+\cos\theta)$




Using $\hat{t} = -\hat{s}\left(1 - \frac{M^2}{\hat{s}}\right)(1-y)$

$$\hat{u} = -\hat{s}\left(1 - \frac{M^2}{\hat{s}}\right) y$$

and

$$\int_0^1 dy \, y^{\alpha} (1-y)^{\beta} = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)} ,$$

we get

$$\hat{\sigma}_{qG} = \sigma^{(0)} \frac{\alpha_s}{4\pi} \cdot \left\{2P^{(1)}_{q\leftarrow g}(\hat{\tau}) \left[\frac{-1}{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} + \ln \frac{M^2(1-\hat{\tau})^2}{4\pi\mu^2\hat{\tau}}\right] + \frac{3}{2} + \hat{\tau} - \frac{3}{2}\hat{\tau}^2\right\},$$

with

$$P^{(1)}_{q\leftarrow g}(\hat{\tau}) = \frac{1}{2}\left[\hat{\tau}^2 + (1-\hat{\tau})^2\right]$$

$$\sigma^{(0)} \equiv \frac{\pi}{12} g_w^2 \frac{1}{\hat{s}}$$