INTRODUCTION TO THE PARTON MODEL AND PERTURBATIVE QCD

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I. The Parton Model and Deeply Inelastic Scattering

II. From the Parton Model to QCD

III. Factorization and Evolution

The Context of QCD
'Fundamental Interactions'

$$\left\{\begin{array}{l} \text{`Strong' - QCD} \\ \left.\begin{array}{l} \text{Weak} \\ \text{Electromagnetic - QED} \end{array}\right\} \text{Electro-weak} \\ \quad\ddots \\ \text{Gravity} \end{array}\right.$$

'Strong' - QCD, Weak and Electromagnetic - QED: Standard Model

- QCD: 'A theory off to a good start'

$$\vec{F} = -\frac{G m_1 m_2 \hat{r}}{r^2} \qquad \mathcal{L}_{QCD} = \bar{q}\not{D}q - \frac{1}{4}F^2$$

$\vec{F} = -\frac{G m_1 m_2 \hat{r}}{r^2}$	$\mathcal{L}_{QCD} = \bar{q}\not{D}q - \frac{1}{4}F^2$
$\downarrow$	$\downarrow$
elliptical orbits in 2-body problem	asymptotic freedom at high energy
$\ddots \ \ddots \ \ddots \ \ddots \ \ddots$	$\ddots$? ? ?

- New fundamental interactions
  $\rightarrow$ new mathematical physics

I. The Parton Model and Deeply Inelastic Scattering

1. Nucleons to Quarks
2. General analysis of DIS
3. Getting at the Quark Distributions
4. Extensions from DIS

1. Nucleons to Quarks

- Protons, neutrons, pions

$\begin{pmatrix} p \\ n \end{pmatrix}$ 'isodoublet' N

p	$m = 938.3$ MeV $S = 1/2$ $I_3 = +1/2$	n	$m = 939.6$ $S = 1/2$ $I_3 = -1/2$

$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$ 'isotriplet' π

$\pi^\pm$	$m = 139.6$ $S = 0$ $I_3 = \pm 1$	π^0	$m = 135.0$ $S = 0$ $I_3 = 0$

Isospin 'Space'

analogue: rotation group

- 'Historic': π as $N\bar{N}$ bound state

$$\pi^+ = (p\bar{n})\ , \quad \pi^- = (\bar{p}n)\ , \quad \pi^0 = \frac{1}{\sqrt{2}}(p\bar{p} + n\bar{n})$$

  - Fermi & Yang 1952
  - Nambu & Jona-Lasinio (1960) (dynamics)

- 'Modern': π, N common substructure: *quarks*
  - Gell Mann, Zweig 1964

- spin $S = 1/2$,
  isospin doublet (u, d) & singlet (s)
  with appoximately equal masses (s heavier);

$$\begin{pmatrix} u\ (Q = 2e/3, I_3 = 1/2) \\ \\ d\ (Q = -e/3, I_3 = -1/2) \\ \\ s\ (Q = -e/3, I_3 = 0) \end{pmatrix}$$

$$\pi^+ = (u\bar{d})\ , \quad \pi^- = (\bar{u}d)\ , \quad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})\ ,$$
$$p = (uud)\ , \quad n = (udd)\ , \quad K^+ = (u\bar{s})\ldots$$

- Requirement for N:
  symmetric spin/isospin wave function (!)
- μ_p/μ_n = -3/2 (good to %)
- and now, six: 3 'light' (u, d, s), 3 'heavy': (c, b, t)

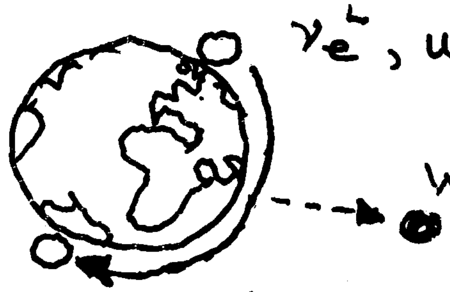
- Quarks in the Standard Model
- "Weak isospin" doublets

$\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad L \rightarrow \overset{s}{\longleftarrow} \bullet \longrightarrow p$

$\begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \qquad$ couple to $W^\pm, Z^0, \gamma$

$\begin{pmatrix} t \\ b' \end{pmatrix}_L \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$

"weak isospin" ≈ "strong isospin"



$$d'^L \approx d^L$$
$$\approx d^L \cos\theta_c + s^L \sin\theta_c$$
$$= d^L V_{ud} + s^L V_{us} + b^L V_{ub}$$

$$V^\dagger V = I \quad (\text{CKM})$$

(Z^0, γ do not change "flavor")

- $e_R, \ldots \tau_R$ couple to Z^0, γ only
  → parity violation
- same mechanism (Spon. Sym. Breaking via Higgs) gives masses to leptons and quarks as to $W^\pm, Z^0$.

- Quarks as Partons
  Seeing Quarks

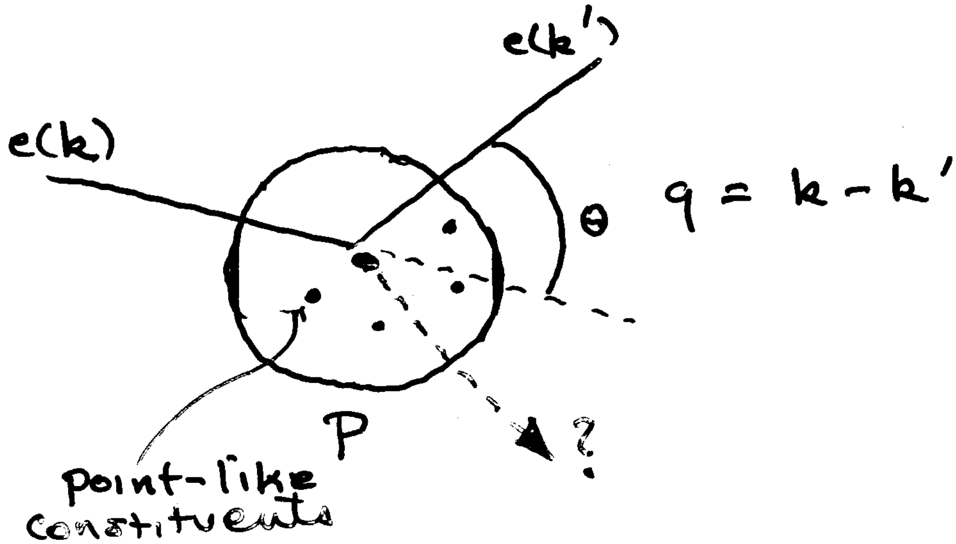
Confinement: no isolated fractional charges observed

Can we still see quarks? (SLAC 1969)

Look closer: do HE e^- bounce off anything hard?

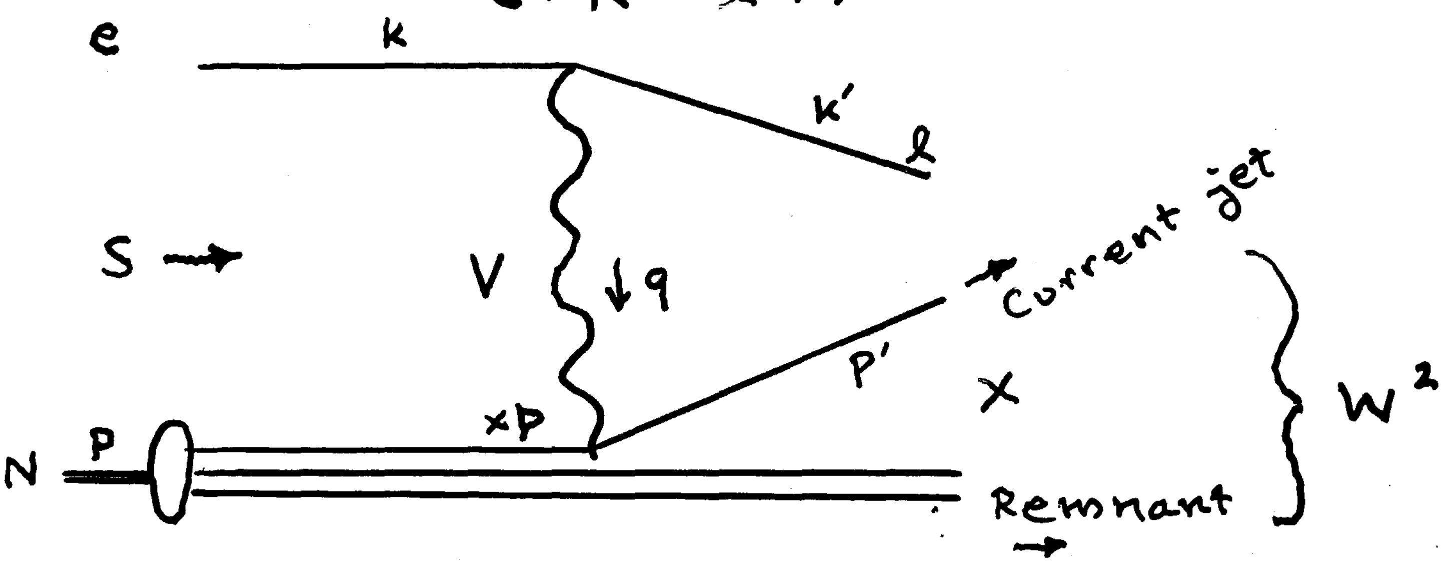
(Rutherford 'prime')

Look for:



θ distribution: information on spin of constituents (if any)

KINEMATICS
$e + N \rightarrow \ell + X$



$$Q^2 = -q^2 = -(k-k')^2 \qquad \text{momentum transfer}$$

$$x = \frac{Q^2}{2p\cdot q} \qquad \text{momentum fraction}$$
$$(p'^2 = (xp+q)^2 = 0)$$

$$y = \frac{p\cdot q}{p\cdot k} \qquad \text{fractional energy transfer}$$

$$W^2 = (p+q)^2 = \frac{Q^2}{x}(1-x)$$
squared final-state mass of hadrons

$$xy = \frac{Q^2}{S}$$

$\ell = e \quad (e^{\pm}) \qquad V = \gamma, Z_0 \qquad$ NC

$\ell = \nu \quad (e^{-}) \qquad V = W^-$

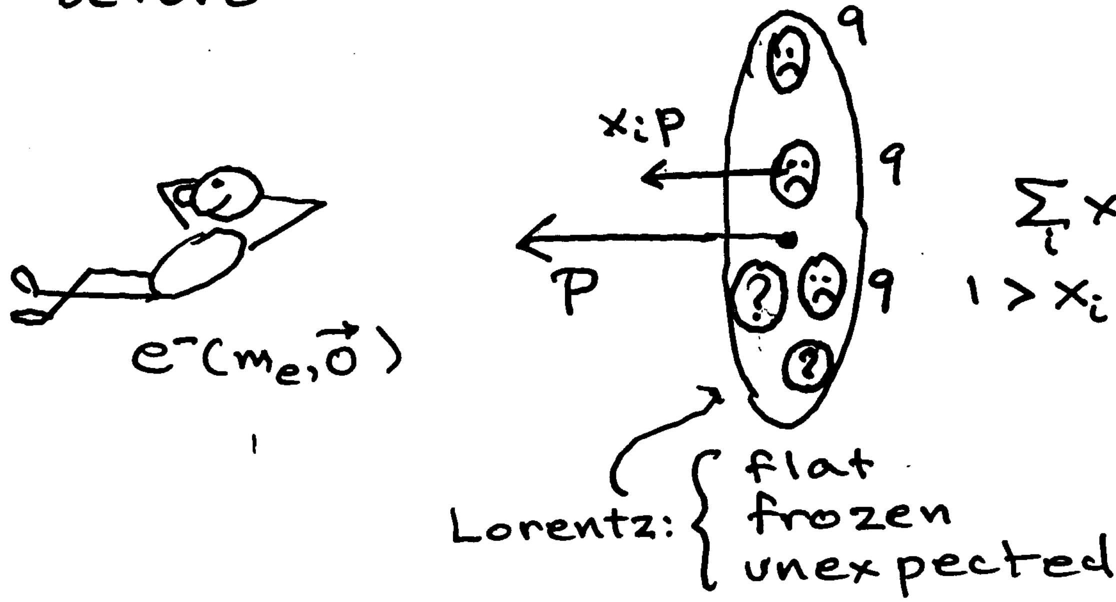
$\qquad\qquad\qquad\qquad\qquad\qquad$ CC

$\ell = \bar{\nu} \quad (e^{+}) \qquad V = W^+$

b) Parton interpretation
(Feynman 1969, 1972)

Look in e^- rest frame:

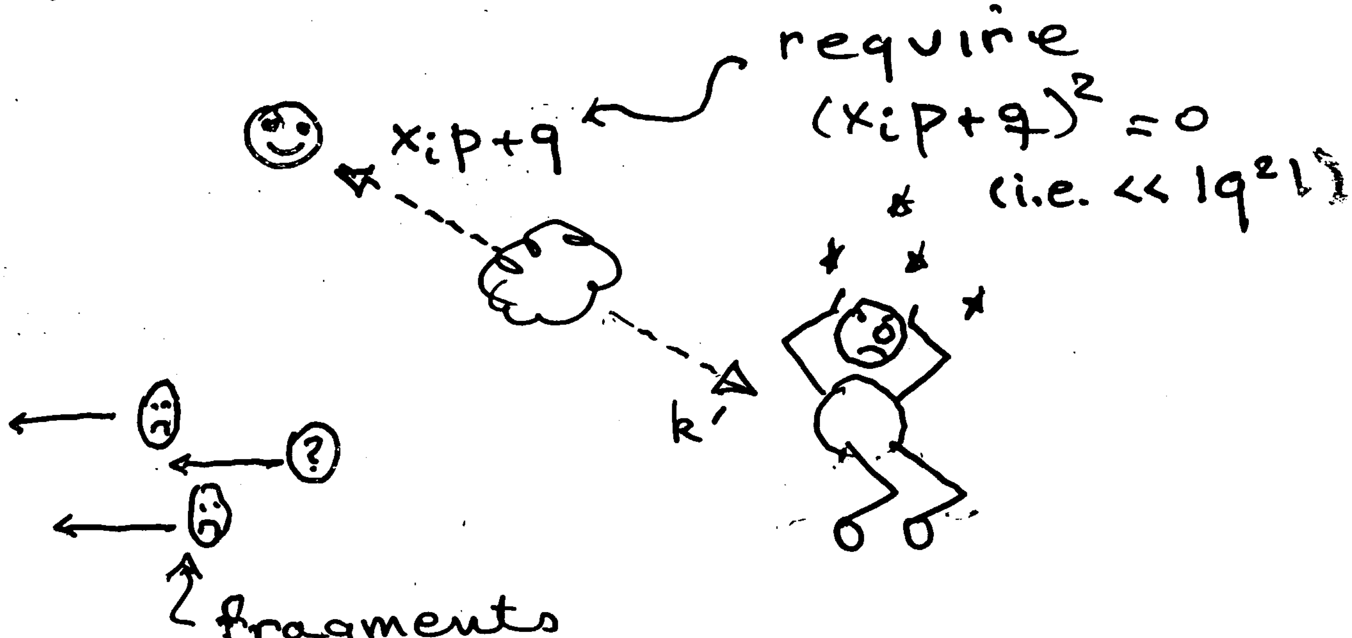
i) before



$\sum_i x_i = 1$

$1 > x_i > 0$

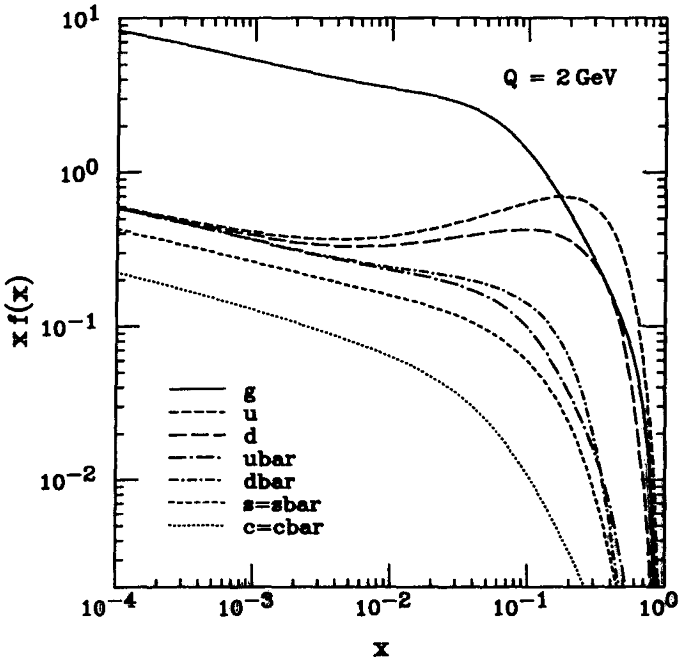
ii) after



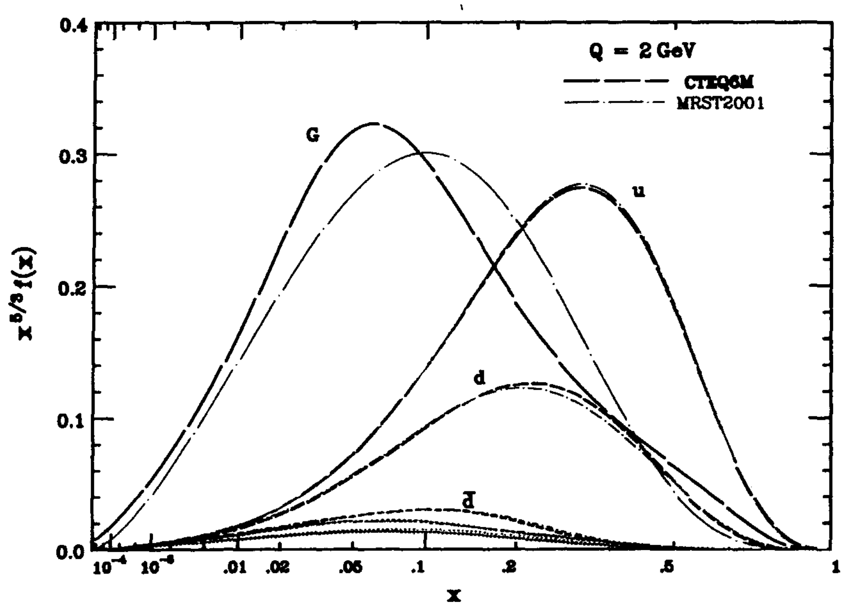
'Deeply Inelastic Scattering'

Two portraits of modern parton distributions

- CTEQ6 as seen at moderate momentum transfer:



- Two modern fits compared (note weighting with x)



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~~3-25~~

• Basic Parton Model Relation:

$$\boxed{\sigma_{eh}^{DIS}(p,q) = \sum_{\substack{\text{partons} \\ a}} \int_0^1 dx\, \hat{\sigma}_{ea}^{el}(xp,q)\, \phi_{a/h}(x)}$$

where: σ_h^{DIS} DIS cross section for h

$$\underset{k}{e} + \underset{p}{h} \rightarrow \underset{(k-q)}{e} + \underset{(p+q)}{X}$$

$(xp+q)^2 = 0$

$\rightarrow x = -\frac{q^2}{2p\cdot q}$

$\hat{\sigma}_{ea}^{el}$ <u>ELASTIC</u> cross section for a

$$\underset{k}{e} + \underset{xp}{a} \rightarrow \underset{(k-q)}{e} + \underset{(xp+q)}{a}$$

$\phi_{a/h}(x)$ Distribution of parton a in hadron h (probability for a to have xp)

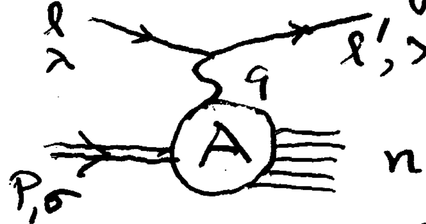
in words:

$\begin{pmatrix}\text{hadronic inelastic}\\ \text{cross section}\end{pmatrix} = \begin{pmatrix}\text{partonic elastic}\\ \text{cross section}\end{pmatrix}$

$\otimes \begin{pmatrix}\text{probability}\\ \text{for } p_a = x p_h\end{pmatrix}$

nontrivial assumption:
Quantum mechanical incoherence of large-q scattering and partonic distribution
(heuristic justification as above)

2. General Analysis of DIS:



any $M_n^2 \geq m_p^2$

- Cross Section & Hadronic Tensor

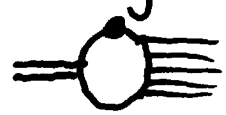
$$A^{(\lambda,\lambda',\sigma;\,q)}_{e+p\to e+n} = \bar{u}_{\lambda'}(\ell')(-ie\gamma_\mu)u_\lambda(\ell)$$

$$\cdot \frac{(-ig^{\mu\mu'})}{q^2} \qquad \} \; q = \ell - \ell'$$

$$\cdot \langle n | e J^{e.m.}_{\mu'}(0) | p, \sigma \rangle$$

$\bar{u}u = 2m$

$J^{e.m.}$



$$d\sigma_{DIS}(q^2) = \frac{1}{2^2}\,\frac{1}{2s}\,\frac{d^3\ell'}{(2\pi)^3 2\omega_{\ell'}} \sum_n \sum_{\lambda,\lambda',\sigma} |A|^2$$

$(8\pi^2)\cdot(8\pi)$ $\quad * \, (2\pi)^4\,\delta^4(p_n + \ell' - p - \ell)$

in $|A|^2$, let

$$L^{\mu\nu} = \frac{e^2}{8\pi^2} \sum_{\lambda\lambda'} (\bar{u}_{\lambda'}\gamma^\mu u_\lambda)^* (\bar{u}_{\lambda'}\gamma^\nu u_\lambda)$$

leptonic tensor

$$= (e^2/2\pi^2)(\ell^\mu \ell'^\nu + \ell'^\mu \ell^\nu - g^{\mu\nu}\,\ell\cdot\ell')$$

hadronic tensor

$$W_{\mu\nu} = \frac{1}{8\pi} \sum_{\sigma,n} \left(\langle n | J_\mu | p,\sigma\rangle^* \langle n | J_\nu | p,\sigma \rangle \right)$$

$$\boxed{2\omega_\ell \frac{d\sigma}{d^3\ell'} = \frac{1}{s(q^2)^2}\, L^{\mu\nu}\, W_{\mu\nu}}$$

to be measured

known from QED

$W_{\mu\nu}$ has 16 components, but ... for instance:

current conservation: $\partial^\mu J_\mu^{e.m.} = 0$

$$\rightarrow \langle n | \partial^\mu J_\mu(x) | p \rangle = 0$$

$$\rightarrow (p_n - p)^\mu \langle n | J_\mu(0) | p \rangle = 0$$

$$\rightarrow q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0$$

+ parity, t-reversal etc...

structure functions

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(x, Q^2)$$

$$+ \left(p_\mu - q_\mu \frac{p \cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p \cdot q}{q^2}\right) W_2(x, Q^2)$$

(as above $^* 1/x^* = -2p \cdot q / q^2 = 2p \cdot q / Q^2$;
$Q^2 = -q^2$) so far a def$^{\underline{n}}$

$$p_n^2 = (p+q)^2 = m^2 + 2p \cdot q + q^2$$

$$= m^2 + Q^2 \frac{(1-x)}{x}$$

dimensionless structure fns

$$F_1 \equiv W_1$$

$$F_2 \equiv p \cdot q \, W_2$$

- Structure Functions in the Parton Model; the Callan-Gross Relation

from "basic formula":

$$\frac{d\sigma^{ep}_{DIS}}{d^3\ell'} = \int_x^1 d\xi \left[\frac{d\sigma^{eq}_{el.}(\xi)}{d^3\ell'}\right]\phi_{q/p}(\xi) \qquad (*)$$

same as in ep (!)

$$\omega_{\ell'}\frac{d\sigma^{eq}_{el}}{d^3\ell'} = \frac{1}{2(\xi s)Q^4} L^{\mu\nu} W^{eq}_{\mu\nu}(\ell+\xi p \to \ell'+p')$$

eq process; lowest-order elastic (Born) $p'^2=0$

3 integrals

$$W^{eq}_{\mu\nu} = \frac{1}{8\pi}\sum_{spin}\int \frac{d^3p'}{(2\pi)^3 2\omega_{p'}} \left| q\ \xi p \to p' \right|^2 (2\pi)^4 \delta^4(p'-q-\xi p)$$

4 deltas

$$= \frac{1}{8\pi}(2\pi)\frac{Q_f^2}{2\xi p\cdot q}\delta\left(1-\frac{x}{\xi}\right)\cdot 4\left[(\xi p+q)_\mu \xi p_\nu + \xi p_\mu(\xi p+q)_\nu - \xi p\cdot q\, g_{\mu\nu}\right]$$

trace

$$= -\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\underbrace{\delta\left(1-\frac{x}{\xi}\right)Q_f^2/2}_{W_1^{eq}}$$

$$+\left(\xi p_\mu - q_\mu\frac{\xi p\cdot q}{q^2}\right)\left(\xi p_\nu - q_\nu\frac{\xi p\cdot q}{q^2}\right)\underbrace{\left(\frac{Q_f^2\,\delta(1-\frac{x}{\xi})}{\xi p\cdot q}\right)}_{W_2^{eq}}$$

- use $\delta(1-\frac{x}{\xi})$
- substitute in (*)

$$F_2(x) = \sum_f Q_f^2\, x\, \phi_{f/p}(x) = 2x F_1(x)$$

"scaling" no Q^2-dep.

Spin of quark → $F_2 = 2x F_1$ Callan-Gross reln.

The C-G reln shows compatibility of quark model; parton model

Scaling follows from point-like nature of quarks

Both work pretty well: SLAC 69

Photon polarizations

in p rest frame can take $\quad \nu = \frac{p \cdot q}{m_p}$

$$q^\mu = (\overset{0}{\nu}; \overset{1}{0}, \overset{2}{0}, \overset{3}{\sqrt{Q^2+\nu^2}})$$

in this frame possible polarizations

$$\epsilon_{R,L}(q) = \frac{1}{\sqrt{2}}(0; 1, \mp i, 0), \quad \epsilon_{long}(q) = \frac{1}{Q}(\sqrt{Q^2+\nu^2}; 0, 0, \nu)$$

(the other one is absent by $q^\mu W_{\mu\nu} = 0$)

can expand (alternate)

$$W_{\mu\nu} = \sum_{\lambda = L, R, long} \epsilon^*_\lambda(q)_\mu \epsilon_\lambda(q)_\nu F_\lambda(x, Q^2)$$

with (e.m.): $\quad F^{e.m.}_{L,R} = F_1$

$$F_{long} = \frac{F_2}{2x} - F_1$$

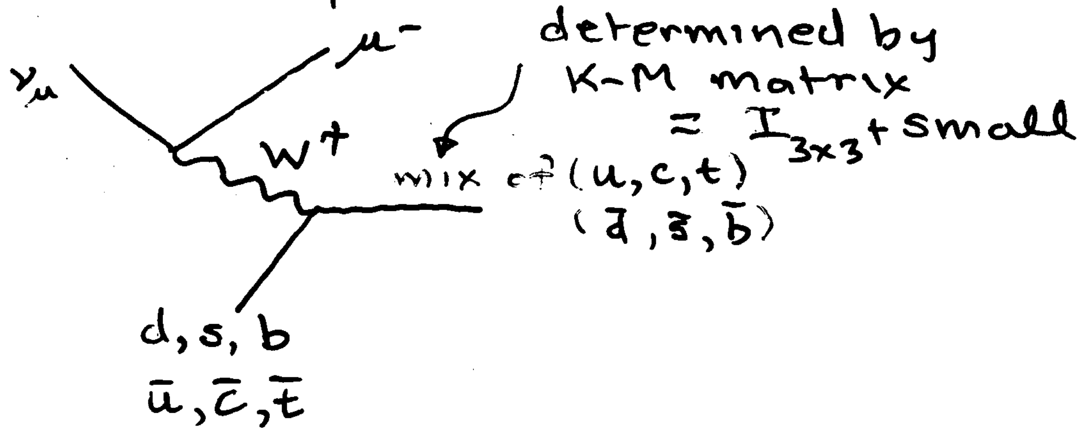
C-G reln:

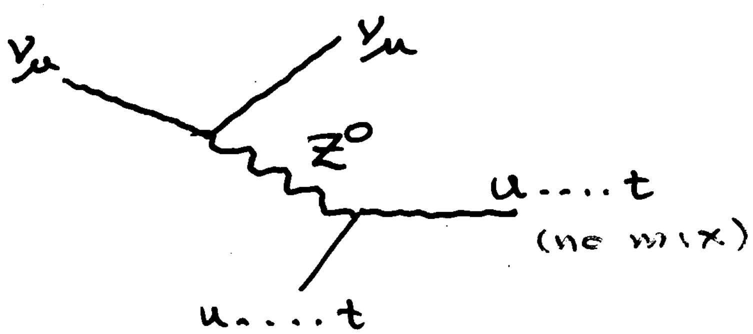
F_{long} vanishes in parton model (later: w/ calculable QCD corrections)

- **Neutrino Scattering**

basic processes change flavor if $W^\pm$ is exchanged; flavor is preserved for Z exchange

for example:





Lack of parity symmetry in $W^\pm$, Z coupling:

$$W_{\mu\nu}^{(\nu h)} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1^{(\nu h)}$$

often-used convention

$$+ \left(p_\mu - q_\mu \frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu \frac{p\cdot q}{q^2}\right) \frac{1}{(m_h^2)} W_2^{(\nu h)}$$

$$- i \epsilon_{\mu\nu\lambda\sigma} p^\lambda q^\sigma \frac{1}{m_h^2} W_3^{(\nu h)}$$

dimensionless in both conventions

$$F_1 = W_1 \; ; \quad F_2 = \frac{p\cdot q}{(m_h^2)} W_2 \; ; \quad F_3 = \frac{p\cdot q}{(m_h^2)} W_3$$

$W_3 = 0$ for e.m. eh scattering

3. Getting at the Quark Distributions

- Structure functions in the PM

with e, ν - scattering, can determine $\phi_{f/N}$:

<u>Simplifying</u> assumptions:

$$\phi_{u/p} = \phi_{d/n} \qquad \phi_{d/p} = \phi_{u/n} \quad \text{(isospin)}$$

$$\phi_{\bar{u}/p} = \phi_{\bar{u}/n} = \phi_{\bar{d}/p} = \phi_{\bar{s}/p} \quad \left(\begin{matrix}\text{isospin-}^{*}\\ \text{symm.}\\ \text{"sea"}\end{matrix}\right)$$

$$\phi_{c/N} = \phi_{b/N} = \phi_{t/N} = 0 \quad \left(\begin{matrix}\text{no "heavy"}\\ \text{quarks}\end{matrix}\right)$$

Parton Model gives *watch out!

$$F_2^{eN}(x) = 2x F_1^{eN}(x) = \sum_{f=u,d,s} Q_f^2\, x\, \phi_{f/N}(x)$$

or $\gamma^* N$

$$F_2^{(W^+N)}(x) = 2x\left(\sum_{D=d,s,b} \phi_{D/N}(x) + \sum_{U=u,c,t} \phi_{\bar{U}/N}(x)\right)$$

$$F_2^{(W^-N)}(x) = 2x\left(\sum_D \phi_{\bar{D}/N}(x) + \sum_U \phi_{U/N}(x)\right)$$

$$F_3^{(W^+N)} = 2\left(\sum_D \phi_{D/N}(x) - \sum_U \phi_{\bar{U}/N}(x)\right)$$

$$F_3^{(W^-N)} = 2\left(-\sum \phi_{\bar{D}/N}(x) + \sum_U \phi_{U/N}(x)\right)$$

Overdetermined with simplifying assumptions: checks consistency

• Further consistency checks: Sum Rules

$$N_{u/p} = \int_0^1 dx\,[\phi_{u/p}(x) - \phi_{\bar{u}/p}(x)] = 2$$

'$\phi_{val}(x)$'

etc. for $N_{d/p} = 1$

$$1 = N_{u/p} - N_{d/p} = \int_0^1 dx\,[\phi_{d/n} - \phi_{d/p}]$$

isospin

$$= \int_0^1 dx\,[\sum_D \phi_{D/n} + \sum_U \phi_{\bar{u}/n}$$

cancel!

cancel, except for $D = d$

$$- \sum_D \phi_{D/p} - \sum_U \phi_{\bar{u}/p}]$$

i.e. (νn)

$$= \int_0^1 \frac{dx}{2x}\,[F_2^{(\nu n)} - F_2^{(\nu p)}]$$

(Adler SR)

similarly

$$3 = N_u + N_d = \int_0^1 \frac{dx}{2x}\,[x F_3^{(\nu n)^*} + x F_3^{(\nu p)^*}]$$

(Gross-Llewellyn Smith)

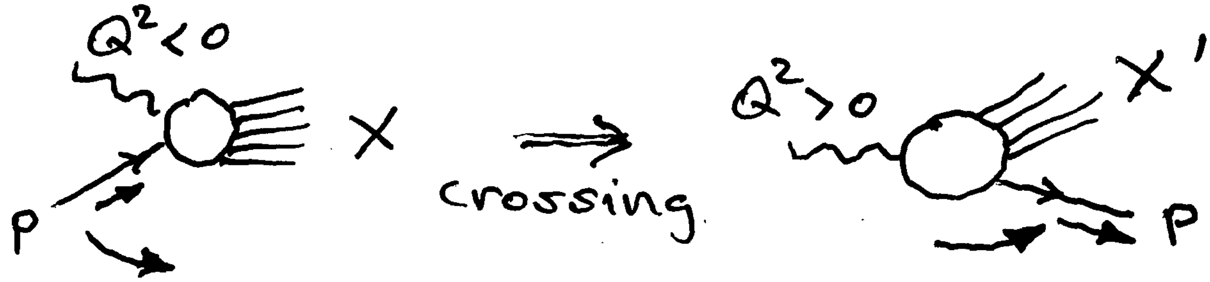
well-verified, but of interest for QCD corrections

some pQCD some NP QCD (SRs that use isospin-symmetric sea fail; see F.O. lecture)

4. Extensions

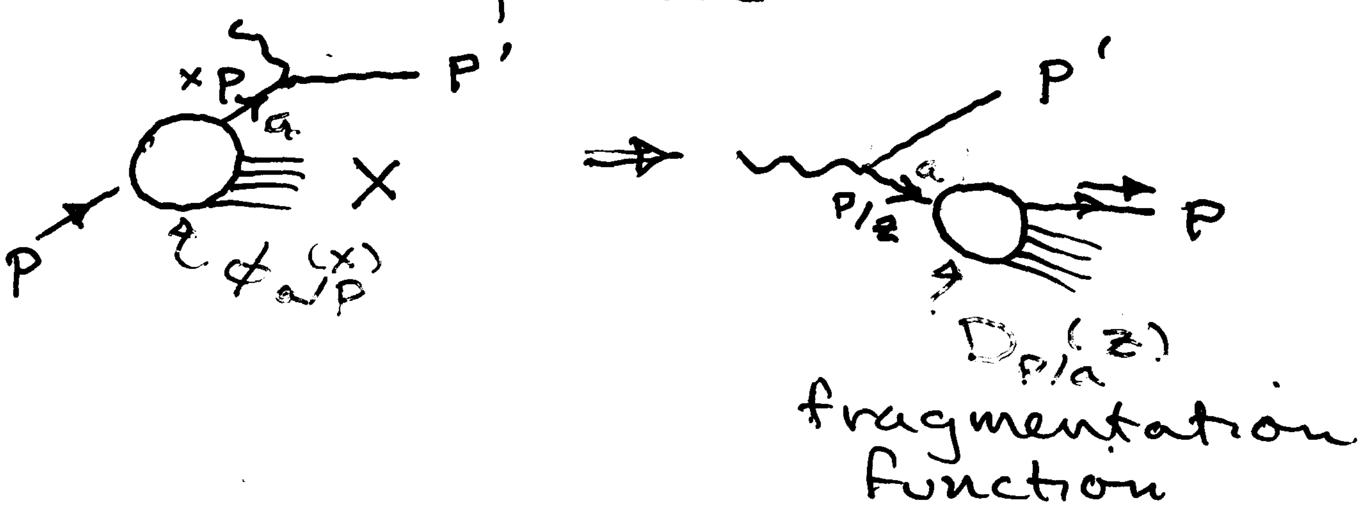
- Fragmentation functions

"cross" DIS



"single-particle inclusive"

cross PM picture

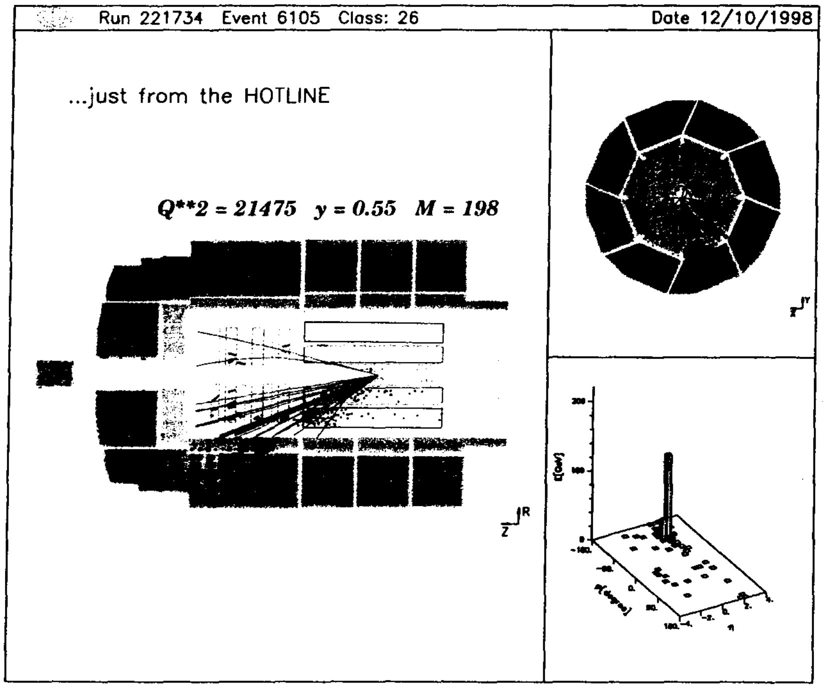


General PM equation for 1PI cross sections

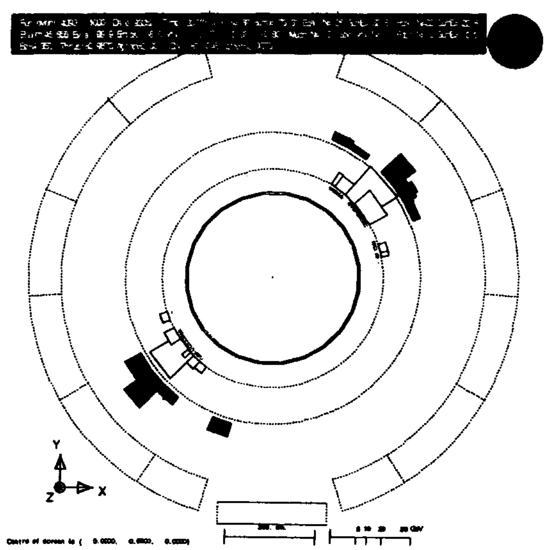
$$\boxed{d\sigma_c(\ell) = \sum_a \int_0^1 dz\; d\hat{\sigma}_a(\ell/z)\, D_{c/a}(z)}$$

Heuristic justification: formation of c from a takes time "τ_0" in rest frame of a but much longer in c.m. frame – thus decouples from $d\hat{\sigma}_a$.

- Fragmentation picture suggests hadrons aligned along parton direction ⇒ Jets
- And that's what happens; DIS:



- And that's what happens; e^+e^-:



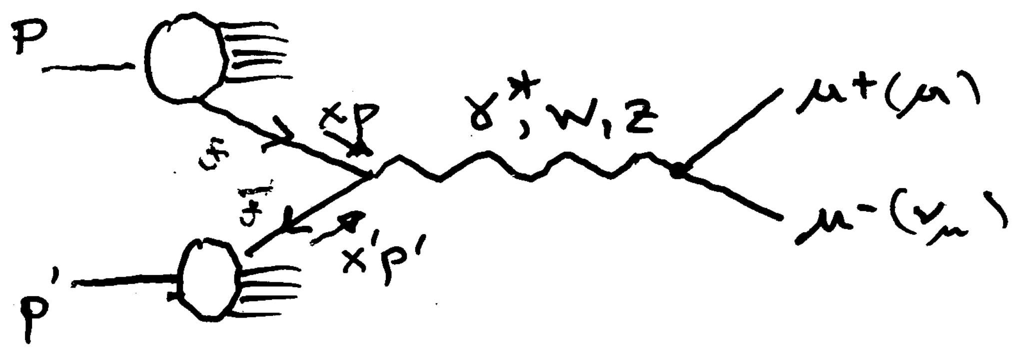
- Drell-Yan (Dilepton) Cross Section in the Parton Model

$$p + p' \to \mu^+\mu^-(q^\mu) + X$$

$$q^2 = +Q^2$$

$e^+e^-, \mu^+\mu^-$ from γ^* or Z; $\mu\nu$ from W

PM picture



PM formula

$$\frac{d\sigma}{dQ^2} = \sum_{f=q,\bar{q}} \int_0^1 dx\, dx'\, \phi_{f/p}(x)\, \frac{d\hat{\sigma}_{f\bar{f}}}{dQ^2}\, \phi_{\bar{f}/p'}(x')$$

$$\frac{4\pi\alpha^2}{9Q^2(xx's)} Q_f^2\, \delta\left(1 - \frac{Q^2}{xx's}\right)$$

(later)

The basis for calculation of W, Z production...

II. From The Parton Model to QCD

1. Color and QCD
2. Field Theory Essentials
3. Infrared Safety
4. Summary

1. Color and QCD

- Enter the Gluon

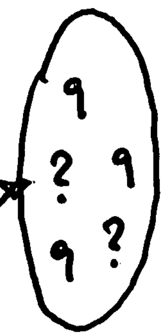
If $\phi_{q/p}(x)$ = prop. for q with momentum xp

Then $F_q = \sum_q \int_0^1 dx\, x\, \phi_{q/p}(x)$

= total fractional momentum carried by quarks

Experiment: $\boxed{F_q \simeq 0.5}$

something else



What else? Quanta of force field that holds N together

'Gluons'

But what are they?

- Color

- Quark model problem

$S_q = 1/2 \rightarrow$ fermion

$\rightarrow$ antisymmetric wave function

(but)

(and) state symmetric in spin/isospin

expect lowest-lying $\psi(\vec{x}_u, \vec{x}_u', \vec{x}_d)$ to be symmetric

where's the antisymmetry?

- Solution (Han Nambu 1968)

Color

q_i, $i = 1, 2, 3$ a new quantum number
b g r

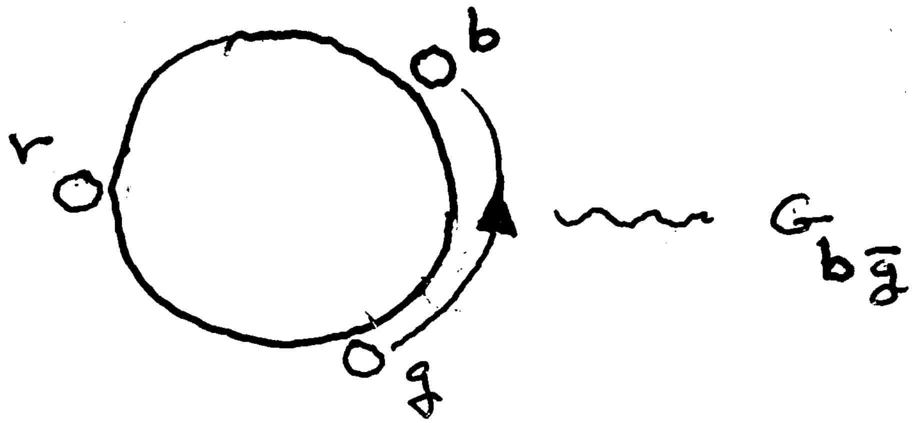
Now can have

$$\psi(u, u, d) = \epsilon_{ijk} \, \psi^{sym}(u_i, u_j, d_k)$$

here's the antisymmetry

- Quantum Chromodynamics: dynamics of Color

Schematic representation: a globe with no North Pole



position on hyperglobe unobservable ($\leftrightarrow$ phase of wave function)

freedom to change 'axes' at different x^μ:

local rotation $\leftrightarrow$ emission of gluon

$S=1$

- Yang Mills 1954

QCD (gluons coupled to color)

- Fritzsch, Gell Mann, Leutwyler
- Weinberg
- Gross, Wilczek

1973

2. Field Theory Essentials

- Fields and Lagrange Density for QCD

$\psi_f(x)$: Quark fields. Dirac fermion (like electron). Color triplet.
$f = u, d, s, c, b, t$

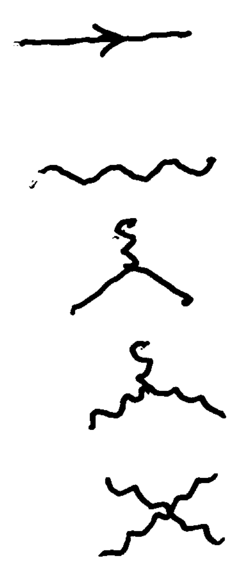
$A(x)$: Gluon field. Vector (like photon). Color octet

$$\mathcal{L}(\psi, A) = \sum_f \bar{\psi}_f \left([i\partial_\mu - g A_\mu^a t_a]\gamma^\mu - m_f \right)\psi_f$$
$$- \frac{1}{4}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g C_{abc} A_\mu^b A_\nu^c \right)^2$$

$$[t_a, t_b] = i C_{abc} t_c$$

- Schematic Pert. Theory Rules

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi$$
$$- \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$
$$- g \bar{\psi} A_\mu^a t_a \gamma^\mu \psi$$
$$- \frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g C_{abc} A^{\mu b} A^{\nu c}$$
$$- \frac{1}{4} g^2 C_{abc} A_\mu^b A_\nu^c C_{ade} A^{\mu d} A^{\nu e}$$



- From Lagrangian to Cross Sections: Outline

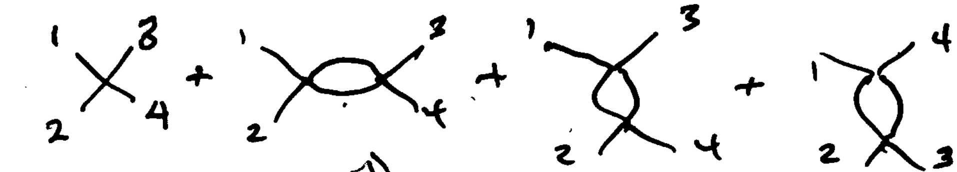
$$
\begin{tikzcd}
\text{Fields} \arrow[r] & \text{Lagrangian} \arrow[d] & \text{Symmetries} \arrow[l] \\
 & \text{Pert. Theo. Rules} \arrow[d] & \\
\text{Renormalization} \arrow[r] & \text{Green Functions} \arrow[d] & \\
 & \text{S-Matrix} \arrow[d] & \\
 & \text{Cross Sections} & \text{Observable}
\end{tikzcd}
$$

- UV Divergences: (Toward Renormalization and The Renormalization Group)

As an example:
Use

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

UV divergences

$M(s,t) =$ 

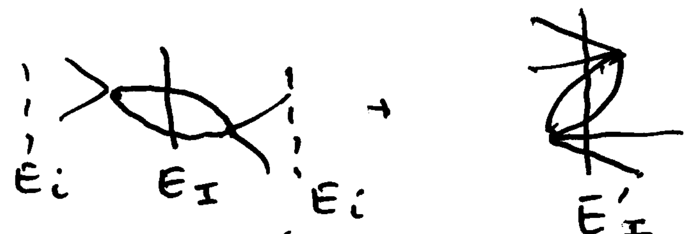
$+ \cdots$

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)} \frac{1}{(p_1+p_2-k)^2-m^2}$$

$$\sim \int^\infty \frac{d^4k}{(k^2)^2} + \cdots$$

Interpretation: states of high 'mass'

Fact:

 $=$ 

$E_s = \sum_{i \text{ in } s} \sqrt{\vec{p}_i^{\,2} + m^2}$

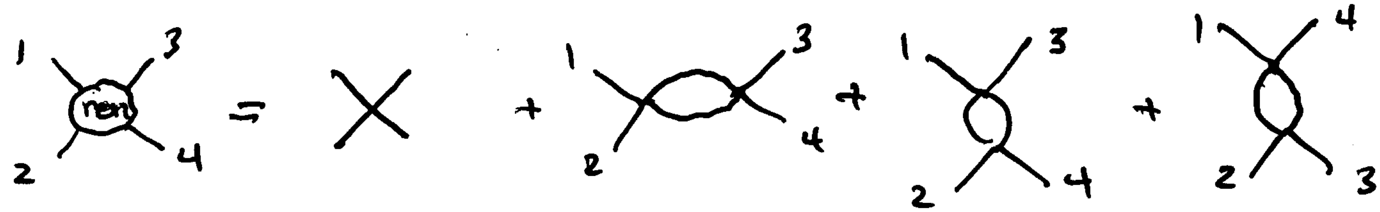
$$\sim \int (PS)_I \left(\frac{1}{E_i - E_I} + \frac{1}{E_i - E'_I} \right)$$

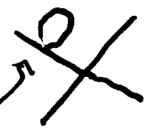
$\to \infty$ from $\vec{P}_I \to \infty$, $E_I \to \infty$

uncertainty $\to$ equivalent to $\Delta t = 0$ 'local' interaction

- Illustration in ϕ^4

4-point: $G = \int_{x_i} \prod_i e^{-ip_i x_i} \langle 0 | T(\prod_1^4 \phi(x_i)) | 0 \rangle$



$+$  $+$ 3 more

p^2-indep.

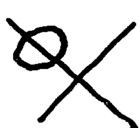
"counterterms"

$+$  $+$  $+$ 3 more

$-i\lambda Z_1^{(1)}$ $\qquad$ $m^2 Z_m^{(1)}$

$=$ UV finite

(Note: at $\mathcal{O}(\lambda)$ no p_i^2 dependence in self-energy)

 $\leftarrow \mathcal{O}(\lambda^3)$

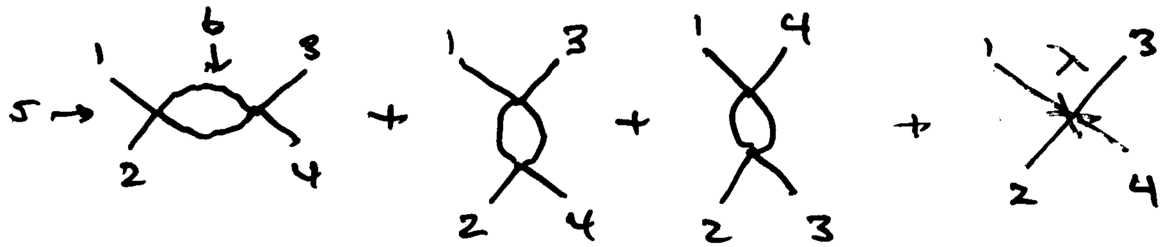
Can choose  $+$  $= 0$ $\quad (m = m_{phys})$

So concentrate on 

What is it going to be?

- Renormalization Schemes

Choose counterterms so that combination



$= \text{finite}$

1 2 3 4

how? for example:

define 1+2+3 by cutting off $\int d^4k$ at $k^2 = \Lambda^2$ (regularization) then

$$1+2+3 = a \ln \frac{\Lambda^2}{s} + b$$

(a, b finite fns. of s, t, u, m^2)

now choose

$$4 = -a \ln \frac{\Lambda^2}{\mu^2}$$

so that

$$1+2+3+4 = -a \ln \frac{s}{\mu^2} + b$$

independent of Λ

Critereon for choosing μ is a "renormalization scheme"

MOM scheme: $\mu = s_0$, some point in mom. space

$\overline{MS}$ scheme: same μ for all graphs

But the value of μ is still arbitrary

μ = renormalization <u>scale</u>

"Modern view" (Wilson):

- Counterterms hide our ignorance of very high-E ($E \gg \mu$) physics
- Very massive ($M \gg \mu$) particles "decouple" at ($E \ll M$)
- Cut-off or $D \neq 4$ regularized theory is an effective theory with the same low energy behavior as the true theory (= SUSY, String...?)

μ-dependence is the price for working with an effective theory

But it has its advantages too...

- The Renormalization Group

As μ changes, mass m and coupling g change in value.

$m = m(\mu)$ $\quad g = g(\mu)$ "renormalized"

but...

Physical quantities can't depend on μ:

$$\mu \frac{d}{d\mu} \underbrace{\sigma}_{\text{invariants}} \left(\frac{t_{ij}}{\mu^2}, \frac{m^2}{\mu^2}, g(\mu), \mu \right) = 0$$

The "group" is the set of all changes in μ.

"RG Equation" $\quad [\sigma] = -\omega$
(let $m \equiv 0$)

$$\boxed{\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \omega \right) \sigma \left(\frac{t_{ij}}{\mu^2}, g(\mu) \right) = 0}$$

$$\boxed{\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu}}$$

- The "Running" Coupling

consider any $(m=0, w=0)$

$$\sigma\left(\frac{t_1}{\mu^2}, \frac{t_2}{t_1} \cdots g(\mu)\right)$$

$$\mu \frac{d\sigma}{d\mu} = 0 \rightarrow \qquad \frac{\partial \sigma}{\partial \ln \mu} = -\beta(g)\frac{\partial \sigma}{\partial g} \qquad (1)$$

in PT:

$$\sigma = g^2(\mu)\,\sigma^{(1)} + g^4(\mu)\left[\sigma^{(2)}\left(\frac{t_2}{t_1}\right) + \tau^{(2)} \ln \frac{t_1}{\mu^2}\right] + \cdots \qquad (2)$$

(2) in (1) $\rightarrow$

$$g^4 \tau^{(2)} = 2g\sigma^{(1)}\beta(g) + \cdots$$

$$\beta(g) = \frac{g^3}{2}\frac{\tau^{(2)}}{\sigma^{(1)}} + \mathcal{O}(g^5)$$

$$\beta(g) \equiv \frac{g^3}{16\pi^2}\beta_1 + \mathcal{O}(g^5)$$

In QCD:

$$\beta_1 = -\left(11 - \frac{2}{3}n_f\right)$$

$\beta_1 < 0 \rightarrow$ g decreases as μ increases

- Asymptotic Freedom

Solution for QCD running coupling
(= effective)
(= renormalized)
(= $g(\mu)$)

$$\mu \frac{\partial g}{\partial \mu} = g^3 \frac{\beta_1}{16\pi^2} \qquad \frac{d\mu}{\mu} \equiv dt$$

$$\frac{dg}{g^3} = \frac{\beta_1}{16\pi^2} dt \qquad \mu_2 = \mu_1 e^t$$

$$\frac{1}{g^2(\mu_1)} - \frac{1}{g^2(\mu_2)} = \frac{\beta_1}{16\pi^2} 2\,dt$$

$$g^2(\mu_2) = \frac{g^2(\mu_1)}{1 - \frac{\beta_1}{16\pi^2} g^2(\mu_1) 2t} \qquad (\beta_1 < 0)$$

$$\xrightarrow[t\to\infty]{} 0 \quad \text{(Asymptotic freedom)}$$

$$g^2(\mu_2) = \frac{g^2(\mu_1)}{1 - \frac{\beta_1}{16\pi^2} g^2(\mu_1) \ln \frac{\mu_2^2}{\mu_1^2}}$$

- Reparameterization : Λ_{QCD}

Effective coupling $\equiv$ renormalized coupling

→ μ_1 and $g^2(\mu_1)$ not independent

→ define $\Lambda = \mu_1 e^{-\beta_1/\alpha_s(\mu_1)}$

independent of μ_1

→ another useful form for $g(\mu)$

$$\boxed{\alpha_s(\mu^2) = \frac{4\pi}{\beta_1 \ln(\mu^2/\Lambda^2)}}$$

'Weak coupling at large momentum scales'

Suggests reln. to parton model in which partons act as if free, at short distances

But how to quantify this observation

3. INFRARED SAFETY

- Would like to choose μ as 'large as possible' in calculations → small $g(\mu)$
- But how large *is* possible?

• Typical S-matrix elt.

$$S\left(\frac{Q_i^2}{\mu^2}, \frac{p_i^2}{\mu^2}, \frac{m_f^2}{\mu^2}, \frac{m_g^2}{\mu^2}, \frac{Q_i^2}{Q_j^2}, g(\mu)\right)$$

$$= \sum_{n=1}^{\infty} a_n\left(\frac{Q_i^2}{\mu^2}, \frac{p_i^2}{\mu^2}, \frac{m_f^2}{\mu^2}, \frac{m_g^2}{\mu^2}, \frac{Q_i^2}{Q_j^2}\right) g^{2n}(\mu)$$

Q_i^2 - large external invariants
p_i^2 - small external masses
m_f - 'light quark' mass
m_g = gluon mass (!!)

The a_n dependend logarithmically on all ratios (standard lore: but see G.S., Phys Rev D18, 2773(78). for detailed discussion, also Collins, Soper, St. in 'pQCD' ed. A. Mueller World S. (1978))

If choose $\mu^2 = Q_i^2$

get function of $x_{ij} \equiv \frac{Q_i^2}{Q_j^2} = O(1)$

but also $\frac{m_f^2}{Q_i^2}, \frac{m_g^2}{Q_i^2} = 0, \frac{p_i^2}{Q_i^2}$

31 Ruins pert. expansion in general

- The Way Out:

  Look for quantities <u>independent</u> of p_i^2, m_f^2, m_g^2

INFRARED SAFE QUANTITIES
(IRS)

RG Eqn for IRS σ

$$\sigma\left(\frac{Q_1^2}{\mu^2}, x_{ij}, g(\mu)\right) = \sigma\left(1, x_{ij}, g(Q_1^2)\right)$$

$$= \sum_{n=1}^{\infty} a_n(x_{ij})\, \alpha_s^n(Q_1^2)$$

$$\alpha_s \equiv g^2/4\pi$$

The majority of applications of PQCD are in the computation of IRS quantities.

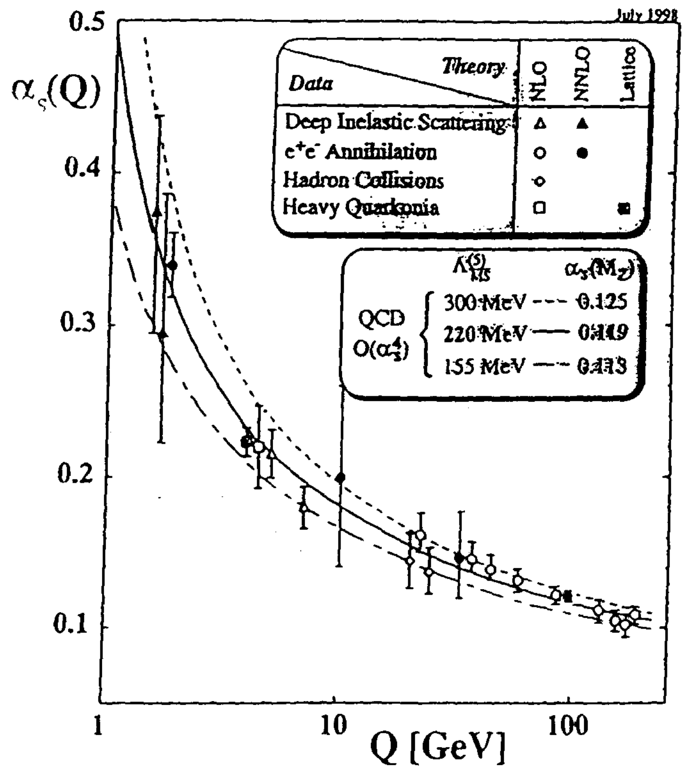
IRS $\leftrightarrow$ momenta $\gg$ masses
"short distance"

MEASURE $\sigma \mapsto$ SOLVE FOR $\alpha_s(Q^2)$

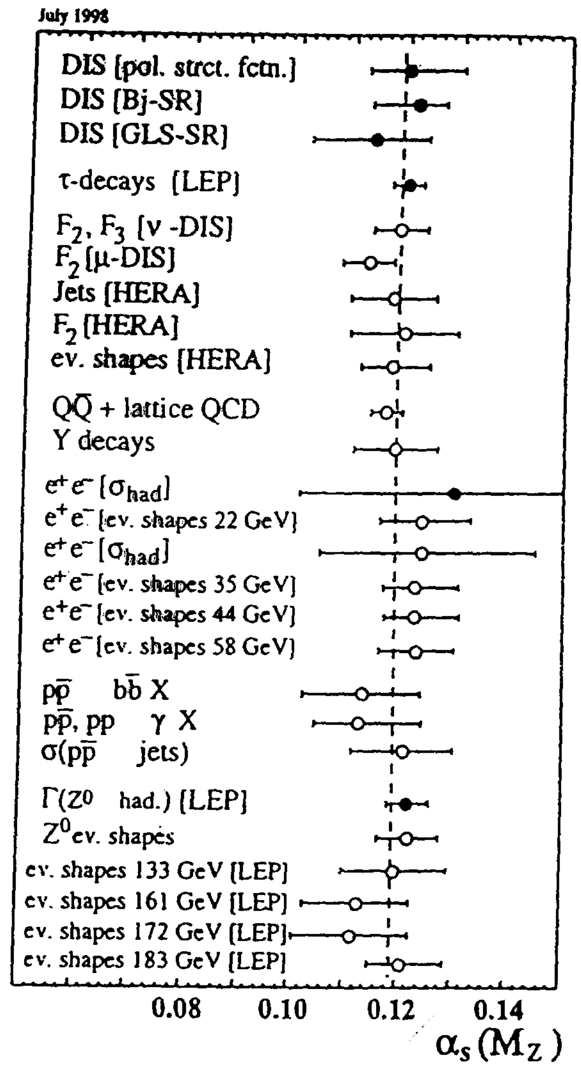
Allows observation of 'running coupling'

The α_s lineup

Given $f's$ (or other NP input where necessary), compare $\hat{\sigma}(\alpha_s)$ to experiment



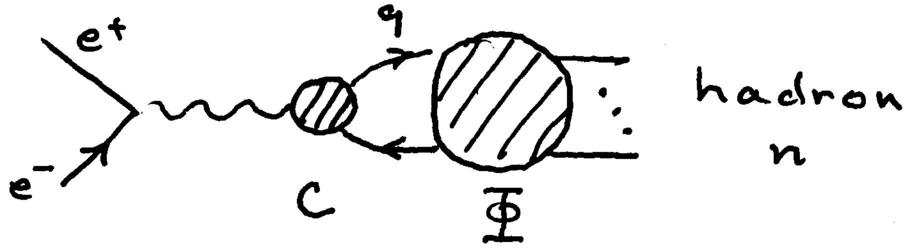
(Bethke 99)



• e^+e^- total cross section

σ_{PT} = IRS ($\Phi = 1$ in notation above)

(a) heuristic picture



'short-distance' C and 'long-distance' Φ have no quantum interference.

$$\Rightarrow \quad P_{e^+e^- \to n} = P_{e^+e^- \to q\bar{q}} * P_{q\bar{q} \to n}$$

classical product of probabilities <u>not</u> amplitudes

$$\Rightarrow \quad \sigma_{tot} = \sum_n P_{e^+e^- \to n}$$

$$= \underbrace{P_{e^+e^- \to q\bar{q}}}_{\text{this is C in this case} \; = \; \sigma_{tot}^{(PT)}} * \boxed{\sum_n P_{q\bar{q} \to n}} \to = 1!$$

$\boxed{\text{KLN Theorem}}$

Note: $\sum_n P_{q\bar{q} \to n} = 1$ is 'unitarity'. Will hold in PT as well as in (hypothetical) exact calculation. But to calculate in PT will need <u>IR REGULATION</u> (compare UV)

- IR Regulation: Why and how?

Test of IR sensitivity:

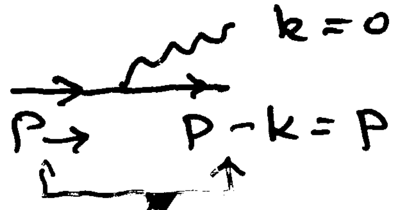
$-\ln \frac{m}{Q} \to \infty$ as $\frac{m}{Q} \to 0$

limit?

$\Rightarrow$ Look for problems in $m=0$ theory

Generic problems at $m^2 = 0 = p^2$

(i)

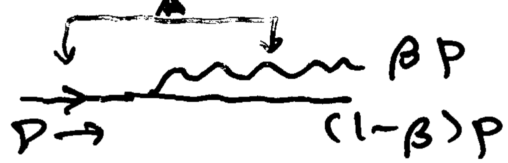


both on-shell

→ long lived states

'infrared divergences'

(ii)



$0 < \beta < 1$

'collinear divergences'

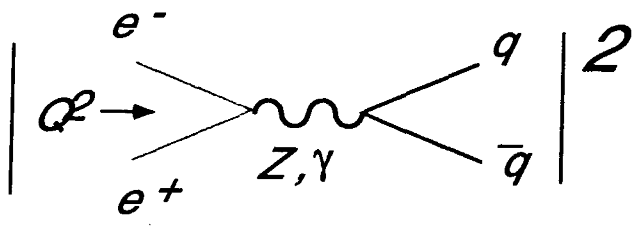
$m=0$ particles are not stable in usual sense. Their interactions just won't quit!

In IR regulated version of theory we 'cut-off' IR (and collinear) divergences by modifying the theory.

Lets see how this works in e^+e^-

Note: IR regulated theory not the same except for IRS quantities

- $e^+e^- \longrightarrow$ hadrons at zeroth order in α_s:
  just $e^+e^- \longrightarrow$ quark + antiquarks

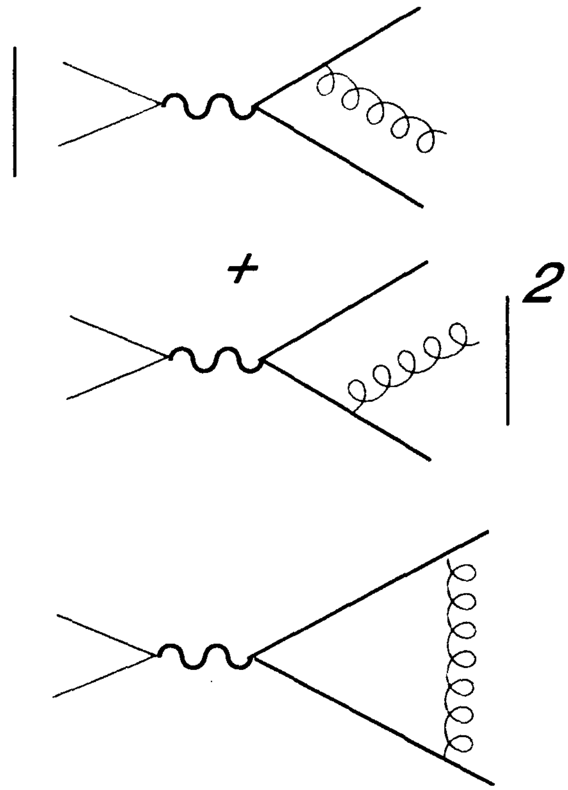


- Total cross section:

$$\sigma^{(0)}_{\mathrm{total}} = N_{\mathrm{colors}} \frac{4\pi\alpha^2_{\mathrm{EW}}}{3Q^2} \sum_f Q_f^2$$

- Nice test of $N_{\mathrm{colors}} = 3$
- But does it really make sense? Yes because σ_{total} is IR Safe
- $e^+e^- \longrightarrow$ hadrons at first order in α_s:
  How does it work?

- The diagrams



- Three-particle final state: proportional to the integral

$$(2\pi)\ \int_0^{Q/2} dk\, k\ \int_0^{\pi} d\theta\ \sin\theta$$
$$\times \left[\frac{Q-(1-u)k^2}{k^2(1-u)^2} + \frac{Q(1+u)}{(Q-2k)(1-u)}\right]$$

$\theta = \theta_{p_1 k}$, $u \equiv \cos\theta$

- Collinear $u \to 0$; infrared (soft gluon) $k \to 0$

$$I_{\mathrm{IR}} \sim (2\pi)\ \int_0 \frac{dk}{k}\ \int_0 \frac{d\theta}{\theta}\ [2Q + 4k]$$

Dimensional regualrization in a nutshell

- (2π): (p_1, k) azimuthal integration
- Interpret: $(2\pi)\, k\, dk \longrightarrow$ volume of ring
- volume of ring $\leftrightarrow$ volume of "sphere" in two dimensions
- Reinterpret

$$(2\pi)\, k \sin\theta \longrightarrow \Omega_d\, (k \sin\theta)^d$$

- where d is ANY complex number

AND Ω is DEFINED BY
$\Omega_d r^d \equiv$ the area of sphere in $d+1$ dimensions.

- For $d > 1$, I becomes finite:

$$\Omega_d \int_0 \frac{dk}{k}\, k^{(d-1)} \int_0 \frac{d\theta}{\theta}\, \theta^{(d-1)}\ [2Q + 4k\,] \sim \frac{1}{(d-1)^2}$$

- Defines a new theory: Dimensionally-regulated QCD
- And $d = 1$ is real QCD
- Often define $\epsilon = 2 - D/2 = (1-d)/2$
  (or $= 4 - D = 1 - d$).
- Do $I_{\rm IR}$, find $1/\epsilon^2$

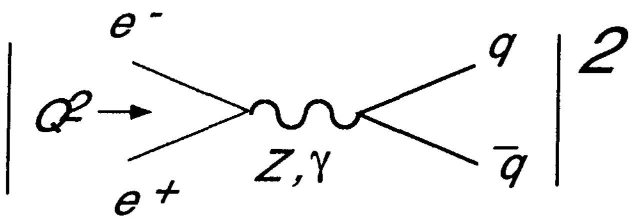
- How it works:

  The UNIQUE solution for Ω_d is:

$$\begin{aligned}\Omega_d &= \frac{2\pi^{(d+1)/2}}{\Gamma\left(\frac{1}{2}(d+1)\right)} \\ &= 2^d \pi^{d/2} \frac{\Gamma\left(\frac{1}{2}d\right)}{\Gamma(d)}\end{aligned}$$

- Check $d = 2, 3, 4$
- *For any integral over virtual or real momenta*
- *a similar "continuation" in dimension can always be found*
- *Sometimes $d = 1$, sometimes $d = 2, = 4 \ldots$*
- *Works as both UV and IR regulator*
- *BUT ONLY IR SAFE QUANTITIES CAN BE EVALUATED AT $\epsilon = 0$, WHERE THEY RETURN TO THE 'REAL' THEORY*

- $e^+e^- \longrightarrow$ hadrons at zeroth order in α_s:
  just $e^+e^- \longrightarrow$ quark + antiquarks



- Total cross section:

$$\sigma^{(0)}_{\text{total}} = N_{\text{colors}} \frac{4\pi\alpha^2_{\text{EW}}}{3Q^2} \sum_f Q_f^2$$

- Nice test of $N_{\text{colors}} = 3$
- But does it really make sense? Yes because σ_{total} is IR Safe
- $e^+e^- \longrightarrow$ hadrons at first order in α_s:
  How does it work?

- IR Regularization Schemes for e^+e^-

  - (i) $\frac{1}{k^2} \rightarrow \frac{1}{k^2 - m_g^2}$ for gluon

  - (ii) dimensional (manifestly preserves gauge invariance)

(i) m_g is 'easy' - all integrals become finite at one loop

find

$$\sigma_3^{(m_g)} = \sigma_0 \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left(2 \ln^2 \frac{Q}{m_g} - 3 \ln \frac{Q}{m_g} + \frac{5}{2} - \frac{\pi^2}{6}\right)$$

$$\sigma_2^{(m_g)} = \sigma_0 \left(1 + \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left(-2 \ln^2 \frac{Q}{m_g} + 3 \ln \frac{Q}{m_g} - \frac{7}{4} + \frac{\pi^2}{6}\right)\right.$$

$$\sigma_{tot} = \sigma_0 \left(1 + \frac{\alpha_s}{\pi}\right) + \mathcal{O}(\alpha_s^2)$$

pretty simple! what about dim. regularization?

Results for Dimensional Regularization for IR and CO divergences!

(for now, just some formulas...)

$$\sigma_3^{(\epsilon)} = \sigma_0 \, \frac{4}{3} \left(\frac{\alpha_s}{\pi} \right) \left(\frac{(1-\epsilon)^2}{(3-2\epsilon)\,\Gamma(2-2\epsilon)} \right)$$

$\epsilon = 2 - n/2$

$$* \left(\frac{4\pi\mu^2}{Q^2} \right)^{2\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{\pi^2}{2} + \frac{19}{2} \right)$$

$$\sigma_2^{(\epsilon)} = -\sigma_0 \, \frac{4}{3} \left(\frac{\alpha_s}{\pi} \right) \left(\frac{(1-\epsilon)^2}{(3-2\epsilon)\,\Gamma(2-2\epsilon)} \right)$$

$$* \left(\frac{4\pi\mu^2}{Q^2} \right)^{2\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{\pi^2}{2} + 4 \right)$$

again, one loop correction is

$$\sigma_0 \left(\frac{\alpha_s}{\pi} \right)$$

lesson σ_{tot} <u>is</u> IRS

(even if σ_2 and σ_3 are very sensitive to long-distance nature of IR regulated theory.)

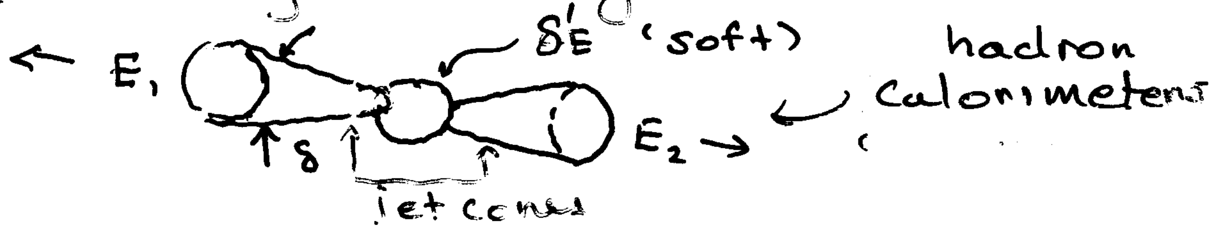
$$\Gamma(z) \equiv \int_0^\infty dx \, x^{z-1} e^{-x}$$

• Jet Cross Sections (e^+e^-)

- heuristic arguments very similar to e^+e^- total
  - note: long-distance interactions possible only for <u>collinear</u> or (long-wavelength) <u>soft</u> particles
    - suggests: summing over states with definite 'jets' of nearly collinear particles + soft particles → IRS cross section in e^+e^-
  - can be made formal using KLN theorem

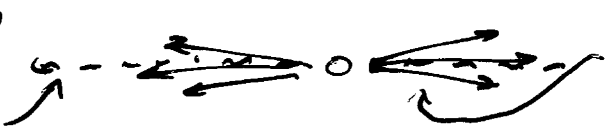
Examples:

(i) energy into angular region



(ii) jet mass, thrust

$$T = \frac{1}{Q} \sum |p_i \cdot \hat{n}_T|$$



Thrust axis $\hat{n}_T$

reconstruct mass from lines

'ancestor' of DURHAM, JADE & related algorithms

Typical Example:

Two-jet cross section in e^+e^-
(begins at α_s^0; dominates as
$Q \to \infty$ since $\alpha_s(Q) \to 0$)

$$\sigma_{2J}^{(PM)} = \frac{3\sigma_0}{8}(1+\cos^2\theta)$$

$$\sigma_{2J}^{(pQCD)} = \frac{3\sigma_0}{8}(1+\cos^2\theta)\left(1 + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n C_n\right)$$

$C_n = C_n(y)$ or $C_n(\delta)$

$y \sim m_J^2/s$

Example: Calorimeter 2-jet cross section

$$\sigma_{2J}^{(Q)} = \frac{3\sigma_0}{8}(1+\cos^2\theta) \cdot \left(1 - \frac{4\alpha_s(Q)}{3\pi}\left(4\ln\delta\ln\delta' + 3\ln\delta + \frac{\pi^2}{3} + \frac{5}{2}\right)\right)$$

only Q dependence

as $Q \to \infty$ $\quad \sigma_{tot} \to \sigma_{tot}^{2J}$

for p-p jets, often use
$\Delta R = \sqrt{\Delta\phi^2 + \Delta\eta^2}$ in place of δ, δ'

4. Classic applications of infrared safety

- Infrared Safe Cross Sections

- Generalizations of PM to Factorized Cross Sections

PM: $\sigma_h^{DIS} = \int d\xi\, \sigma_{Born,a}(Q,\xi)\, \phi_a(\xi)$

pQCD $\sigma_h^{DIS} = \int d\xi\, H_a(Q,\xi)\, \phi_a(\xi, Q)$

IRS: $H_a(Q,\xi) = \sigma_{Born} + \alpha_s(Q) H^{(1)} + \cdots$ note

$\phi_a(\xi, Q)$ depends on Q

etc. for DY

- Evolution

$\phi_a(\xi, Q)$ obeys eq. of form

$$\frac{\partial}{\partial \ln Q} \phi_a(\xi, Q) = \int_\xi^1 d\xi' \, P_{ab}\left(\frac{\xi}{\xi'}, \alpha_s(Q)\right) \phi_b(\xi', Q)$$

$P_{ab}(\xi/\xi', \alpha_s(Q))$ IRS

Allows us to compute Q-dependence (scale breaking) of DIS structure functions, DY cross sections, etc...