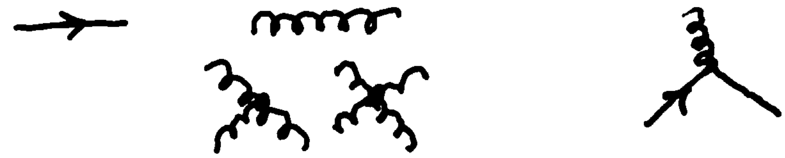
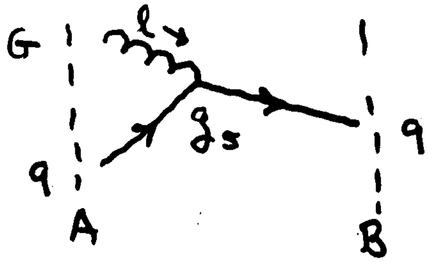
RENORMALIZATION SCHEMES & SCALES ⟶ RUNNING COUPLING
(A MICROINTRODUCTION)

- $$\mathcal{L}_{QCD} = \underset{\downarrow}{\mathcal{L}_q} + \underset{\downarrow}{\mathcal{L}_G} - \underset{\downarrow}{g_s \bar{q}\, \gamma \cdot A\, q}$$

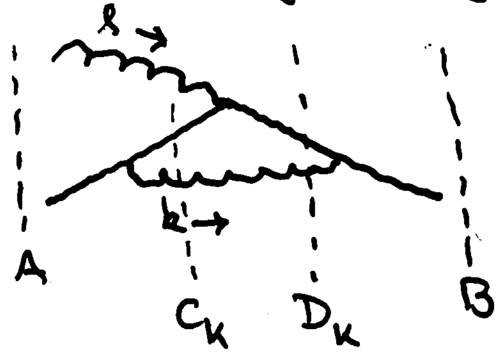


- Every Amplitude Tells a Story
  "state-to-state"



$$M^{(0)}_{A\to B}(\ell) \sim g_s \underset{\text{i.e. } |\vec{\ell}|}{\ell}$$

QM: "every story you can think of"



$$\sum_k M^{(1)}_{A\to C_k \to D_k - B}(\ell, k) \equiv M^{(1)}_{A\to B}(\ell)$$

$$M^{(1)}_{k\gg\ell} \sim \int_\ell^\infty dk\, \underbrace{k^2}_{\#\text{ paths}} \cdot \underbrace{\frac{1}{k^3}}_{\substack{\text{relative}\\ \text{norm}}} \cdot \underbrace{(g_s k)^2}_{M^{(0)}_{A\to C_k}\; M^{(0)}_{D_k\to B}} \cdot \underbrace{\frac{1}{k} \cdot \frac{1}{k}}_{\substack{\text{lifetimes}\\ C_k,\, D_k}} \cdot \underbrace{(g_s \ell)}_{\text{LO}}$$

$$\sim (g_s \ell)\, g_s^2 \int_\ell^\infty \frac{dk}{k}$$

1a' UV Divergence

- Regularization

Change the theory to cut down on high-energy paths!

Dimensional regularization: change the counting of states!

$$\int_{\ell}^{\infty} \frac{dk}{k} \xrightarrow[\text{DR}]{} \int_{\ell}^{\infty} \frac{dk}{k}\left(\frac{k^2}{\mu^2}\right)^{-\epsilon} = \left(\frac{\ell^2}{\mu^2}\right)^{-\epsilon}\frac{1}{2\epsilon}$$

new scale

$$= \frac{1}{2\epsilon} - \ln\frac{\ell}{\mu} + \mathcal{O}(\epsilon)$$

Or - just give $k > \Lambda$ weight zero!

$$\int_{\ell}^{\infty} \frac{dk}{k} \xrightarrow[\text{cutoff}]{} \int_{\ell}^{\Lambda} \frac{dk}{k} = \ln\frac{\Lambda}{\ell}$$

$$\Lambda = e^{\frac{1}{2\epsilon}}\mu$$

DR form

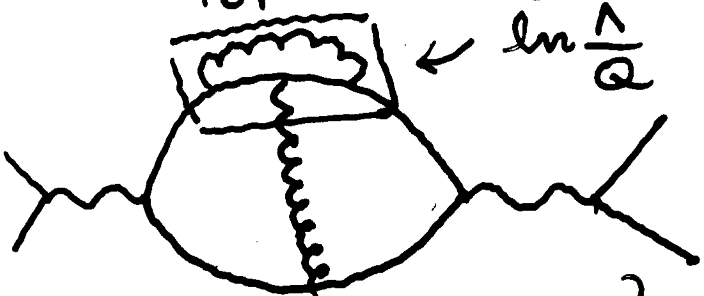
Relation:

$$\left(\frac{k^2}{\mu^2}\right)^{-\epsilon} \approx 1 \quad \text{for } k < \mu e^{\frac{1}{2\epsilon}} \sim \Lambda$$

Coupling of regularized theory:

$$g_s^0 \to \alpha_s^0 \equiv \frac{g_s^{0\,2}}{4\pi} \quad (\text{"bare"})$$

- UV Divergence in $\sigma^{e^+e^-}_{tot}$

$\sigma_{tot} \sim \mathrm{Im}\Big\{$ 

$+ \text{ friends} \Big\}$

In regularized theory:

$$\sigma_{tot} = \sigma_B\left(1 + \frac{\alpha_s^0}{\pi} + \left(\frac{\alpha_s^0}{\pi}\right)^2\left[b_0 \ln\frac{\Lambda}{Q} + s_0\right] + \cdots\right)$$

- Enter the Renormalized Coupling

$$\alpha_s^0 \equiv \alpha_s^R\left(1 - b_0\frac{\alpha_s^0}{\pi}\left[\ln\frac{\Lambda}{\mu} + c_0\right] + \cdots\right)$$

(✱)

MUST have a new <u>scale</u>

THIS is the <u>scheme</u>

change $\mu \longleftrightarrow$ change c_0

In renormalized theory:

$$\sigma_{tot} = \sigma_B\left(1 + \frac{\alpha_s^R}{\pi} + \left(\frac{\alpha_s^R}{\pi}\right)^2\left[b_0 \ln\frac{\mu}{Q} - c_0 + s_0\right] + \cdots\right)$$

★ Independent of Λ!

★ $\alpha_s^R = \alpha_s^R(\mu_0, c_0) \equiv \alpha_s^R(\mu)$

★ Generalizes to all orders

• Systematize: Renormalized $\mathcal{L}$

$$\mathcal{L}_R(\alpha_s^R) = \mathcal{L}(\alpha_s^0)$$

$$= \mathcal{L}_q + \mathcal{L}_G - g_s^R\, \bar q\, \gamma\cdot A\, q$$


g_s

$$-(g_s^0 - g_s^R)\, \bar q\, \gamma\cdot A\, q \leftarrow \text{"counter-term"}$$

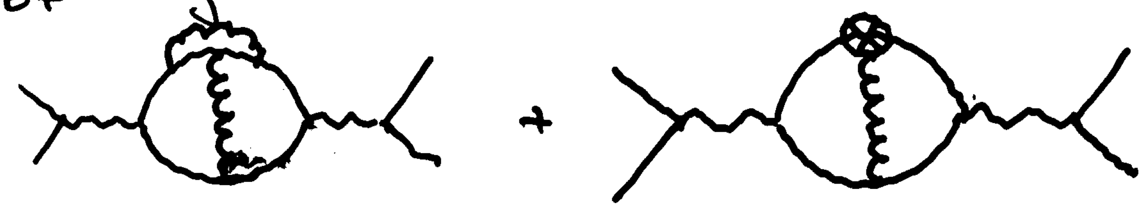

$g_s^0 - g_s^R$

$$g_s^0 - g_s^R = \sqrt{4\pi\alpha_s^0} - \sqrt{4\pi\alpha_s^R}$$

$$= \sqrt{4\pi\alpha_s^R\left(1 - \alpha_s^R\left[b_0 \ln\frac{\Lambda}{\mu} + c_0\right]\right)} - \sqrt{4\pi\alpha_s^R}$$

$$\cong -\frac{1}{2} g_s^R \alpha_s^R \left[b_0 \ln\frac{\Lambda}{\mu} + c_0\right] + \cdots$$

σ_{tot} again: finite!



• The β-function:

$$\beta(\alpha_s) \equiv \mu \frac{\partial}{\partial \mu} \alpha_s^R(\mu)$$

solve (*) previous page

$$= \mu \frac{\partial}{\partial \mu} \frac{\alpha_s^0}{1 - b_0 \alpha_s^0 \left(\ln\frac{\Lambda}{\mu} + c_0\right) + \cdots}$$

$$= -\alpha_s^0 \frac{+b_0(\alpha_s^0/\pi)}{(\alpha_s^0/\alpha_s^R)^2} + \cdots$$

$$= -\frac{b_0}{\pi} \alpha_s^{R\,2} + \cdots$$

↑ indep. of c_0 even to $\alpha_s^{R\,3}$ (if no μ in c_0!)

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