Resummation: Vector Bosons, the Higgs and Beyond (I)

CTEQ Summer School
2006

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- Resummation: Organization of soft and collinear radiation to all orders in PT

- Why resum?
  - Tests of perturbative stability for inclusive cross sections.
  - The only way to calculate certain critical distributions for W, Z transverse momentum & jet event shapes.
  - As such, tests of QCD to *all* orders: LO, NLO . . .
  - A window to the perturbative/nonperturbative transition.
  - An analytic complement to, stimulus for & test of parton shower techniques and tools.
  - Nice formulas (a matter of taste).

- Depends on very some general concepts too.

OUTLINE

Lecture 1

I. Review: how we get away with pQCD

II. The physical basis of factorization

III. Vector bosons: Q_T & and its factorization

Lecture 2

IV. Threshold resummation

V. Resummed jet shapes and $1/Q$ corrections

VI. Resummation and the Higgs

VII. Generalizations and limitations

I. Review: how we get away with perturbative QCD

The sorrows of QCD perturbation theory:

1. **Confinement**

$$\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[\phi_a(x)\dots]\,|0\rangle$$

   has no $q^2 = m^2$ pole for any field (particle) ϕ_a that transforms nontrivially under color (confinement)

2. **The pole at $p^2 = m_\pi^2$**

$$\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[\pi(x)\dots]\,|0\rangle$$

   is not accessible to perturbation theory (χSB etc., etc.)

- **And yet we use infrared safety & asymptotic freedom:**

$$Q^2\ \hat{\sigma}_{\mathrm{SD}}(Q^2,\mu^2,\alpha_s(\mu)) = \sum_n c_n(Q^2/\mu^2)\ {\alpha_s}^n(\mu) + \mathcal{O}\left(1/Q^p\right)$$
$$= \sum_n c_n(1)\ {\alpha_s}^n(Q) + \mathcal{O}\left(1/Q^p\right)$$

- **What are we really calculating? PT for color singlet operators**

  – $\int \mathrm{e}^{-iq\cdot x}\langle 0|\, T[J(x)J(0)\ldots]\,|0\rangle$ **for color singlet currents**

  $\mathrm{e}^+\mathrm{e}^-$ **total, sum rules etc. "no scale"**

– **Another class of color singlet matrix elements:**

$$\lim_{R\to\infty}\int dx_0 \int d\hat{n}\, f(\hat{n})\, \mathrm{e}^{-iq\cdot y}\langle 0|\, J(0) T[\hat{n}_i \Theta_{0i}(x_0, R\hat{n}) J(y)]\, |0\rangle$$

With Θ_{0i} the energy momentum tensor

– **These are what we really calculate: jet cross sections, etc.**

If the "weight" $f(\hat{n})$ introduces no new dimensional scale,

and all $d^k f/d\hat{n}^k$ bounded, then

individual final states have IR divergences, but these cancel in sum over collinear splitting/merging & soft parton emission because they respect energy flow.

We regularize these divergences dimensionally (typically) and "pretend" to calculate the long-distance enhancements only to cancel them in infrared safe quantities

It is this intermediate step that makes the calcualtions tough, and is part [not all] of why higher-order calculations are hard!

The goals of experiment are remarkably similar – to control late stage interactions in calorimeters.

Resummation organizes large, or potentially large, terms from high orders in α_s at the short-distance scale.

- **Onward to factorization**

$$Q^2\sigma_{\rm phys}(Q,m) = \omega_{\rm SD}(Q/\mu,\alpha_s(\mu)) \otimes f_{\rm LD}(\mu,m) + \mathcal{O}\left(1/Q^p\right)$$

  - μ = **factorization scale;** m= **IR scale (**m *may* **be perturbative)**
  - **New physics in** $\omega_{\rm SD}$**;** $f_{\rm LD}$ **"universal"**
  - **ep DIS inclusive,** $\rm pp \rightarrow$ **jets,** $Q\bar{Q}$**,** $\pi(p_T)$ **. . .**
  - **Exclusive decays:** $B \rightarrow \pi\pi$
  - **Exclusive limits:** $\rm e^+e^- \rightarrow JJ$ **as** $m_J \rightarrow 0$

- **Whenever there is factorization, there is evolution**

$$0 = \mu \frac{d}{d\mu} \ln \sigma_{\mathrm{phys}}(Q, m)$$

$$\mu \frac{d \ln f}{d\mu} = -P(\alpha_s(\mu)) = -\mu \frac{d \ln \omega}{d\mu}$$

PDF f <u>or</u> Fragmentation D

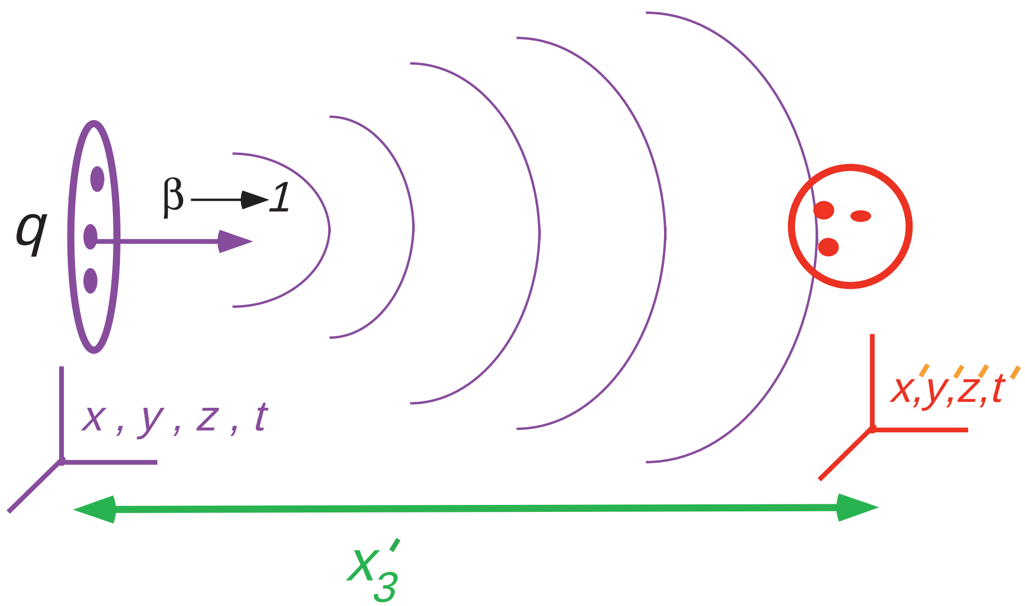
- **Wherever there is evolution there is resummation**

$$\ln \sigma_{\mathrm{phys}}(Q, m) = \exp \left\{ \int_q^Q \frac{d\mu'}{\mu'} P\left(\alpha_s(\mu')\right) \right\}$$

- **Infrared safety & factorization proofs**:
  - **(1) $\omega_{\rm SD}$ incoherent with long-distance dynamics**
  - **(2) Mutual incoherence when $v_{\rm rel} = c$:**
    Jet-jet factorization Ward identities.
  - **(3) Wide-angle soft radiation sees only total color flow:**
    jet-soft factorization Ward identities.
  - **(4) Dimensionless coupling and renormalizability**
    $\Leftrightarrow$ no worse that logarithmic divergence in the IR:
    fractional power suppression $\Rightarrow$ finiteness

II. The physical basis of factorization

Classical picture



$$\Delta \equiv \beta c t' - x_3'$$

– **Why a classical picture isn't far-fetched . . .**

The correspondence principle is the key to the origin of IR divergences.

– **Any accelerated charge must produce classical radiation,**

and infinite numbers of soft gluons are required to make a classical field.

Transformation of a scalar field:

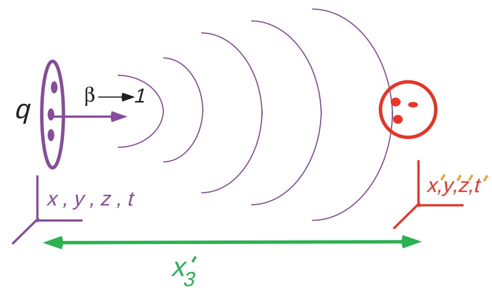
$$\phi(x) \;=\; \frac{q}{\sqrt{x_T^2 + x_3^2}} \;=\; \phi'(x') \;=\; \frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$$

From the Lorentz transformation: $x_3 = \gamma(\beta c t' - x'_3) \equiv \gamma \Delta'$.

Closest approach is at $\Delta' = 0$, **i.e.** $t' = \frac{1}{\beta c} x'_3$.

The scalar field transforms "like a ruler": At any fixed $\Delta' \neq 0$, the field decreases like $1/\gamma = \sqrt{1-\beta^2}$.

Why? Because when the source sees a distance x_3, the observer sees a much larger distance.



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
gauge $(-)$	$A^-(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^-(x') = \frac{-q\gamma(1+\beta)}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{q}{\lvert\vec{x}\rvert^2}$	$E_3'(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}}$
Gauge fields :	$A_3 \sim \gamma^0,$	$E_3 \sim \gamma^{-2}$

- The “gluon” $\vec{A}$ is enhanced, yet is a total derivative:

$$A^\mu = q\frac{\partial}{\partial x_\mu}\ \ln\left(\beta c t' - x_3\right) + \mathcal{O}(1-\beta) \sim A^-$$

- A^- is an unphysical polarization & can be removed by a gauge transformation!

- The “force” $\vec{\mathrm{E}}$ field of the incident particle does not overlap the “target” until the moment of the scattering.

- “Advanced” effects are corrections to the total derivative:

$$1-\beta \ \sim\ \frac{1}{2}\left[\sqrt{1-\beta^2}\right]^2 \ \sim\ \frac{m^2}{2E^2}$$

- Power-suppressed! These are corrections to factorization.

- Initial-state interactions decouple from hard scattering
- Summarized by multiplicative factors: the parton distributions
- Interactions after the scattering are too late to affect large momentum transfer, creation of heavy particle, etc.
- Fragmentation of partons to jets too late to know details of the hard scattering: factorization of fragmentation functions.
- $\Rightarrow$ Cross section for hard scattering is IR safe,
- with power-suppressed corrections.

- **The gauge-theory analog of our classical argument is the universal soft-parton factor:**

  For soft gluon k emitted by fast quark p, Dirac eq. gives:

$$\bar{u}(p)\,(-ig_s\,\gamma^\mu\,)\,\frac{\not p + \not k\ +\ m}{(p+k)^2\ -\ m^2}\ =\ \bar{u}(p)\,(-ig_s\,)\,\frac{p^\mu}{p\cdot k}\ +\ (IR\ finite)$$

  In a diagram p^μ will be contracted with a gluon propagator,

  and in $p \cdot A = 0$ gauge, this term vanishes!

$$G^{\nu\mu}(k) = -\left(g^{\nu\mu}\ -\frac{p^\nu\,k^\mu\ +\ k^\nu\,p^\mu}{p\cdot k}\ +\ p^2\,\frac{k^\nu\,k^\nu}{(p\cdot k)^2}\right)$$

- **Notice this gauge depends on the momentum p.**

- **The origin of the "universality" of soft gluon interactions.**

- **But it is the same for every parton in a jet.**

A good example is (we'll come back to this in resummation) pions at measured transverse momentum. PDFs $\otimes$ hard scattering $\otimes$ fragmentation functions:

$$\frac{p_T^3\, d\sigma(x_T)}{dp_T} = \sum_{a,b,c} \int_0^1 dx_1\, f_{a/H_1}\left(x_1, \mu_F^2\right) \int_0^1 dx_2\, f_{b/H_2}\left(x_2, \mu_F^2\right)$$
$$\times \int_0^1 dz\, z^2\, D_{h/c}\left(z, \mu_F^2\right)$$
$$\times \int_0^1 d\hat{x}_T\, \delta\left(\hat{x}_T - \frac{x_T}{z\sqrt{x_1 x_2}}\right) \int_{\hat{\eta}_-}^{\hat{\eta}_+} d\hat{\eta}\, \frac{\hat{x}_T^4\, \hat{s}}{2}\, \frac{d\hat{\sigma}_{ab\to cX}(\hat{x}_T^2, \hat{\eta})}{d\hat{x}_T^2 d\hat{\eta}}$$

with

$$x_T^2 = \frac{4p_T^2}{S}$$
$$\hat{\eta}_+ = -\hat{\eta}_- = \ln\left[(1+\sqrt{1-\hat{x}_T^2})/\hat{x}_T\right]$$

III. Vector bosons: Q_T and its factorization

Every final state from a hard scattering carries the imprint of QCD dynamics from at all distance scales

– One loop corrections: talk by L. Reina

– Look at transverse momentum distribution at order α_s

$$q(p_1) + \bar{q}(p_2) \longrightarrow \gamma^*(Q) + g(k)\,,$$

– Treat this $2 \to 2$ process at lowest order (α_s) "LO"
in factorized cross section, so that $\mathbf{k} = -\mathbf{Q}_T$

– **Factorized cross section at fixed $\mathbf{Q}_T$:**

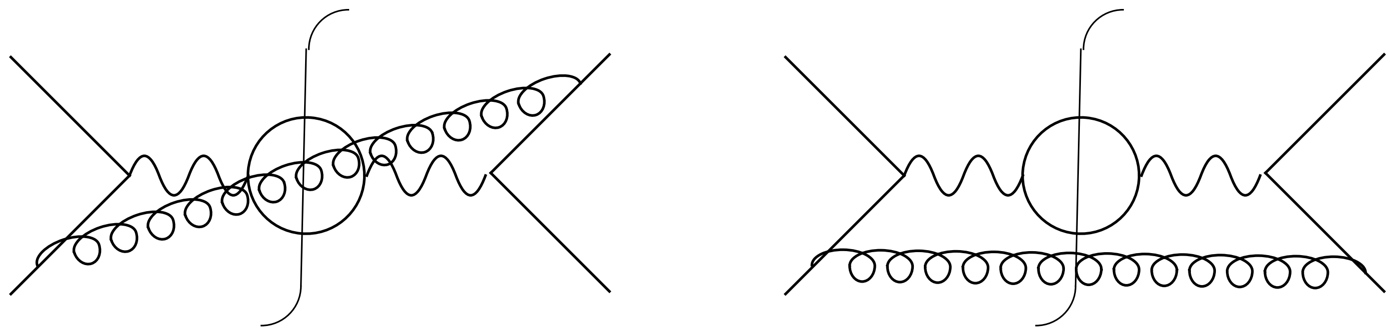
$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,p_1,p_2)}{dQ^2 d^2\mathbf{Q}_T} = \int_{\xi_1,\xi_2} \sum_{a=q\bar{q}} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu,\xi_1 p_1,\xi_2 p_2,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} \times f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

– **Recall:** μ **is the factorization scale that separates IR (f) from UV ($d\hat{\sigma}$) in quantum corrections.**

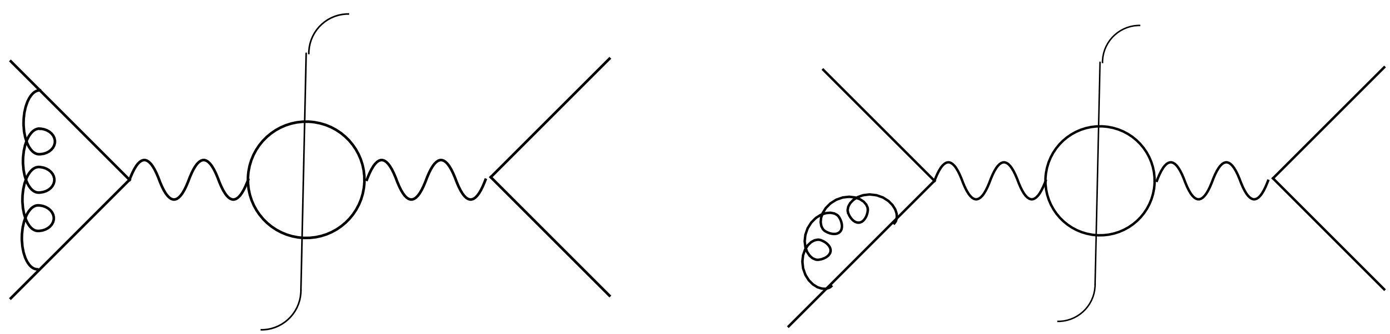
– μ **appears in** $\hat{\sigma}$ **through** $\alpha_s(\mu)$ **and** $\ln(\mu/Q)$ **so choosing** $\mu \sim Q$ **can improve perturbative predictions.**

– **Evolution:** $\mu df(x,\mu)/d\mu = \int_x^1 P(x/\xi)\, f(\xi,\mu)$ **makes energy extrapolations possible.**

– **The diagrams at order** α_s
Gluon emission contributes at $Q_T \neq 0$



Virtual corrections contribute only at $Q_T = 0$



– **The result is finite for** $\mathbf{Q}_T \neq 0$ **. . .**

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}\to\gamma^* g}}{dQ^2\, d^2\mathbf{Q}_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left(1 - \frac{4\mathbf{Q}_T^2}{(1-z)^2 \xi_1 \xi_2 S}\right)^{-1/2}$$

$$\times \left[\frac{1}{\mathbf{Q}_T^2} \frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2}\right]$$

as long as $\mathbf{Q}_T \neq 0$, $z = Q^2/\xi_1\xi_2 S \neq 1$.

Q_T **integral** $\rightarrow \frac{\ln(1-z)}{1-z}$; z **integral** $\rightarrow \frac{\ln \mathbf{Q}_T^2}{\mathbf{Q}_T^2}$.

Both singularities cancel in the inclusive cross section.
Both inspire resummation of higher order corrections.

The leading singularity in $\mathbf{Q}_T$

– **As we'll see later:** $1 - z \sim 2k_0/Q \geq 2|\mathbf{k}_T|/Q$

– z **integral: If Q^2/S not too big, PDFs nearly constant:**

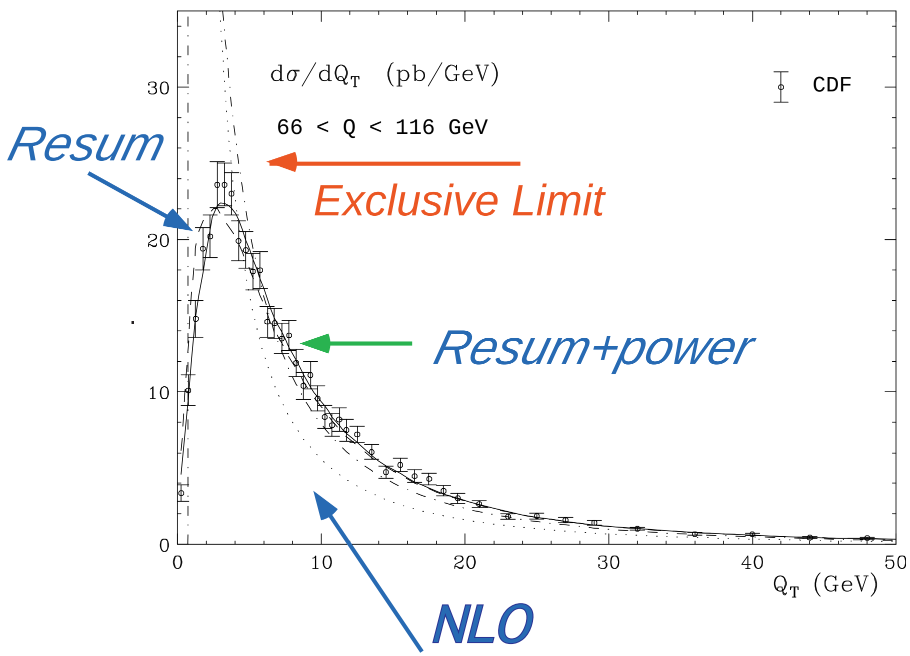
$$\frac{1}{\mathbf{Q}_T^2} \int_{1-Q^2/S}^{1-\mathbf{Q}_T^2/Q^2} \frac{dz}{1-z} = \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$\Rightarrow$ **Prediction for Q_T dependence:**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\alpha_s C_F}{\pi} \frac{1}{\mathbf{Q}_T^2} \ln\left[\frac{Q^2}{\mathbf{Q}_T^2}\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2} \, f_{a/N}(\xi_1,\mu) \, f_{\bar{a}/N}(\xi_2,\mu)$$

– **Compare to: Z p_T from Run I**



(from Kulesza, G.S., Vogelsang (2002))

– $\ln Q_T/Q_T$ **works pretty well for large** Q_T

– **At smaller** Q_T **reach a maximum, then a decrease near "exclusive" limit (parton model kinematics)**

– **Most events are at "low"** $Q_T \ll Q = m_Z$.

Getting to $Q_T \ll Q$: Transverse momentum resummation

(Logs of Q_T)/Q_T to all orders

How? Variant factorization and separation of variables

q and $\bar{q}$ "arrive" at point of annihilation with transverse momentum of radiated gluons in initial state.

q and $\bar{q}$ radiate independently (fields don't overlap!).

Final-state QCD radiation too late to affect cross section

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T}$$

Summarized by: Q_T-factorization:

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} = \int d\xi_1 d\xi_2\ d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}_{sT}\, \delta\left(Q_T - k_{1T} - k_{2T} - k_{sT}\right)$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})\ U_{a\bar{a}}(k_{sT}, n)$$

The $\mathcal{P}'s$: new Transverse momentum-dependent PDFs

Also need U: "soft function" for wide-angle radiation

Symbolically:

$$\frac{d\sigma_{NN\to QX}}{dQd^2Q_T} \qquad H \times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, k_{1T})\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, k_{2T})$$

$$\otimes_{\xi_i, k_{iT}}\; U_{a\bar{a}}(k_{sT}, n)$$

We will solve for the k_T dependence of the $\mathcal{P}$'s.

New factorization variables: n^μ apportions gluons k:

$$p_i \cdot k < n\cdot k \;\Rightarrow\; k \;\in \mathcal{P}_i$$

$$p_a\cdot k, p_{\bar{a}}\cdot k \;>\; n\cdot k \;\Rightarrow\; k \in U$$

Convolution in $k_{i,T}$s $\Rightarrow$ Fourier $\mathrm{e}^{i\vec{Q}_T\cdot\vec{b}}$

The factorized cross section in "impact parameter space":

$$\frac{d\sigma_{NN\to QX}(Q,b)}{dQ} = \int d\xi_1 d\xi_2$$
$$\times\ H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar{a}\to Q+X}$$
$$\times \mathcal{P}_{a/N}(\xi_1, p_1\cdot n, b)\, \mathcal{P}_{\bar{a}/N}(\xi_2, p_2\cdot n, b)\ U_{a\bar{a}}(b,n)$$

Now we can resum by separating variables!

the LHS independent of $\mu_{\rm ren}$, n $\Rightarrow$ two equations

$$\mu_{\rm ren}\frac{d\sigma}{d\mu_{\rm ren}} = 0 \qquad n^\alpha \frac{d\sigma}{dn^\alpha} = 0$$

Method of Collins and Soper, and Sen (1981)

Change in $\mathcal{P}$ must cancel change in (UV) H and (IR) U:

$$p\cdot n\,\frac{\partial}{\partial p\cdot n}\,\ln\,\mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(b\mu)$$

G matches H, K matches U. Renormalization indep. of n^μ:

$$\mu\,\frac{\partial}{\partial\mu}\left[\,G(p\cdot n/\mu) + K(b\mu)\,\right] = 0$$

$$\mu\,\frac{\partial}{\partial\mu}G(p\cdot n/\mu) \;=\; A(\alpha_s(\mu)) \;=\; -\,\mu\,\frac{\partial}{\partial\mu}K(b\mu)$$

Solve this one first.

$$G(p\cdot n/\mu) + K(b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A_a(\alpha_s(\mu'))$$

Notice the scale in the coupling is now a variable.

Now the consistency equation is

$$p\cdot n \frac{\partial}{\partial p\cdot n} \ln \mathcal{P}(p\cdot n/\mu, b\mu) = G(p\cdot n/\mu) + K(\mu/p\cdot n) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'} A(\alpha_s(\mu'))$$

Integrate $p\cdot n$ **and get double logs in** $b \to \alpha_s^n \frac{\ln^{2n-1}(Q/Q_T)}{Q_T}$**.**

Transformed solution back to Q_T: all the (Logs of Q_T)/Q_T:

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[\, E_{a\bar{a}}^{\mathrm{PT}}(b,Q,\mu)\,\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\, f_{a/N}(\xi_1,1/b)\, f_{\bar{a}/N}(\xi_2,1/b)$$

"Sudakov" exponent suppresses large $b \leftrightarrow$ small Q_T:

$$E_{a\bar{a}}^{\mathrm{PT}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T))\,\ln\left(\frac{Q^2}{k_T^2}\right) + 2B_q(\alpha_s(k_T))\right]$$

With $B = 2(K+G)_{\mu=p\cdot n}$, and lower limit: $1/b$ (NLL)

* **Leading log: fixed $\alpha_s(Q)$, $A^{(1)}(\alpha_s/\pi)$ only**

$$\frac{d\sigma_{NN\mathrm{res}}}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}} \exp\left[-\ A^{(1)}(\alpha_s(Q)/\pi)\ \ln^2(bQ)\right]$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\ f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

* **If ignore evolution of the f's, get an overall factor**

$$\frac{d\sigma_{NN\to\mu^+\mu^-+X}(Q,\mathbf{Q}_T)}{dQ^2 d^2\mathbf{Q}_T} = \frac{\partial}{\partial Q_T^2}\ e^{-\left[\ A^{(1)}(\alpha_s(Q)/\pi)\ \ln^2(Q^2/Q_T^2)\right]}$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1\xi_2} \frac{\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\ f_{a/N}(\xi_1,\mu)\, f_{\bar{a}/N}(\xi_2,\mu)$$

$*$ **Comments:**

The functions $A_i(\alpha_s)$ and $B_i(\alpha_s)$ are anomalous dimensions.

And can be calculated by comparison to low orders.

In particular, $A_i(\alpha_s)$ is the numerator of the $1/(1-x)$ term in splitting function $P_{ii}(x)$

because it's the infrared divergent $(x \to 1)$ coefficient of the collinear $b \to \infty$ singularity.

$*$ $A_q(\alpha_s) = \frac{\alpha_s}{\pi} C_q \left(1 + \frac{\alpha_s}{\pi} K + \ldots\right)$, $K = C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5n_F}{9}$

$*$ **Logs from LO, NLO in** $A_q = A_q^{(1)}(\alpha_s/\pi) + \ldots$, **LO in** B_q

$$E_{q\bar{q}} = -\,2\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\ \left[A_q(\alpha_s(k_T))\ \ln\left(\frac{Q^2}{k_T^2}\right)\ +\ B(\alpha_s(k_T))\right]$$

$$\sim 2C_i\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\ \left[\left\{\frac{\alpha_s(k_T)}{\pi} + K\,\frac{\alpha_s(k_T)}{\pi}\right\}\ln\left(\frac{Q^2}{k_T^2}\right)\ + 2\frac{\alpha_s(k_T)}{\pi}\right]$$

$$\sim 2C_i\,\frac{\alpha_s(Q)}{\pi}\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2}\left[\ \left\{1 + \left(\frac{\alpha_s(Q)}{\pi}\right)\ (K -\ \beta_0)\right\}\ln\left(\frac{Q^2}{k_T^2}\right)\right.$$

$$\left. +2\,\frac{\alpha_s(Q)}{\pi}\right]$$

* **The pattern:**

$$
\begin{aligned}
2C_i \frac{\alpha_s(Q)}{\pi} \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \Bigg[\left\{ 1 + \left(\frac{\alpha_s(Q)}{\pi}\right) \left(K - \frac{\beta_0}{4\pi}\right) \right\} \ln\left(\frac{Q^2}{k_T^2}\right) \\
+2\frac{\alpha_s(Q)}{\pi} \Bigg] \\
\sim \alpha_s \ln^2(bQ)(1 + \alpha_s \ln(bQ) + \dots) \\
+\alpha_s \ln(bQ)(1 + \alpha_s \ln(bQ) + \dots) \\
+ \dots
\end{aligned}
$$

* **These are LL($A^{(1)}$), NLL ($B^{(1)}$, $A^{(2)}$), and so on**

* **NLL is good so long as $\alpha_s(Q) \ln bQ \leq 1$.**

* Evaluating a resummed cross sections: re-enter NPQCD.

We start with:

$$E^{\rm PT} = - \int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2A_q(\alpha_s(k_T)) \ln\left(\frac{Q^2}{k_T^2}\right) + B_q(\alpha_s(k_T)) \right]$$

With running coupling:

$$\alpha_s(k_T) = \frac{\alpha_s(Q)}{1 + \frac{\alpha_s(Q)}{4\pi}\beta_0 \ln\left(\frac{k_T^2}{Q^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{k_T^2}{\Lambda^2_{\rm QCD}}\right)}$$

Singularity in integral at $b^2 = Q^2\ \exp[-4\pi/\beta_0\alpha_s(Q)] \sim \frac{1}{\Lambda^2}$.

* **Problem:** **how to do the inverse transform with the running coupling when $k_T^{\min} \sim 1/b$ gets small?**

* **At least four approaches:**

1) Work in Q_T-space directly to some approximation
The originals: Dokshitzer, Diakanov & Troyan
Revived by Ellis & Veseli Kulesza & Stirling
who re-derived it from b-space.

2) Insert a "soft landing" on the k_T integral by replacing

$$1/b \to \sqrt{1/b^2 + 1/b_*^2}$$

for some fixed b_*. (CS, CSS "b_*" prescription, ResBos)

3) Extrapolation of $E^{\rm PT}$ into NP region (Qiu, Zhang).

4) Minimal: avoid the singularity at $1/b = \Lambda_{\rm QCD}$
by monkeying with the b-space contour integral.
(This technique introduced in threshold resummation;
then adapted by Laenen, GS and Vogelsang,
and Bozzi, Catani, de Florian and Grazzini.)

Any of these “define” PT. All will fit the data
qualitatively, and with a little work quantitatively.

But all require new parameters for quantitative fit.
This is not all bad . . . let’s see why.

A bit more consideration generalizes (for the A-term) for small k_T to some upper limit μ_I:

$$E^{\rm soft} = \int_0^{\mu_I^2} \frac{dk_T^2}{k_T^2}\, A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right)\, \left(e^{i\mathbf{b}\cdot\mathbf{k}_T} - 1\right)$$

$$\sim -\int_0^{\mu_I^2} \frac{dk_T^2}{k_T^2}\, (\mathbf{b}\cdot\mathbf{k}_T)^2 A_q(\alpha_s(k_T))\, \ln\left(\frac{Q^2}{k_T^2}\right) + \cdots$$

$\theta(k_T - 1/b) \Rightarrow (e^{i\mathbf{b}\cdot\mathbf{k}_T} - 1)$**; in fact, correct to all orders,**

but the expansion is for b " small enough" only.

What is $\int_0^{\mu_I^2} dk_T^2\ \alpha_s(k_T)$?

Don't really know, but suggests a nonperturbative correction of the form (exhibiting the μ_I is unconventional)

$$E^{\rm NP} = b^2\mu_I^2\left(g_1\ \ln\left(\frac{Q}{\mu_I}\right) + g_2\right)$$

Since this is an exponent, whatever the definition of the pertrubative resummed cross section, it is smeared with a Gaussian whose width in b (k_T) space decreases (increases) with $\ln Q$.

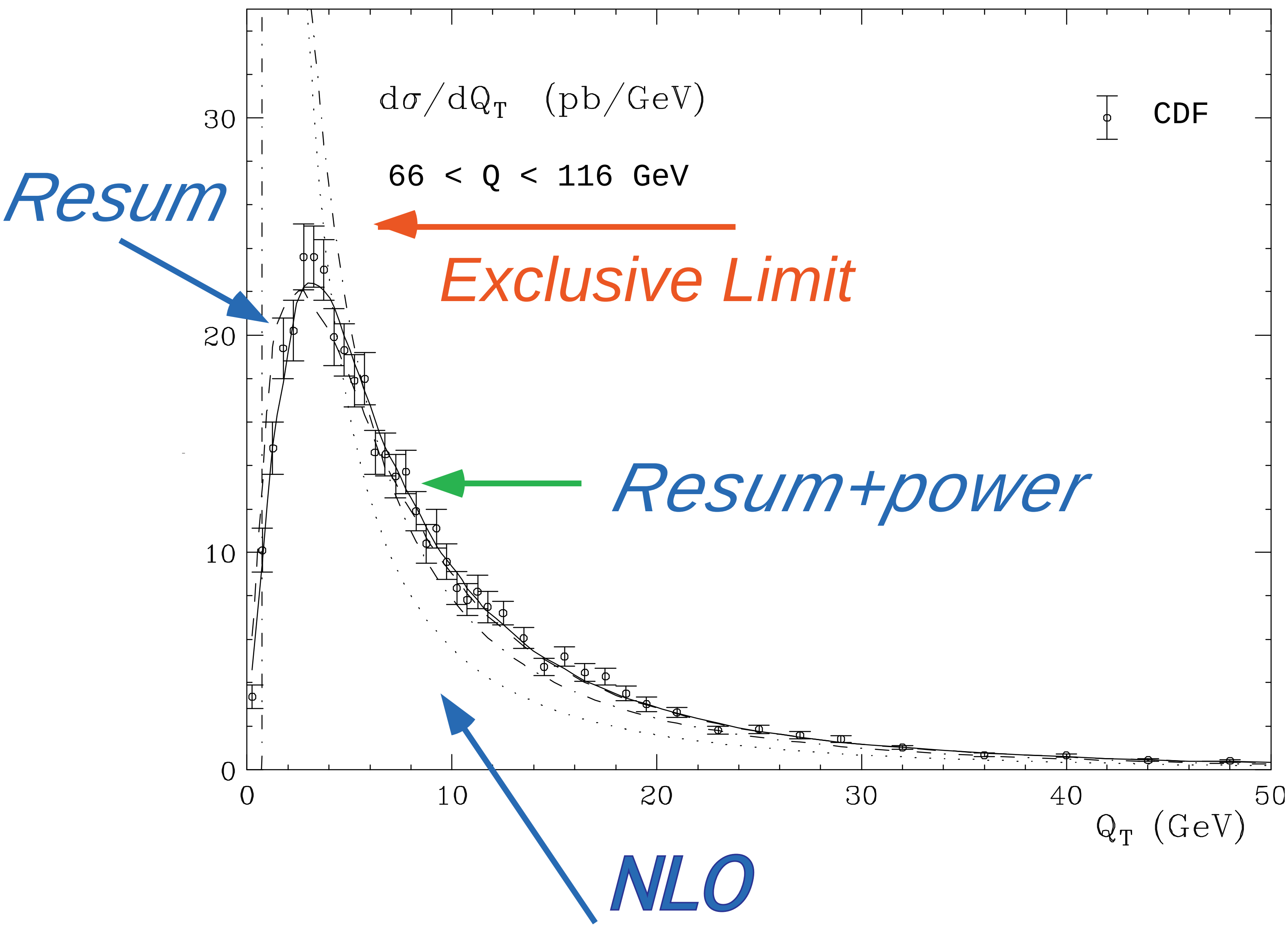
In summary

$$\frac{d\sigma(Q_T)}{dQ^2\, d^2\vec{Q}_T} = \sum_a H_{a\bar{a}}(\alpha_s(Q^2)) \int \frac{d^2 b}{(2\pi)^2}\, e^{i\vec{Q}_T\cdot\vec{b}}\, e^{E^{\mathrm{PT}}_{a\bar{a}}(b,Q,\mu)}\; e^{\,-\,\mu_I^2 b^2 \left(g_1\, \ln\left(\frac{Q}{\mu_I}\right) + g_2\right)}$$

$$\times \sum_{a=q\bar{q}} \int_{\xi_1 \xi_2} \frac{d\hat{\sigma}_{a\bar{a}\to\mu^+\mu^-(Q)+X}(Q,\mu)}{dQ^2}\; f_{a/N}(\xi_1, 1/b)\, f_{\bar{a}/N}(\xi_2, 1/b)$$

$$= \;\pi \int d^2\mathbf{k}_T\; \frac{e^{-k_T^2/4[\mu_I^2(g_2 \ln(Q/k_T)+g_2)]}}{\mu_I^2(g_2 \ln(Q/k_T) + g_2)}\; \frac{d\sigma_{NN}(\mathbf{Q_T} - \mathbf{k_T})}{dQ^2\, d^2\vec{Q}_T}$$

Which gives curves like the one we saw before.



Successful phenomenology for W and Z.
In principle, can also fit to fixed-target Drell-Yan with the same set of NP parameters.

Qiu and Zhang show that NP corrections are dominant for that range of Q^2.

Next – what about those $1/(1-z)$ (soft gluon energy) singularities?

$\ast$ Continue with threshold resummation . . .