Introduction to Perturbative QCD
— Foundation and Simple Applications

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Wu-Ki Tung
I : Basic Ideas

- What is Quantum Chromodynamics (QCD)?
- Why do we believe in Quarks and Gluons?
  - Long-distance physics: the constituent QM;
  - Short-distance physics: the Parton Picture.
  QCD provides the foundation for both.
- How can a Strong-interaction theory, QCD, give rise to the simple Parton Picture?
  Separation of distance scales (Factorization):
  - Ultra-violet Renormalization and Asymptotic Freedom; (very short distance)
II: PQCD at Work: $e^+e^-$ Annihilation

- Order $\alpha_s^0$ (LO) process and the Parton Model;
- Order $\alpha_s^1$ perturbative calculation (NLO QCD correction):
  - Colinear and Soft Singularities;
  - Infra-red Safe (IRS) Physical Observables;
  - Factorization of (non-IRS) Physical Observables into IRS (short-distance, calculable) and Universal (long-distance) pieces.
- General Statement of the Factorization Theorem and its physical interpretation.
III: PQCD at Work: Deep Inelastic Scattering

- Cross Sections and Structure Functions;
- Order $\alpha_s^0$ (LO) processes and the Parton Model;
  - Parton Distributions; Sum Rules.
- Order $\alpha_s^1$ (NLO) QCD corrections:
  A Simple Illustrative Example:
  - Physical origin of the Colinear Singularity—relation to mass singularities;
  - The Structure Function as a convolution of a (IRS) Wilson Coefficient and a (universal) Parton Distribution Function.

- Factorization at NLO.
IV: General Formalism of PQCD

- Factorization Theorem to all orders of the perturbative expansion (with non-zero quark masses);
- The importance of Scales—Factorization and Renormalization;
- General definition of Parton Distribution Functions (PDF);
- The Three Faces of the Magical Factorization Master Formula;
- Scale dependence of PDFs and QCD Evolution;
- Scale dependence (and independence) of Physical Predictions.
V: PQCD and Collider Physics

- Physical xSections and Partonic Interactions:
  - Probing the Standard Model and New Physics;
- Q-\overline{Q} Annihilation and the Drell-Yan process:
  - Lepton-pair prod., W/Z/Γ prod., … etc.
- Gluon dominated process:
  - Direct Photon Production
  - Jet Cross Section: IRS and Jet Definition
- Heavy Quark Production
- PQCD and Top and Higgs Physics
- PQCD and Beyond the SM Physics.

Factorization and Global QCD Analysis of Parton Distribution Functions
Basic Elements of Quantum Chromodynamics (QCD) - a Non-abelian Gauge Field Theory with SU(3) color Gauge Symmetry

Fields: Quarks $\psi^\text{color}_\text{flavor}$ and Gluon $G^\text{color}(A\cdot T, g)$

$$G_{\mu\nu} \cdot t = (\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot t - i g [\vec{A}_\mu \cdot t, \vec{A}_\nu \cdot t]$$

Basic Lagrangian:

$$\mathcal{L}_{\text{class}} = \bar{\psi}(i \slashed{\partial} - g A \cdot t - m)\psi - \frac{1}{4} Tr G_{\mu\nu} G^{\mu\nu}$$

- $g$: gauge Coupling Strength
- $m_i$: quark masses
- $t$ & $T$: color SU(3) matrices in the fundamental and adjoint representations.

Group factors:

$$C_F \left(= \frac{4}{3}\right), \quad C_A \left(= N_c = 3\right), \quad \text{and} \quad T_R \left(= \frac{1}{2}\right)$$
Quantum #’s of Mesons given by: \( L=0,1,2 \)

\( SU(3)_{flav} \) Octets (nonets) of q-qbar bound states.

Addition of Charm Q.N. \( \Rightarrow SU(4) \)

**Pseudo-scalar mesons**

**Vector mesons**

cf. PDG
Combining of $SU(3)_{\text{flavor}}$ and $SU(2)_{\text{spin}} \Rightarrow SU(6)$

Quantum #’s of Baryons given by: L=0,1,2

$SU(3)_{\text{flv}}$ Octets & decuplets of $q\bar{q}-q$ bound states.

"Octet" Baryons

"Decuplet" baryons
Experimental Foundation of QCD II
Short Distance Physics: Deep Inelastic Scattering, $e^+e^-$ Annihilation, and the Parton Model (~1969-72)

Evidences for the existence of Partons:

“direct”: Most Hard Sc. events contain visible
“jets” ⇒ fragments of underlying partons?
Are they point-like? “Rutherford expts”

* (Bjorken) Scaling in DIS;
* annihilation into hadrons;
* Hadron-hadron scattering, ....
Two-jet Events in $e^+e^-$ Annihilation

— Evidence for Quark - Anti-quark Production

An elementary particle event

$e^+ e^- \rightarrow \mu^+ \mu^-$

A typical event in $e^+e^- \rightarrow$ hadron final state

Parton process underlying 2-jet events

Feynman diagram

CM configuration
Partons are Point-like

Modern day “Rutherford Scattering”: In high energy inclusive $e^+e^-$ annihilation and DIS, cross sections are hard (no form-factor like drop off for large $Q$)

First SLAC results on DIS (~1969)

$\frac{\sigma}{\sigma_{pt}}$ (pre-LEP)

$E_{CM}$ (“$Q$“)

$Q^2$

$\frac{\sigma}{\sigma_{pt}}$ (elastic Sc. (FF))

$R$ in $e^+e^-$ Collisions

$10^5$

$10^4$

$10^3$

$10^2$

$10^1$

$10^0$

$10^{-1}$

$10^{-2}$

$10^{-3}$

$10^{-4}$
Properties of Partons:

2-Jet angular distributions in $e^+e^-$, DIS, DY proc. are the same as for scattering into leptons $\Rightarrow$ underlying partons are fermionic

Expts.: EM & Weak Isospin couplings of partons $=$ those of leptons $\Rightarrow$ “Current Quarks”

3-jet events and their detailed properties prove the existence, and spin of gluons $\Rightarrow$ QCD-parton Model complete.
Measuring the Spin of the Quarks

- Use the angular distribution of 2-jet events

In DIS: $F_L \sim 0$ (Callan-Gross) $\Leftrightarrow \sigma \sim (1 + \cosh^2 \psi)$ (cf. later)

Quarks are spin $\frac{1}{2}$ fermions
Measuring the Spin of the Gluon

Use the angular distributions of 3-jet events

Experimental result from SLD:

- (a) 
- (b) 
- (c) 
- (d)
Very important question:
(central to PQCD, and to this course)

How could the **simple parton picture** (with almost non-interacting partons) possibly hold in **QCD** (—a strongly interacting quantum gauge field theory)?
Answer: 3 distinctive Features of QCD

- Asymptotic Freedom:
  A strongly interacting theory at long-distances (even confining) can become weakly interacting at short distances (due to scale dependence implied by the RGE).

- Infra-red Safety:
  There are classes of “infra-red safe” (IRS) quantities which are independent of long-distance physics, hence are calculable in PQCD.

- Factorization:
  There are an even wider class of physical quantities (inclusive cross sections) which can be factorized into long distance components (not calculable, but universal) & short-distance components (process-dependent, but infra-red safe, hence calculable).

The bulk of this course is devoted to exploring the ideas behind these features of QCD.
The importance of *Scales* -- Renormalization and Factorization

- Upper end of exptl. energy
- $10^{19}$ GeV (Planck Scale) $10^{-20}$ fm
- Range of physical interest
- $\sim 1-2$ GeV
- MeV (Nuclear Scale) $100$ fm
- $M$ (huge)
- $\mu$ or $Q$ (large/hard)
- $m$, $\Lambda$ (soft/confinement)
What does renormalization do?

Say, MS renormalization introduces a ren. scale $\mu_R$. In principle, $\mu_R$ is arbitrary; in practice, $\mu_R$ is chosen $\sim$ a physical scale $Q$, or $\sqrt{s}$.

* Physics of scales $|t| \ll 1/\mu$ removed from perturbative calculation; renormalization hides:
  - the ugly: ultra-violet divergences; and
  - the beautiful: short-distance physics $< \frac{1}{\sqrt{s}}$

$(New\ Physics:\ Q.\ Gravity,\ GUT,\ Super-xx,\ ....)$

* For QCD, $\alpha_s(\mu)$ decreases as $\mu$ increases — Asymptotic freedom.
The $\mu$ dependence of $\alpha(\mu)$ is controlled by the renormalization group equation:

$$\frac{d}{d \ln(\mu^2)} \alpha_s(\mu) = -\beta(\alpha_s(\mu)) = -\beta_0 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 - \beta_1 \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 + \ldots.$$ 

Solution of the RGE to 1-loop order sums leading quantum fluctuations to all orders of the fixed-coupling perturbative expansion.

$$\alpha_s(\mu) \approx \alpha_s(M) - \ln\left( \frac{\mu^2}{M^2} \right) \alpha_s^2(M) + \left( \frac{\beta_0}{\pi} \right)^2 \ln\left( \frac{\mu^2}{M^2} \right) \alpha_s^3(M)$$

$$= \frac{\alpha_s(M)}{1 + \frac{\beta_0}{\pi} \alpha_s(M) \ln\left( \frac{\mu^2}{M^2} \right)} = \frac{\pi}{\beta_0 \ln\left( \frac{\mu^2}{\Lambda^2} \right)}.$$

$\alpha_s(M)$, or $\Lambda$, is a parameter in the solution.

$\beta > 0 \implies \alpha(\mu)$ decreases as $\mu$ increases—QCD is asymptotically free.
Asymptotic Freedom

Universal (running) coupling:

\[ \alpha_s(Q) = \frac{g^2}{4\pi} = \frac{b}{\ln(Q/\Lambda)}(1 + \ldots) \]
How is $\alpha_s(\mu)$ measured in the variety of hadronic processes listed in the previous slide?

In general, how can one relate PQCD calculations (on leptons, quarks and gluons) to physical observables measured in the lepton-hadron world?

Answer: (i) IRS; and (ii) Factorization ...
II: PQCD at Work: $e^+e^-$ Annihilation

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$e^+e^-\text{ Annihilation into Hadrons:}$

Leading Order in PQCD

Total cross section, normalized to point-like cross section:

$$R = \frac{\sigma_{e^+e^-\rightarrow\text{hadrons}}}{\sigma_{e^+e^-\rightarrow\mu^+\mu^-}}(E) = \sum_f q_f^2$$

Angular distribution:

$$\frac{d\sigma}{d\cos\theta} \propto (1 - \cos^2\theta)$$
e^+ e^- Annihilation into Hadrons: Next to Leading Order (NLO) in PQCD

Kinematics:

\[ x_i = \frac{E_i}{\sqrt{s}/2} = \frac{2 q \cdot p_i}{s} \]

\[ i = 1, 2, 3 \]

\[ \sum x_i = \frac{2 q \cdot \sum p_i}{s} = 2 \]

\[ 2(1 - x_1) = x_2 x_3 (1 - \cos \theta_{23}), \text{ cycl.} \]

Differential cross section at the parton level:

\[ \frac{d\sigma}{\sigma_0 \, dx_1 \, dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \]
Colinear and Soft Singularities

\[ \frac{d\sigma}{\sigma_0 \, dx_1 \, dx_2} = \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \]

diverges when:

- \( x_i \to 1 \) (colinear)
- \( x_i \to 0 \) (soft)

In both configurations, the virtual propagator line goes on mass shell: \( k^2 \to 0 \)
Moral: Singularities occur at boundaries of phase space (colinear/soft) where 2-->3 kinematics collapses to 2-->2 and the 4-mom. \( k \) of the internal line goes on-mass shell.

In general (and for theory students):

(These singularities correspond to solutions to the Landau equations for pinch surfaces of the Feynman diagrams.)

\textit{cf. TASI lecture notes of Sterman}
Separation of Short- and Long-distance Interactions
Space-time Picture

Null Plane coordinates:
\[ k^\pm = \frac{k^0 \pm k^3}{\sqrt{2}} \quad ; \quad k^2 = 2k^+ k^- - \vec{k}_T^2 \]

Space-time connection:
\[ \int d^4 x \ e^{i x \cdot k} \cdots \]
\[ x \cdot k = x^- k^+ + x^+ k^- - \vec{x}_T \cdot \vec{k}_T \]

On mass shell:
\[ k^- = \frac{\vec{k}_T^2 + m^2}{2k^+} \]

High-energy interaction:
\[ k^+ \to \infty \Rightarrow k^- \to 0 \Rightarrow x^+ \to \infty \]
Correspondence between singularities in momentum space and the development of the system in space-time:

Consider the fourier transform.

\[ S_F(k) = \int dx^+ dx^- dx \exp(i[k^+ x^- + k^- x^+ - k \cdot x]) \, S_F(x). \]

Contributing values of \( x \) have small \( x^- \) large \( x^+ \).
Moral: Singularities associated with divergent perturbative X-sec \(<--\) interactions a long time after the creation of the initial quark-antiquark pair.

Question:

What to do with the long-distance physics associated with these collinear/soft singularities?
Infra-red Safe Physical Observables

IRS observables are those that are insensitive to the colinear and soft singularities of the perturbative calculation, hence the long distance behavior of QCD.

Example: the total cross section $\sigma_T(e^+e^-\rightarrow\text{hadrons})$

$$\sigma_{\text{tot}}(s) = \sigma_0(s)[1 + \alpha_s(s) c_1 + \ldots]$$

*Block – Nordsieck Thm* $\rightarrow c_{1,2,...}$ are finite, i.e. IRS (unitarity)

**Order $\alpha_s$:**

- Cancellation of the colinear/soft singularities between real and virtual diagrams

Once the quark-antiquark pair is produced (at short distance), the probability for them to turn into some hadron state is unity, independent of the long-distance behavior of the theory.
Other Examples of IRS Observables

- Sterman-Weinberg Jet cross section and its modern variants
  - Jade, Cone, $k_T$, ...
  cf. lecture on jets
- Shape Variables
  - Thrust, sphericity, aplanarity, oblateness, ...
  cf. lecture on $e^+e^-$
- Energy-energy Correlations
- ...

Essential feature of a general IRS physical quantity:

the observable must be such that it is insensitive to whether $n$ or $n+1$ particles contributed -- if the $n+1$ particles has $n$-particle kinematics.

e.g. a IRS "jet algorithm"
IRS observables are inclusive quantities

\[ I = \frac{1}{2!} \int d\Omega_2 \frac{d\sigma[2]}{d\Omega_2} S_2(p_1^\mu, p_2^\mu) \\
+ \frac{1}{3!} \int d\Omega_2 dE_3 d\Omega_3 \frac{d\sigma[3]}{d\Omega_2 dE_3 d\Omega_3} S_3(p_1^\mu, p_2^\mu, p_3^\mu) \\
+ \frac{1}{4!} \int d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4 \times \frac{d\sigma[4]}{d\Omega_2 dE_3 d\Omega_3 dE_4 d\Omega_4} S_4(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) \\
+ \ldots. \]

where Sn's satisfy, for \(0 < \lambda < 1\) (colinear) or \(\lambda = 0\) (soft):

\[ S_{n+1}(p_1^\mu, \ldots, (1 - \lambda)p_n^\mu, \lambda p_n^\mu) = S_n(p_1^\mu, \ldots, p_n^\mu). \]
Example 1: $\sigma_{tot}$

$$S_n(p_1^\mu, \ldots, p_n^\mu) = 1.$$ IRS condition obviously satisfied.

Example 2: Thrust

$$S_n(p_1^\mu, \ldots, p_n^\mu) = \delta \left( T - T_n(p_1^\mu, \ldots, p_n^\mu) \right).$$

$$T_n(p_1^\mu, \ldots, p_n^\mu) = \max_{\vec{u}} \sum_{i=1}^{n} \frac{\left| \vec{p}_i \cdot \vec{u} \right|}{\sum_{i=1}^{n} |\vec{p}_i|}.$$ cf. lectures on e+e-
Check the IRS criterion:

\[ T_n(p_1^\mu, \ldots, p_n^\mu) = \max_{\bar{u}} \frac{\sum_{i=1}^{n} |\vec{p}_i \cdot \bar{u}|}{\sum_{i=1}^{n} |\vec{p}_i|}. \]

- Contribution from a particle with \( \vec{p} \to 0 \) drops out.

- Replacing one particle by two collinear particles doesn’t change the thrust:

\[ |(1 - \lambda) \vec{p}_n \cdot \bar{u}| + |\lambda \vec{p}_n \cdot \bar{u}| = |\vec{p}_n \cdot \bar{u}|, \]

and

\[ |(1 - \lambda) \vec{p}_n| + |\lambda \vec{p}_n| = |\vec{p}_n|. \]
Are applications of PQCD confined to IRS physical observables?

(Most physical observables are not IRS!)

Fortunately not. In fact, the “QCD Parton Model” for lepton-lepton, lepton-hadron and hadron-hadron scattering cross sections at high energies provides a much more powerful framework for applying PQCD to study a vast range of SM and New Physics processes:

The basic idea behind this class of applications is the factorization of short-distance physics (of leptons, quarks, gluons, new particles) from long-distance physics (of hadrons).
Factorization in $e^+e^-$ interaction at high energies

Example: One particle inclusive cross-section

Fragmentation function: $D_a^h(z, \mu)$
- Long-distance physics;
- Universal.

Hard scattering: $\hat{\sigma}^a(s, z, \mu)$
- Short distance physics;
- IRS, perturbatively cal.

$$\sigma(s, z) = \int_z^1 \frac{d\zeta}{\zeta} \hat{\sigma}^a\left(\frac{s}{\mu^2}, \frac{z}{\zeta^2}, \alpha_s(\mu)\right) \cdot D_a(\zeta, \mu)$$

Details will be discussed in the context of DIS (to follow).
Factorization and the Parton Picture at High Energies

(Requires at least one large scale $Q$—hard scattering)

lepton-lepton
$e^+ e^-$

lepton-hadron
$e, \mu, \nu$

DIS
$p, d, A$

"factorization"

PEP, PETRA, Cornell, LEP, SLD, NLC

SLAC, FNAL, CERN, HERA

short \{ distance
long
hadron-hadron

$W/Z/\gamma/\mu^+\mu^-$

$Q$

$b/t/H/h/\ldots$

$P, \bar{P}, A$

$P, d, A$

FNAL, CERN, Tevatron, RHIC, LHC

short

long

\{ distance \}