CTEQ Symposium

Nov 7-9, 1996

S. Brodsky

Towards a Precise QCD Calculus:

Scale-Setting as Resummation

"BLM" Method: SJB, G.P. Lepage, P. Mackenzie
Phys. Rev. D28, 228 (1983)

Commensurate Scale Relations:
SJB + H.J. Lu
Phys. Rev. D51, 3652 (1995)

Precision tests of a theory:

relate one observable to another

- all theoretical conventions irrelevant

e.g.
- calculational method
- renormalization procedure
- choice of cutoffs
- ⋮

$$\mathcal{L}_{QCD} \Rightarrow \mathcal{L}^{(\Lambda)}\left[m_0(\Lambda), \alpha_0(\Lambda)\right]$$

UV regularization

Many possible renormalization schemes

Choice of Λ irrelevant

Convention: dim regularization

$$\mu^{2\epsilon} \int d^{4-2\epsilon} k$$

$$\alpha_{\overline{MS}}(\mu), \quad m_{\overline{MS}}(\mu)$$

Physical quantities indep of UV scheme, renorml. scale

$$\frac{d}{d\mu} \mathcal{O} = 0, \quad \mathcal{O}_A \Rightarrow \mathcal{O}_B$$

Parameters in $\mathcal{L}_{eff} = \mathcal{L}^{(\Lambda)}$

are theoretical constructs

- dependent on cutoff convention

Relations between observables must be

* independent of theoretical conventions

* Use observables to fix parameters

Use physical observables to define "effective charge" $\alpha_A(Q^2)$

$$\overline{R}_{e^+e^-}(s) \equiv R_0(s)\left[1 + \frac{\alpha_R(s)}{\pi}\right]$$

$\overline{R}_{e^+e^-}(s)$: smeared; $R_0(s)$: $\sum e_q^2$

$$\int_0^1 dx \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] \equiv \frac{g_A}{6}\left[1 - \frac{\alpha_{g_1}}{\pi}\right]$$

Bj sum rule

$$V(Q^2) \equiv -4\pi C_F \frac{\alpha_V(Q^2)}{Q^2}$$

$V(Q^2)$: Heavy Quark Potential

"Running couplings"

QD: $\boxed{\alpha_A(Q_A^2) \Rightarrow \alpha_B(Q_B^2)}$

Indep of μ, renorm. scheme!

Effective Charges / Running Couplings obey

$$\frac{d}{d\ln Q^2}\alpha_A(Q^2) = \beta_A(\alpha_A(Q^2))$$

* First two terms β_0, β_1 universal!

QCD Program: choose

any effective charge

as "standard candle"

e.g. $\alpha_V(Q_0^2)$

→ predict all other observables in terms of $\alpha_V(Q^2)$

* 1) No μ, scheme dependence

* 2) No scale ambiguity

$\alpha_V(Q^2)$: $Q^2 = -t = -q^2$

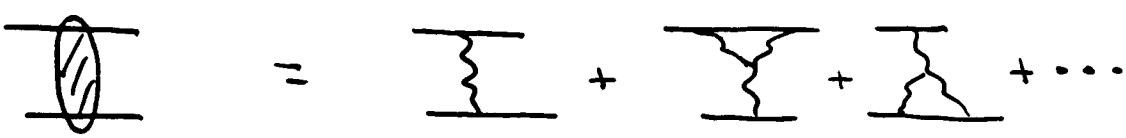
Quarkonium Physics:

Scales and Schemes

Natural renormalization scheme $\alpha_V(Q)$

$$V(Q^2) \equiv -\frac{4\pi C_F \alpha_V(Q)}{Q^2}$$

Potential for infinitely heavy test charges

$t = -Q^2$  $= \cdots$

Obeys standard RGE:

* $$\frac{\partial}{\partial \ln Q} \alpha_V(Q) = \beta\left[\alpha_V(Q)\right]$$

β_0, β_1 universal

quark mass dependence as in QED

* $\alpha_V(Q)$: sums all V.P. contributions (no scale ambiguity!)

Input from LGT:

$$\alpha_V^{(n_f=3)}[Q = 8.2\,\text{GeV}]$$

$$= \begin{cases} 0.1945\,(30) & \Upsilon \\ 0.1940\,(67) & \psi. \end{cases}$$

Davies, et al

$$\alpha_{\text{Lattice}} \Rightarrow \alpha_P \tilde{\Rightarrow} \alpha_V$$

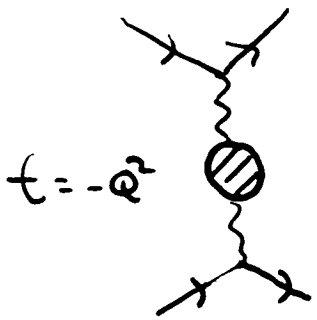
Use BLM

$$\alpha_{\overline{MS}}(M_Z) = \begin{cases} 0.117\,(2) & \text{NNLO L+W.} \\ 0.115\,(2) & \text{O} \end{cases}$$

Lüsher + Weist NNLO $(n_f = 0)$

$\alpha_{\overline{MS}} \to \alpha_P$

Application of BLM to QED



Coulomb scatt of heavy lepton

$$V(Q^2) \equiv -\frac{4\pi\alpha_V(Q^2)}{Q^2}$$

$$Q^2=0: \quad \alpha_V^{-1} = 137.035989(6).$$

$$\alpha_V(Q^2) = \frac{\alpha_V(Q_0^2)}{1-\Pi(Q^2,Q_0^2)}$$

↑ sums all VP from

Compute observables in terms of α_V

Lautrup de Rafael BLM

$$a_l = \frac{\alpha(Q^*)}{2\pi} + C_{20}\frac{\alpha^2(Q^{**})}{\pi^2} + \ldots$$

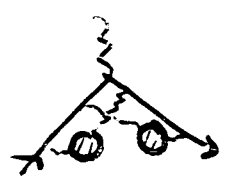
↑ Universal Form!

J. H. Field

$$Q^* = e^{-5/4} m_l \qquad \text{(also MVTh)}$$

C_{20} indep of n_F^{VP}, conformal coef

Sums all VP



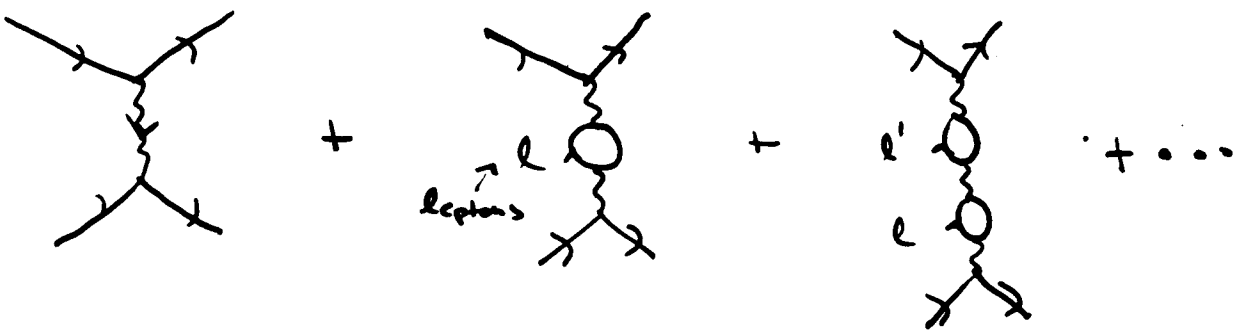
* In QED: no scale ambiguity!

Measure in Coulomb Scattering of Heavy Charges

$$\alpha(q^2) = \frac{\alpha(q_0^2)}{1 - \pi(q^2; q_0^2)}$$

"running" coupling sums all Vac. Pol.

$t = q^2$



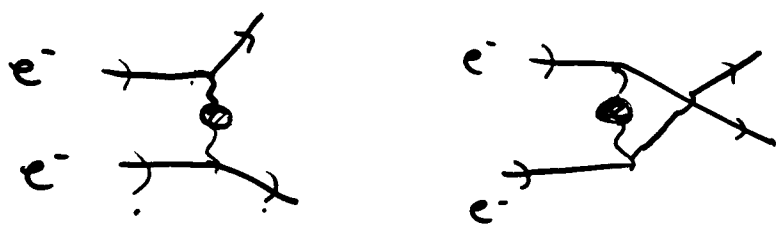
$$\pi(q^2, q_0^2) = \sum_l \frac{\alpha(q_0^2)}{3\pi} \log q^2/q_0^2 + \cdots$$

$(|q^2|, |q_0^2| \gg 4m_l^2)$

Example: $e^- e^- \to e^- e^-$ scattering

$$\mathcal{M}^{\uparrow\uparrow \to \uparrow\uparrow} = 8\pi \frac{s}{t} \alpha(t) + 8\pi \frac{s}{u} \alpha(u)$$

two scales in $d\sigma/dt$



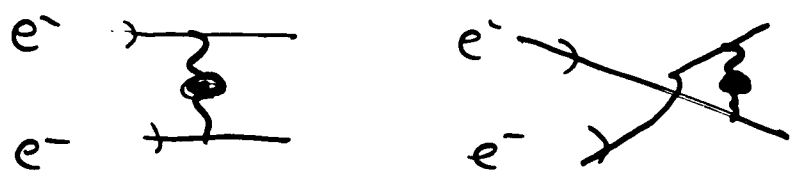
"Dressed Skeleton" Expansion

$\alpha(t)$ ↑ unambiguous $\alpha(u)$ →

In general QCD processes

involve many invariants.

many scales



$$\mathcal{M}^{\uparrow\uparrow\to\uparrow\uparrow}(s,t) = 8\pi \frac{s}{t}\alpha(t) + 8\pi \frac{s}{u}\alpha(u$$

must use t, u otherwise $\alpha \neq$ V.P.

Cannot use single scale p_T^2

$\alpha(Q^2)$ sums all V.P

to all orders.

Relate Observable to Observable

No dependence on renormalization scheme or scale at any order of pert. theory

* $\alpha_A(Q_A) = \alpha_B(Q_B)\left[1 + c_1 \frac{\alpha_B}{\pi} + \ldots\right.$

coefficients $\{c_n\}$

* identical to theory with $\beta = 0$

"conformal coefficients"

* all $\beta \neq 0$ terms contained in α_B

Commensurate Scale Relations:

$$* \quad \alpha_B(Q_B) = \alpha_A(Q_A)\left[1 + r_1^{B/A}\frac{\alpha_A}{\pi} + \cdots\right]$$

($r_1^{B/A}$: conformal coeff.)

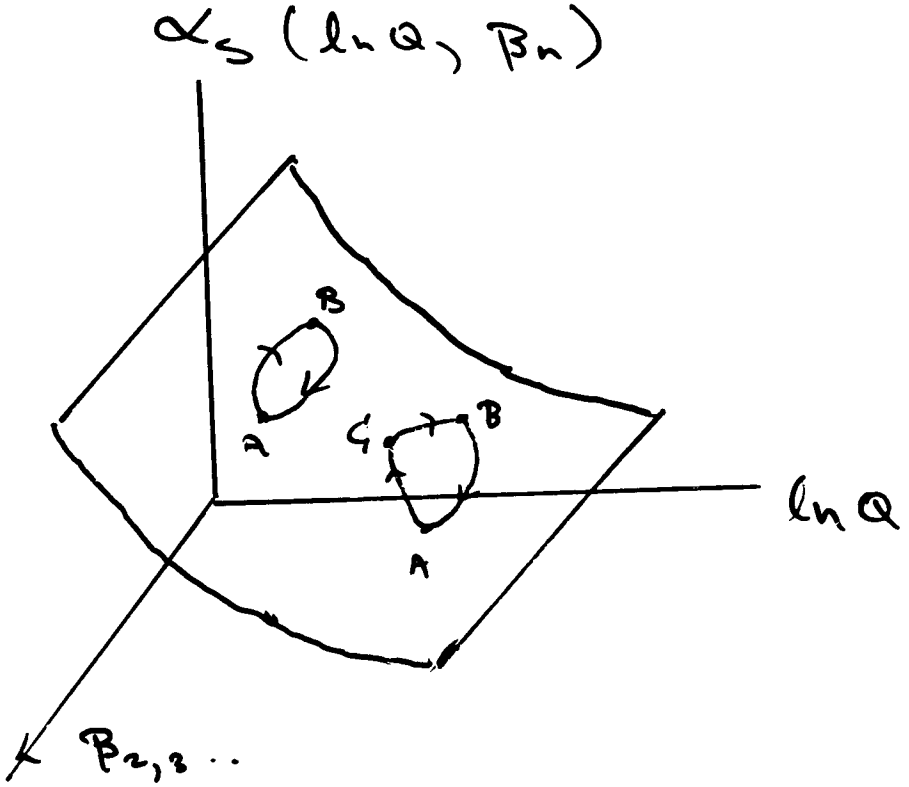
$$Q_B/Q_A = \lambda_{B/A}$$

Peterman Stückelberg Renormalization "Group":

$\lambda_{B/A} = \lambda_{B/C}/\lambda_{A/C}$ — transitive

$\lambda_{B/A} = \lambda^{-1}_{A/B}$ — symmetry

$\lambda_{A/A} = I$ — identity



EdB
Lepage
Mackenzie

EdB + MJL
Lepage + Mackenzie
J. Rathsman

BLM :

* All $\beta \neq 0$ contributions resummed into scale of running coupling

* coefficients of perturbative series same as conformally-invariant theory.

Commensurate Scale Relations:

* Relations between observables

* no scheme, scale ambiguity

* generalized Crewther relation

$\beta_0 = 11 - 2/3\, n_f$

BLM scale fixing

(any effective charge / scheme)

$$\alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(Q_0)\ln\frac{Q}{Q_0} + \cdots}$$

Given: $$R(Q) = k\,\alpha_s(\mu)\left[1 + \frac{\alpha_s(\mu)}{\pi}(An_f + B) + \cdots\right.$$

work $$An_f + B = -3/2\,\beta_0 A + \left(\frac{33}{2}A + B\right)$$

* shift $$\mu \to \mu e^{3A(\mu)} = Q^*$$

↑ μ indep

Then

* $$R(Q) = k\,\alpha_s(Q^*)\left[1 + \frac{\alpha_s(Q^*)}{\pi}C + \cdots\right.$$

* $$C = \frac{33}{2}A + B$$ conformal coeff.

Note: $$Q^* = \mu e^{3A}$$

determined from n_f coeff

Proof of BLM Scale-Setting

In α_V-scheme, all VP summed in $\alpha_V(Q$

$\Rightarrow$ coefficients n_f-independent!

any eff. charge

$$\alpha_A(Q_A) = \alpha_V(\mu)\left[1 + (A_{VP} n_f + B)\frac{\alpha_V}{\pi} + \cdots\right]$$

must shift argument of α_V:

$$Q_V = e^{3A_{VP}(\mu)}\,\mu$$

$$\alpha_A(Q_A) = \alpha_V(Q_V)\left[1 + r_1^{A/V}\frac{\alpha_V}{\pi} + \cdots\right]$$

* then $r_1^{A/V} = B + \frac{33}{2}A_{VP}$

is independent of n_f.

* $\lambda_{A/V} = \frac{Q_A}{Q_V}$ fixed, unique

* All n_f-VP summed into α_V resummation

Repeat for another effective charge

$$\alpha_B(Q_B) = \alpha_V(Q_V)\left[1 + r_1^{B/V}\frac{\alpha_V}{\pi} + \cdots\right]$$

$$\alpha_A(Q_A) = \alpha_V(Q_V)\left[1 + r_1^{A/V}\frac{\alpha_V}{\pi} + \cdots\right]$$

∴

$$\boxed{\alpha_B(Q_B) = \alpha_A(Q_A)\left[1 + r_1^{B/A}\frac{\alpha_A}{\pi} + \cdots\right]}$$

$$r_1^{B/A} = r_1^{B/V} - r_1^{A/V}$$ indep of n_f, β_0

$$Q_B/Q_A = \lambda_{B/A} = \frac{\lambda_{B/V}}{\lambda_{A/V}}$$ transitivity reflects Abelian

* Alternatively, compute Q_B/Q_A by requiring $r_1^{B/A}$ to be n_f-indep. ⇒ BLM.

* Result is independent of intermediate scheme P+S

* Can use $\overline{MS}$:

$$\alpha_V(Q) = \alpha_{\overline{MS}}\left(e^{-5/6}Q\right)\left[1 - 2\frac{\alpha_{\overline{MS}}}{\pi} \cdots\right]$$

S. Brodsky + H. J. Lu

Example of Commensurate Scale Relation:

"effective charge"

$$R_{e^+e^-}(Q^2) \equiv R^0_{e^+e^-}(Q^2)\left[1 + \frac{\alpha_R(Q)}{\pi}\right]$$

$$\Gamma^{p-n}(Q^2) = \int_0^1 dx \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right]$$

$$\equiv \frac{g_A}{3}\left[1 - \frac{\alpha_{g_1}(Q)}{\pi}\right]$$

$$\boxed{\alpha_{g_1}(Q) = \alpha_R(0.52Q)\left[1 - \frac{\alpha_R}{\pi} + \cdots\right]}$$

Relate physical observable to physical observable

Relative scale fixed

No scale or scheme ambiguity

Expansion in finite quantity, not

$$\alpha_{\overline{MS}}(Q) = \frac{4\pi}{\beta_0 \ln Q^2/\Lambda^2} = \infty \text{ at } Q$$

Define "Effective Charge" $\alpha_R(s)$

$$R_{e^+e^-}(s) \equiv R_0(s)\left[1 + \frac{\alpha_R(s)}{\pi}\right]$$

$R_0(s) \uparrow 3\sum_i e_i^2 \quad (s \gg m_i^2)$

$$\boxed{\frac{\alpha_R(s)}{\pi} \equiv \frac{R_{e^+e^-}(s) - R_0(s)}{R_0(s)}}$$

* $\alpha_R(s)$ determined from expt.

* $\alpha_R(s)$ obeys usual RGE in PQCD (important test of theory)

* $\alpha_R(s)$ defines a renormalization scheme if we use $\alpha_R(s)$ as expansion

* Other perturbatively-calculable expt. quantities can be used to define effective charge

$$R(Q) \equiv 3 \sum_f Q_f^2 \left[1 + \frac{\alpha_R(Q)}{\pi} \right]$$

$\mu \Rightarrow Q$

$$\frac{\alpha_R(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2 \left[\left(\frac{41}{8} - \frac{11}{3}\zeta_3 \right) C_A - \frac{1}{8} C_F + \left(-\frac{11}{12} + \frac{2}{3}\zeta_3 \right) f \right]$$

$$+ \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3 \left\{ \left(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5 - \frac{121}{432}\pi^2 \right) C_A^2 \right.$$

$$+ \left(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5 \right) C_A C_F - \frac{23}{32} C_F^2$$

$$+ \left[\left(-\frac{970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5 + \frac{11}{108}\pi^2 \right) C_A \right.$$

$$\left. + \left(-\frac{29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5 \right) C_F \right] f$$

$$+ \left(\frac{151}{162} - \frac{19}{27}\zeta_3 - \frac{1}{108}\pi^2 \right) f^2$$

$$\left. + \left(\frac{11}{144} - \frac{1}{6}\zeta_3 \right) \frac{d^{abc} d^{abc}}{C_F d(R)} \frac{\left(\sum_f Q_f \right)^2}{\sum_f Q_f^2} \right\}$$

assumes
$\mu = Q$

$\uparrow$
$d_{abc} d^{abc} - 6 \sum - \sum$

$d(R) = N$

$C_A = N$

Gorishny
Kataev
Larin

$d_{abc} d^{abc} = \frac{40}{3}$

$C_F = \frac{N^2 - 1}{2N}$

$\sum_f Q_f = 2/3 - 1/3 - 1/3 = 0$

Surguladze
Samuel

$= 1/2$

$$\int_0^1 dx\,\left[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)\right] \equiv \frac{1}{3}\left|\frac{g_A}{g_V}\right|\left[1 - \frac{\alpha_{g_1}(Q)}{\pi}\right]$$

$$\begin{aligned}\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} &+ \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2\left[\frac{23}{12}C_A - \frac{7}{8}C_F - \frac{1}{3}f\right] \\ &+ \left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3\left\{\left(\frac{5437}{648} - \frac{55}{18}\zeta_5\right)C_A^2 + \left(-\frac{1241}{432} + \frac{11}{9}\zeta_3\right)C_AC_F + \frac{1}{32}C_F^2\right. \\ &\quad + \left[\left(-\frac{3535}{1296} - \frac{1}{2}\zeta_3 + \frac{5}{9}\zeta_5\right)C_A + \left(\frac{133}{864} + \frac{5}{18}\zeta_3\right)C_F\right]f \\ &\quad \left. + \frac{115}{648}f^2\right\}.\end{aligned}$$

Assumes:
$\mu = Q$

S. A. Larin

J. A. M. Vermaseren

NLO BLM formulas in terms of beta functions

Any effective charge in perturbative QCD can be written in the following form

$$\frac{\alpha_1(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \left(A_1 + B_1\beta_0\right)\left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2 + \left(C_1 + D_1\beta_0 + E_1\beta_0^2 + \tfrac{1}{4}B_1\beta_1\right)\left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3 + \cdots. \tag{1}$$

where the light-by-light contributions should be included in the C_1 coefficient. Notice the appearance of the $\frac{1}{4}B_1\beta_1$ term in the NNLO coefficient.

Similarly, given a second effective charge $\alpha_2(Q)$, we can put it in the form

$$\frac{\alpha_2(Q)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \left(A_2 + B_2\beta_0\right)\left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^2 + \left(C_2 + D_2\beta_0 + E_2\beta_0^2 + +\tfrac{1}{4}B_2\beta_1\right)\left(\frac{\alpha_{\overline{\rm MS}}(Q)}{\pi}\right)^3 + \cdots. \tag{2}$$

The two effective charges $\alpha_1(Q)$ and $\alpha_2(Q)$ are related by the following series,

$$\frac{\alpha_1(Q)}{\pi} = \frac{\alpha_2(Q)}{\pi} + \left(A_{12} + B_{12}\beta_0\right)\left(\frac{\alpha_2(Q)}{\pi}\right)^2 + \left(C_{12} + D_{12}\beta_0 + E_{12}\beta_0^2 + +\tfrac{1}{4}B_{12}\beta_1\right)\left(\frac{\alpha_2(Q)}{\pi}\right)^3 + \cdots, \tag{3}$$

where the coefficients $A_{12}, B_{12}, C_{12}, D_{12}$ and E_{12} are given by:

$$\begin{aligned} A_{12} &= A_1 - A_2 \\ B_{12} &= B_1 - B_2 \\ C_{12} &= C_1 - C_2 - 2(A_1 - A_2)A_2 \\ D_{12} &= D_1 - D_2 - 2(A_1B_2 + A_2B_1) + 4A_2B_2 \\ E_{12} &= E_1 - E_2 - 2(B_1 - B_2)B_2. \end{aligned} \tag{4}$$

Applying the BLM procedure we obtain

$$\frac{\alpha_1(Q)}{\pi} = \frac{\alpha_2(Q^*)}{\pi} + A_{12}\left(\frac{\alpha_2(Q^{**})}{\pi}\right)^2 + C_{12}\left(\frac{\alpha_2(Q^{***})}{\pi}\right)^3 + \cdots, \tag{5}$$

↑ ↑

conformal[1] coefficients

where the renormalization scales are

$$\log\left(Q^{*2}/Q^2\right) = -4\ B_{12} + 4\ \beta_0\ \left(B_{12}^2 - E_{12}\right)\left(\frac{\alpha_2(Q)}{\pi}\right),$$
$$\log\left(Q^{**2}/Q^2\right) = -4\ \frac{D_{12}}{A_{12}}. \tag{6}$$

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \frac{3}{4}C_F\left(\frac{\alpha_R(Q^{**})}{\pi}\right)^2$$

0 for 3 flavors

$$+\left[\frac{9}{16}C_F^2 - \left(\frac{11}{144} - \frac{1}{6}\zeta_3\right)\frac{d^{abc}d^{abc}}{C_F N}\frac{\left(\sum_f Q_f\right)^2}{\sum_f Q_f^2}\right]\left(\frac{\alpha_R(Q^{***})}{\pi}\right)^3$$

$$Q^* = Q\exp\left[\frac{7}{4} - 2\zeta_3 + \left(\frac{11}{96} + \frac{7}{3}\zeta_3 - 2\zeta_3^2 - \frac{\pi^2}{24}\right)\left(\frac{11}{3}C_A - \frac{2}{3}f\right)\frac{\alpha_R(Q)}{\pi}\right]$$

$$Q^{**} = Q\exp\left[\frac{523}{216} + \frac{28}{9}\zeta_3 - \frac{20}{3}\zeta_5 + \left(-\frac{13}{54} + \frac{2}{9}\zeta_3\right)\frac{C_A}{C_F}\right]$$

Note. Abelian light-by-light not summed.

$$\widehat{\alpha}_{g_1}(Q) = \widehat{\alpha}_R(Q^*) - \widehat{\alpha}_R^2(Q^{**}) + \widehat{\alpha}_R^3(Q^{***})$$

where

$$\widehat{\alpha}_{g_1}(Q) = \frac{3C_F}{4\pi}\alpha_{g_1}(Q)$$

$$\widehat{\alpha}_R(Q) = \frac{3C_F}{4\pi}\alpha_R(Q)$$

Independent of N_c!

See also:
Single Scale
Kataev, et al
Rathsman

* why is the relation between

α_R and α_{g_1}

so simple?

{ Kataev
H.J. Lu
813

Consider conformal limit $\beta_0 \Rightarrow 0$, $\beta_1 \Rightarrow 0$

$$CSR \Rightarrow (1+\hat{\alpha}_R)(1-\hat{\alpha}_{g_1}) = 1$$

* Follows from Crewther relation!

$\beta = 0$
chiral

$$3\underset{\substack{\uparrow \\ \pi^0 \to \gamma\gamma}}{S} = \underset{\substack{\uparrow \\ \text{Bj} \\ \text{GLS}}}{k}\,\underset{\substack{\uparrow \\ R^A_{e^+e^-}}}{R'} = k\left(\tfrac{4}{3}\underset{\substack{\uparrow \\ R^V_{e^+e}}}{R}\right)$$

Deviations from Crewther Relation

proportional to β

Kataev
Broadhurst

Test of Commensurate Scale Relation

* Generalized Crewther Relation

$$\frac{R_{e^+e^-}(s)}{3\sum e_q^2} \cdot \int_0^1 dx \; \frac{F_3^{\nu p}(x,Q^2) + F_3^{\bar{\nu} p}(x,Q^2)}{6}$$

$$= 1 + \Delta\beta_0 \frac{\alpha_R^3}{\pi^3} \quad \sim 7\times 10^{-4}$$

where $\sqrt{s} = Q^* \simeq 0.38\, Q$ commensurate scales

$O[\alpha_s^3]$ correction comes from $Q^{**} \neq Q^*$

* Check at $Q^2 = 3\ \text{GeV}^2$:

$$\underset{\uparrow}{1.20} \cdot \frac{\underset{\uparrow}{2.50 \pm 0.13}}{3} = 1.00 \pm 0.0$$

$1 + \hat{\alpha}_R(s)$: from R smoothed Mattingly + Stevenson, $\sqrt{s} = .38\, Q = .66$ GeV

$1 - \hat{\alpha}_{F_3}(Q)$: from CCFR measurement of GLLS, $Q = \sqrt{3}$ GeV

$1 + \Delta\beta_0 \hat{\alpha}^3$

SOB
Kataev
Lu
Gabadze

$$\frac{1}{3\sum_f Q_f^2} R_{e^+e^-}(\sqrt{s} = 5.0 \text{ GeV}) = 1.08 \pm 0.0$$

$$\therefore \quad \frac{\alpha_R^{expt}}{\pi}(\sqrt{s} = 5.0 \text{ GeV}) = 0.08 \pm 0.0$$

Predict:

$$\frac{\alpha_{g_1}}{\pi}(Q = 12.33 \pm 1.20 \text{ GeV})$$

$$= \frac{\alpha_{GLS}}{\pi}(Q = 12.33 \pm 1.20 \text{ GeV})$$

$$= 0.074 \pm 0.026$$

Expt:

$$\frac{\alpha_{GLS}}{\pi}(Q = 12.25 \text{ GeV}) = 0.093 \pm 0.0$$

Using BLM to fix scale:

we can relate PQCD observables

$$\boxed{\alpha_{g_1}(Q) = \alpha_R(0.520\,Q)\left(1 - \frac{\alpha_R}{\pi} + \cdots\right)}$$

* test QCD by tracking Q^2-dep, normalization

$\alpha_{g_1}(Q)$, $\alpha_R(Q)$ "effective charges"

scheme choices

$11 - 2/3\, n_f$

obey RGE for $\alpha_s(Q^2)$ $\quad \underbrace{\beta_0, \beta_1}_{\text{scheme-indep}}, \beta_2 \cdots$

* No scale ambiguity = physical scale

* thresholds at heavy quarks correct

* Eliminate $\overline{MS}$: unphysical scheme, scale

* Some Higher-Twist also related

Conformal Coefficients: $(\beta = 0)$

$$\frac{\alpha_A}{\pi} = \frac{\alpha_B}{\pi} + C_1 \frac{\alpha_B^2}{\pi^2} + \cdots$$

Contain no IR renormalon

$$n! \beta_0^n \alpha_s^n$$

May be well-behaved

e.g Generalized Crewther Relation

$$\frac{\alpha_g}{\pi} = \frac{\alpha_R}{\pi} - \frac{\alpha_R^2}{\pi^2} + \frac{\alpha_R^3}{\pi^3} +$$

Geometric Series

Relation between $R(s)$ and $\Gamma_{\tau \to \text{hadrons} + \nu_\tau}$

$$R_\tau = 3\left[1 + \frac{\alpha_\tau}{\pi}(m_\tau)\right]$$

$$R(Q) = 3\sum_F Q_F^2\left[1 + \frac{\alpha_R(Q)}{\pi}\right]$$

$$\alpha_\tau(Q) = \frac{\alpha_{\overline{MS}}(Q)}{\pi}$$

Gorishny
Kataev
Larin

$$+ \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^2\left[\left(\frac{947}{144} - \frac{11}{3}\zeta_3\right)C_A - \frac{C_F}{8} + \left(\frac{-85}{72} + 2\zeta_3\right)\right.$$

$$+ \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^3\left\{\left(\frac{559715}{10368} - \frac{2591}{72}\zeta_3 - \frac{55}{18}\zeta_5 - \frac{121}{432}\pi^2\right)C\right.$$

$$+ \left(\frac{-1733}{576} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5\right)C_A C_F - \frac{23}{32}C_F^2$$

$$+ \left[\left(\frac{-24359}{1296} + \frac{73}{6}\zeta_3 + \frac{5}{9}\zeta_5 + \frac{11\pi^2}{108}\right)C_A\right.$$

$$\left. + \left(\frac{-125}{288} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5\right)C_F\right]f$$

$$\left. + \left(\frac{3935}{2592} - \frac{19}{18}\zeta_3 - \frac{\pi^2}{108}\right)f^2\right\} + \cdots$$

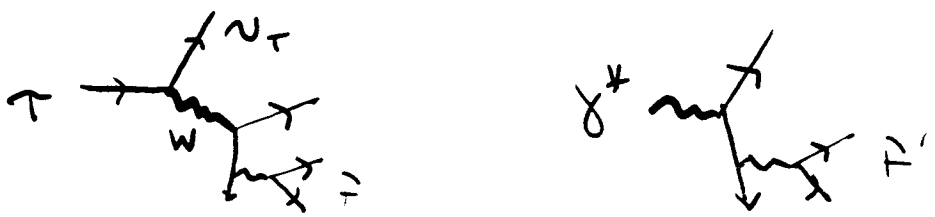
Apply NLO BLM: [f=3]

$$\frac{\alpha_\tau(m_\tau)}{\pi} = \frac{\alpha_R(Q^*)}{\pi}$$

$$Q^* = m_\tau \exp\left[-\frac{19}{24} - \frac{169}{128}\frac{\alpha_R(m_\tau)}{\pi}\right]$$

Note that all β_3, β_5, π^2 disappear!

Up to NNLO: α_τ, α_R related by scale shift



* $\Gamma_{\tau \to \nu_\tau \text{ hadrons}}$ ~ integral over $R_{e^+e^-}(Q)$

* MVT: $\frac{Q^*}{m_\tau} = e^{-19/24} + \cdots$ adjusts for $f \neq f'$ threshold

* BLM: discover hidden conformal relation

$$\alpha_\tau(m_\tau) = \alpha_R(c\, m_\tau)$$

$$c m_\tau = Q^*_{BLM}.$$

Suppose $c \Rightarrow c' = c + \delta c$

$$\alpha_\tau(m_\tau) = \alpha_R(c'\, m_\tau)$$

$$+ \sum_n n!\, B_0^n\, \alpha_s^n (\delta c)$$

IR renormalon induced by

choosing incorrect scale.

Infrared Renormalons

$$\rho = \sum_n c_n \alpha_s^n(\mu)$$

$$c_n \sim n!\, \beta_0^n \quad !$$

Divergent series generated by β_0 [V.P.] insertions

Eliminated iff $\mu = Q^*_{BLM}$

Remaining series $\Rightarrow$ conformal theory

Q^*_{BLM} expansion may be divergent

QCD: Padé, Borel sum

BLM + Resummation

BLM:

$$Q = \sum_{n=0}^{\infty} c_n \alpha_r^n(Q_n^*)$$

Observable

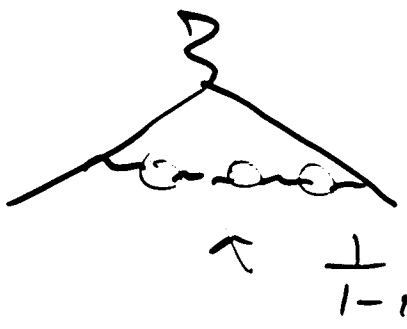
c_n : identical to coefficients in conformal theory $\beta = 0$

Any other scale: $Q_n \neq Q_n^*$

$n! (\beta_0 \alpha_s)^n$ $\quad$ Q_n^* resums $\beta_0, n_f, \ldots$

Renormalon growth from

$$\int dk^2 \, \alpha(k^2) \, f(k^2)$$



$Q_n^* \Leftrightarrow$ MVT

resums all β terms

BLM equivalent to <u>MVT</u>

$$\int d^4 l \, F(l) \, \alpha_V(l^2)$$

$$= \int d^4 l \, F(l) \, \alpha_V(Q^{*2})$$

$$\alpha_{\overline{MS}}(Q^2) = \alpha_V(e^{5/3} Q^2)\left[1 + 2\frac{\alpha_V}{\pi}(Q^2)\right]$$

See also: Braun + Beneke

Neubert

BLM

Lepage + Mackenzie LGTs.

BLM

Lepage + Mackenzie
PRD 48, 2250 (

Mean value theorem approach

Want $\int d^4q\, f(q)\, \alpha_V(q) = \alpha_V(q^*) \int d^4q\, f(q)$

Recall: $\alpha_V(q) = \alpha_V(\mu)\left[1 + \beta_0 \ln\frac{\mu^2}{q^2}\frac{\alpha_V(\mu)}{4\pi} + \ldots\right]$

$\therefore$ $\int d^4q\, f(q) \ln\frac{\mu^2}{q^2} = \ln\frac{\mu^2}{q^{*2}} \int d^4q\, f(q)$

$*$ $\ln q^{*2} = \dfrac{\int d^4q\, f(q) \ln q^2}{\int d^4q\, f(q)}$

sets physical scale in α_V scheme.

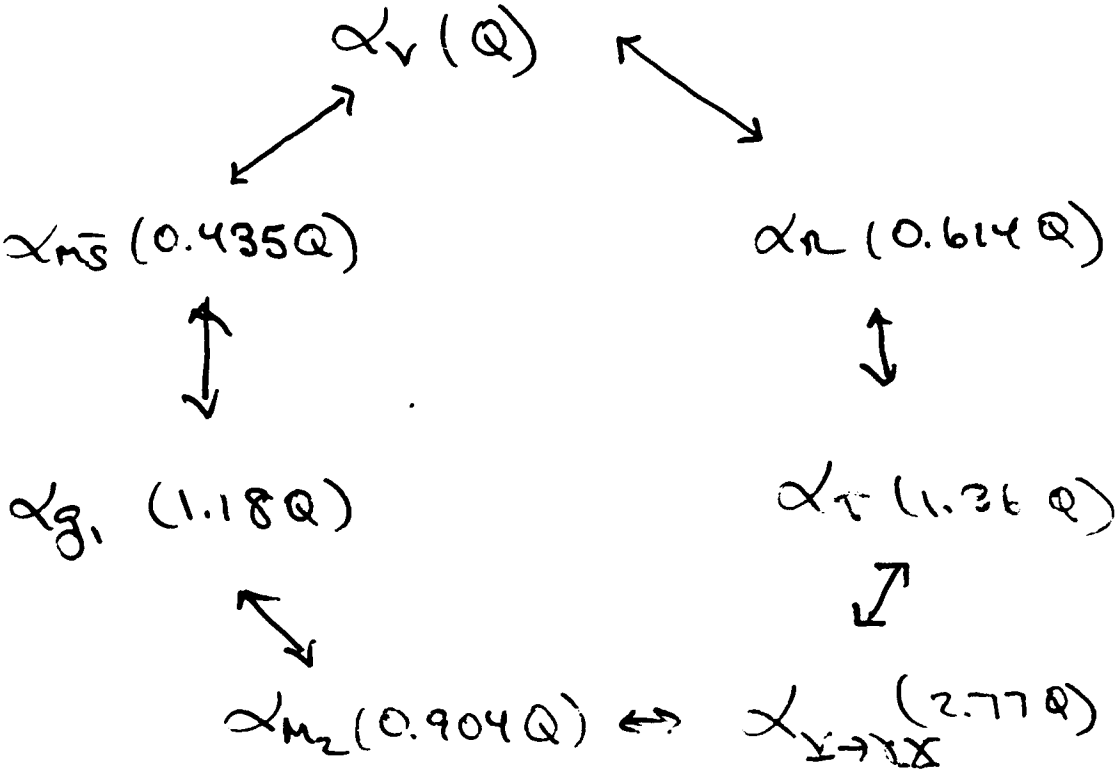
$\Rightarrow$ all n_f summed in $\alpha_V(q^*)$

not in coeffs

BLM

HLW

Commensurate Scaling Relations



e.g.

$$\alpha_{g_1}(Q) = \alpha_R(0.520Q)\left(1 - \frac{\alpha_R}{\pi} + \cdots\right)$$

transitive
reflexive
symmetric
identity

} scheme transformations

Renormalization Group

Peterman / Stueckelberg

We can define a "running coupling"

for $\overline{MS}$:

SJB
Gill
Mirabelli

$$\widetilde{\alpha}_{\overline{MS}}(Q^2) \equiv \alpha_V\left(e^{5/3}\, Q^2\right)$$

$$\left[1 + 2\frac{\alpha_V}{\pi} + C_2 \frac{\alpha_V^2}{\pi^2}\right.$$

↑
computed by M. Peter

analytic extension $n_f(Q)$ contin

Q^2 scale determined : no ambiguity

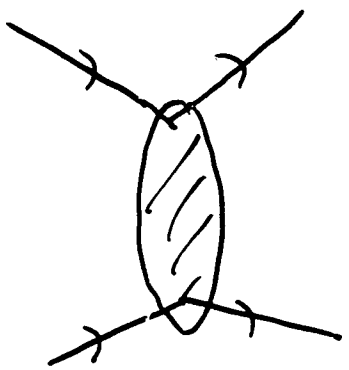
flavor thresholds unambiguous

$e^{5/3}$ due to $\overline{MS}$ convention

$2\frac{\alpha_V}{\pi} = 2\frac{C_A}{3}\frac{\alpha_V}{\pi}$ non-Abelian

* Use BaBar to determine

fundamental QCD coupling $\alpha_V(Q^2)$



$$V(Q^2) \equiv -\frac{4\pi C_F \alpha_V(Q^2)}{Q^2}$$

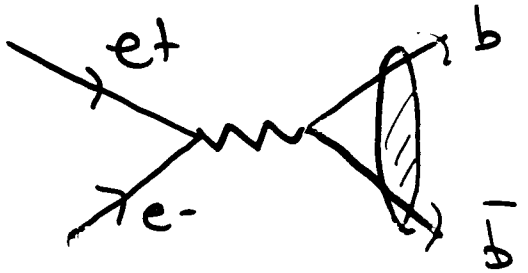
($C_F = 4/3$)

potential for scattering of heavy quark test charge

$\alpha_V(Q^2) = \alpha_{QED}(Q^2)$

(Abelian theory)

unification



sensitive to

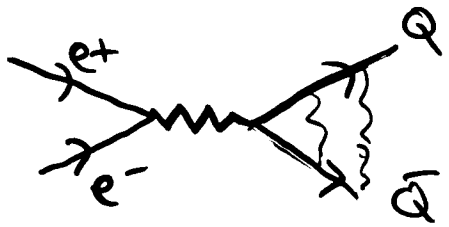
$\alpha_V(\beta^2 s)$

$$\beta^2 = 1 - \frac{4m_Q^2}{s}$$

Hoang. Tuchmann
Kühn SDB

PQCD corrections for slow heavy quarks

SJB
A. H. Hoang
J. H. Kühn
T. Teubner
M. Voloshin



$$\beta^2 = \frac{P_{cm}^2}{s} \ll 1$$

radiation suppressed

Strong corrections near threshold - relate to $\alpha_V(Q^2$

$$|F_1(s) + F_2(s)|^2 = \frac{x}{1-e^{-x}}\left[1 - 4C_F \frac{\alpha_V(m^2 e^{3/4})}{\pi}\right]$$

Use BLM scale-fixing

$$x = \frac{\pi C_F \alpha_V(4m^2\beta^2)}{\beta}$$

$$|F_1(s)|^2 = \frac{x'}{1-e^{-x'}}\left[1 - 3C_F \frac{\alpha_V(m^2 e^{7/6})}{\pi}\right]$$

$$x' = \frac{\pi C_F \alpha_V(4m^2\beta^2/e)}{\beta}$$

$\frac{dN}{d\Omega} \propto (1 + A\cos^2\theta)$: Strong corrections to Anisotrop. $A(\beta)$

$$\boxed{\frac{dN}{d\cos^2\Theta} = 1 + A(\beta^2)\cos^2\Theta}, \qquad A = \frac{\tilde{A}}{1-\tilde{A}}$$

$e^+e^- \to Q\bar{Q}$ $\quad (\beta^2 << 1)$

double resummation $\Rightarrow (\beta_0 \alpha_s)^n$

Coulomb rescattering

$$* \quad \tilde{A}_{QCD}(\beta^2) = \frac{\beta^2}{2} \; \frac{1 - 4\frac{\alpha_V(m^2 e^{7/6})}{\pi}}{1 - \frac{16}{3}\frac{\alpha_V(m^2 e^{3/4})}{\pi}} \; \frac{\frac{x'}{1-e^{-x'}}}{\frac{x}{1-e^{-x}}}$$

$$x = \frac{4\pi}{3}\frac{\alpha_V(4m^2\beta^2)}{\beta}, \qquad x' = \frac{4\pi}{3}\frac{\alpha_V(4m^2\beta^2/e)}{\beta}$$

* Relates observable to observable

$$\underline{A(\beta^2) \Longleftrightarrow V(Q^2)}$$

measured in HQLGTh

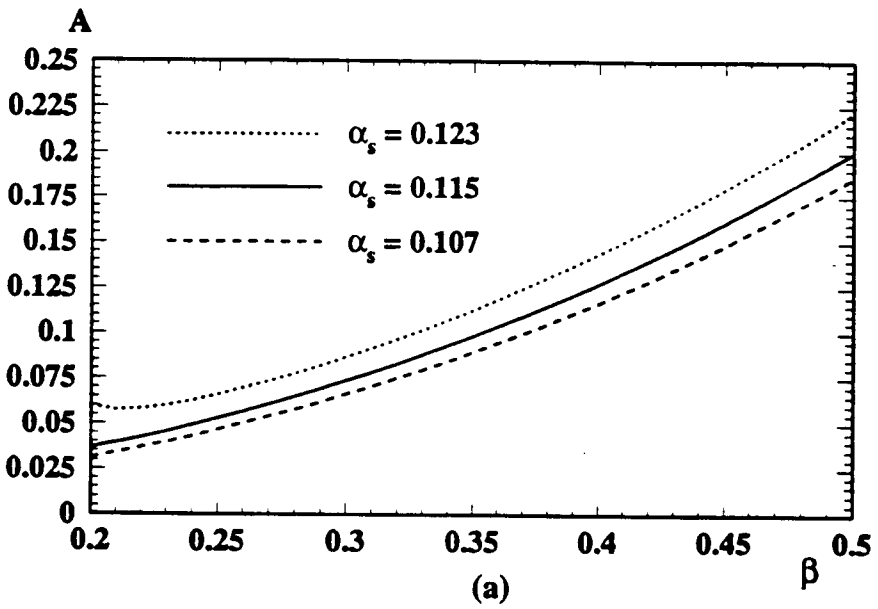
Davies et al

* No scale & scheme ambiguity! - four scales!

* Measure α_V at very low scales!

$$4m^2\beta^2 \to 0$$

$$\frac{dN}{d\cos\theta}\left(e^+e^- \to b\bar{b}\right) \propto 1 + A(\beta^2)\cos^2\theta$$



(a)

$$A = \frac{\tilde{A}}{1-\tilde{A}}$$

$$\alpha_S = \alpha_{\overline{MS}}^{(5)}(M_Z)$$

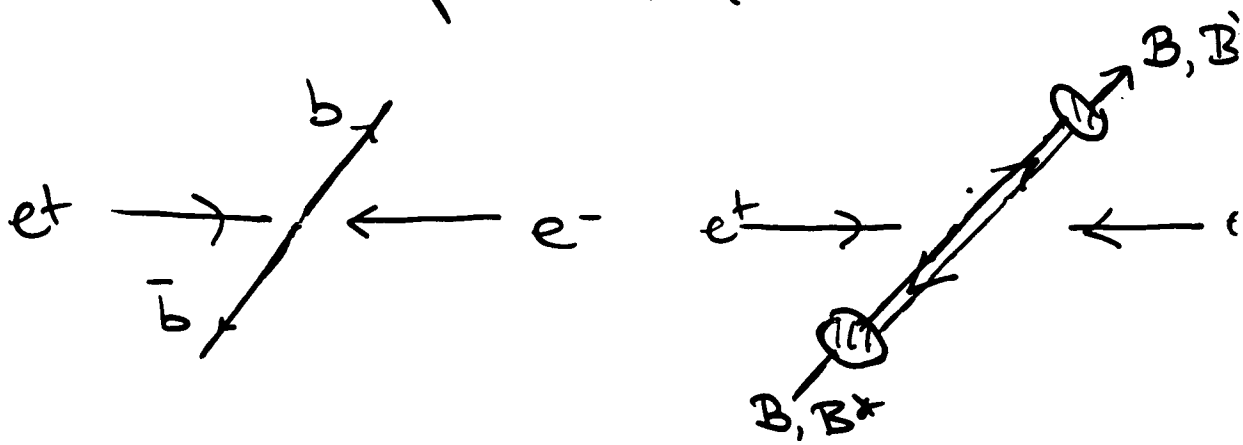
$$\tilde{A} = \frac{\beta^2}{2}\,\frac{\left(1-4\frac{\alpha_V(M^2e^{7/6})}{\pi}\right)}{\left(1-\frac{16}{3}\frac{\alpha_V(M^2e^{3/4})}{\pi}\right)}\,\frac{1-e^{-x}}{1-e^{-x'}}\,\frac{\alpha_V(4M^2\beta^2/e)}{\alpha_V(4M^2\beta^2)}$$

$$x = \frac{4\pi}{3}\frac{\alpha_V(4M^2\beta^2)}{\beta}$$

$$x' = \frac{4\pi}{2}\alpha_V(4M^2\beta^2/e)$$

* Ansatz : HQET

heavy hadrons follow direction of heavy quarks

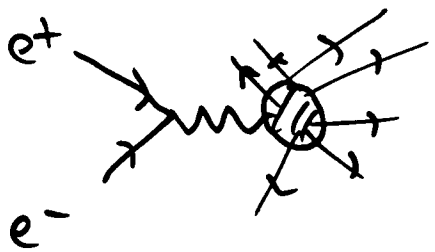


$e^+e^- \to b\bar{b}$ $\qquad$ $e^+e^- \to B\bar{B}, B\bar{B}^*, B$

Use sum over channels to determine

$$\frac{dN}{d\cos\theta} \propto 1 + A(\beta^2)\cos^2\theta$$

$$A(\beta^2) \Rightarrow \alpha_v(\beta^2 s)$$



$$(p_i + p_j)^2 < y s$$

$$f_n = \frac{\sigma_{n\text{-jet}}(y, s)}{\sigma_{TOT}(s)}$$

$$f_2(s, y) = 1 - c_{21}(y)\,\alpha_s(\mu) + c_{22}\left(y, \ln \frac{s}{\mu^2}\right)$$

$$f_3(s, y) = c_{31}(y)\,\alpha_s(\mu) + c_{32}\left(y, \ln \frac{s}{\mu^2}\right)\alpha_s^2$$

$$f_4(s, y) = c_{42}(y)\,\alpha_s^2(\mu) + \ldots$$

G. Kramer + B. Lampe Z. Phys A. 339, 189(1

$\alpha_s = \alpha_{\overline{MS}}$ ren. scheme

μ "arb. trary" renormalization scale

* BLM principle :

running coupling sums all V.P.

$\therefore$ coefficients must be independent of n_f, β_0

This fixes $\mu = Q^*(s, y)$

$$n_f \left[-\tfrac{2}{3} \ln \frac{s}{\mu^2} C_{21}(y) + \frac{C_F}{2\pi} Z_T(y) \right] = 0$$

$$Q^* = \sqrt{s} \exp\left(\frac{\frac{3}{4} \frac{C_F}{2\pi} Z_T(y)}{C_{21}(y)} \right)$$

special $\overline{MS}$

$$\cong \tfrac{1}{2}\left[\ln y - \tfrac{11}{12} + \right.$$

small

$$\therefore \quad Q^*(y) \cong \sqrt{ys}\, e^{-\frac{11}{12}}$$

for small y

See also Ingelman Rotsma

Physically correct! only sensitive to produced flavor

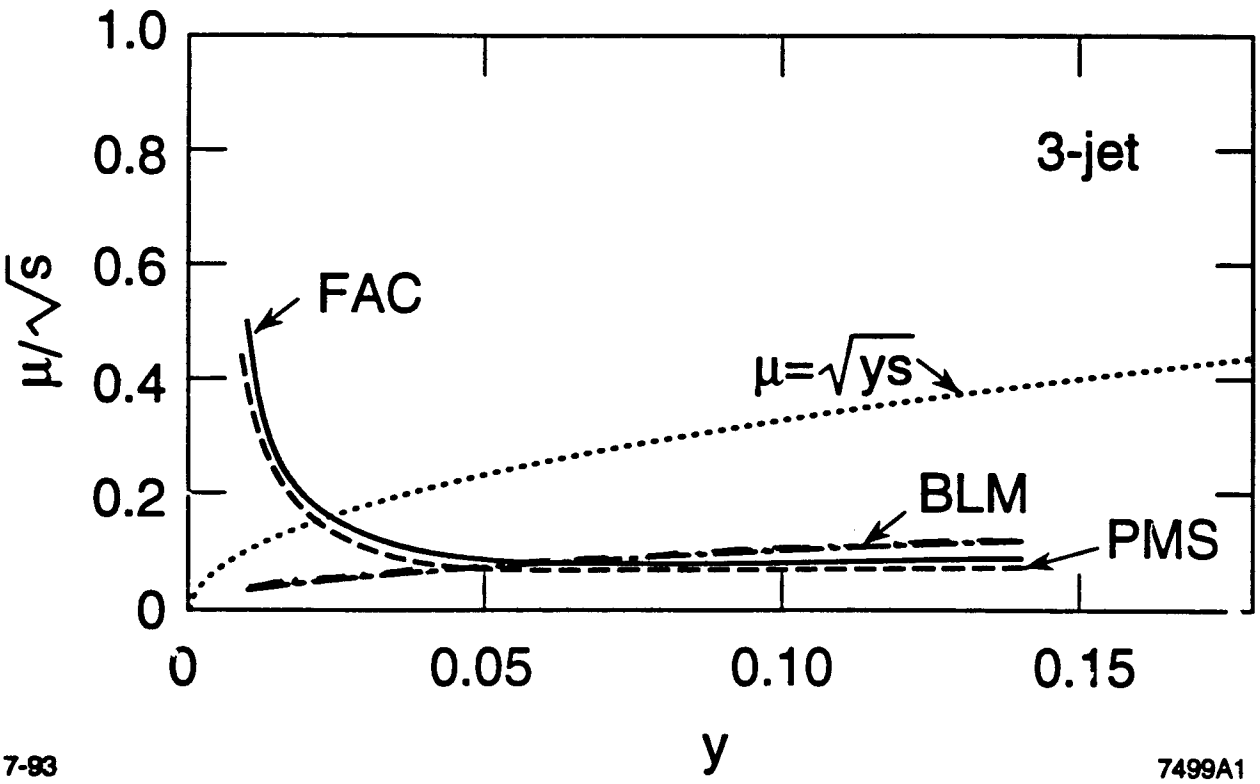
Kramer + Lampe

$\overline{MS}$ Sch

$(p_i + p_j)^2 < y s$

$$\bullet \quad \frac{\sigma_{3\text{-jet}}(y)}{\sigma_{TOT}} = C\,\alpha_s(\mu)\left[1 + (A n_f + B)\frac{\alpha_s}{\pi} + \ldots\right]$$

$C(y)$, $A(y)$, $B(y)$

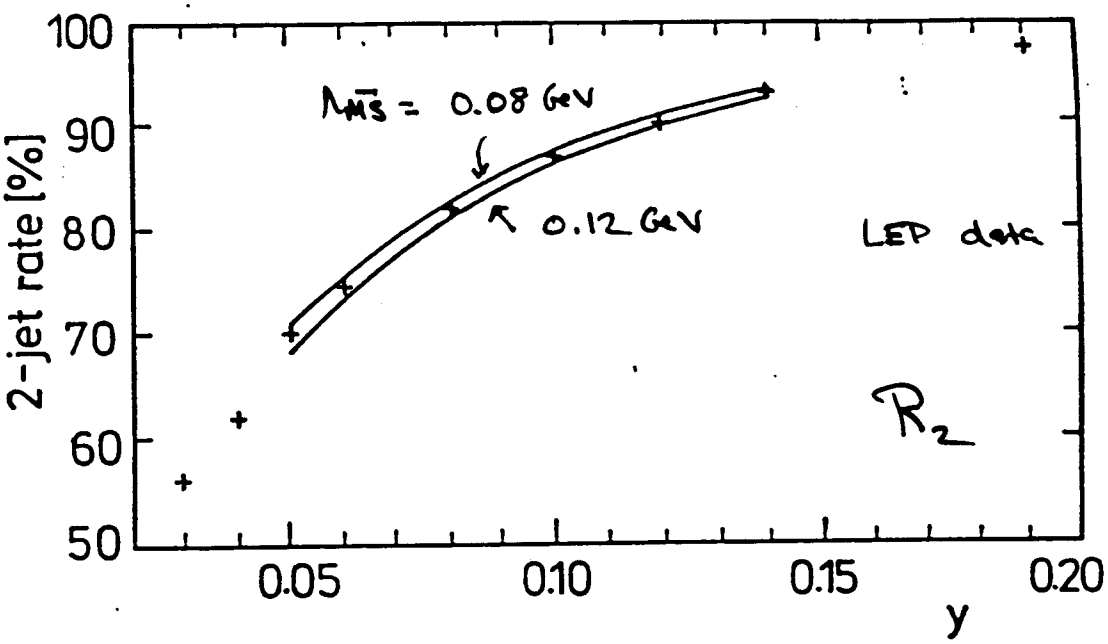


$$\mu_{BLM}(y) = \mu\, e^{3A(y)} \propto \sqrt{y s}$$

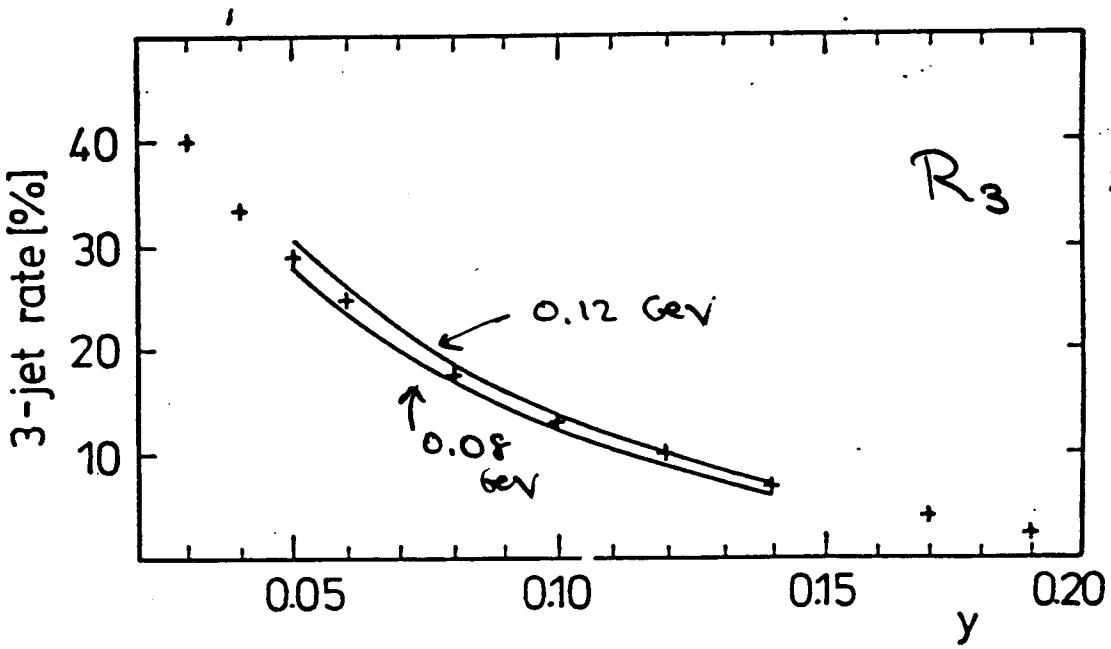
$\Rightarrow$ all n_f dep in α_s

Kramer + Lampe

Jet fractions: $(p_i - p_j)^2 < y s \Rightarrow$ same jet



* $\Lambda_{\overline{MS}} = 0.10 \pm 0.0$ GeV



* $\alpha_{\overline{MS}}(M_Z) = 0.107 \pm .00$

3LM: $\frac{\sigma_{3\text{-jet}}^{(y)}}{\sigma_{tot}} = K_3 \frac{\alpha_{\overline{MS}}(Q^*)}{\pi} \left[1 + C_3 \frac{\alpha_{\overline{MS}}}{\pi} + \cdots \right]$ (n_f indep.)

$M_{3LM} = Q^* \propto \sqrt{ys}$ for $y \to 0$

Two-Jet Fraction at $y \to 0$

$$f_2(s,y) = 1 - c_{21}(y)\,\alpha_s(\mu) + c_{22}\,\alpha_s^2 + \cdots$$

$$= 1 - \frac{C_F}{2\pi} 2\ln^2 \frac{1}{y}\,\alpha_s(\mu) + \frac{C_F^2}{(2\pi)^2} 2\ln^4 \frac{1}{y}\,\alpha_s^2 + \cdots \qquad y \to 0$$

$$= e^{-\frac{C_F}{\pi}\ln^2 \frac{1}{y}\,\alpha_s(\mu)}$$

* Sudakov form factor:

  probability of two jets $\Rightarrow 0$

  for $(p_i - p_j)^2 < y s \qquad y \to 0$

* Correct physics even if $\beta = 0$.

* BLM: $\alpha_s(\mu) \Rightarrow \alpha_s(Q^*)$

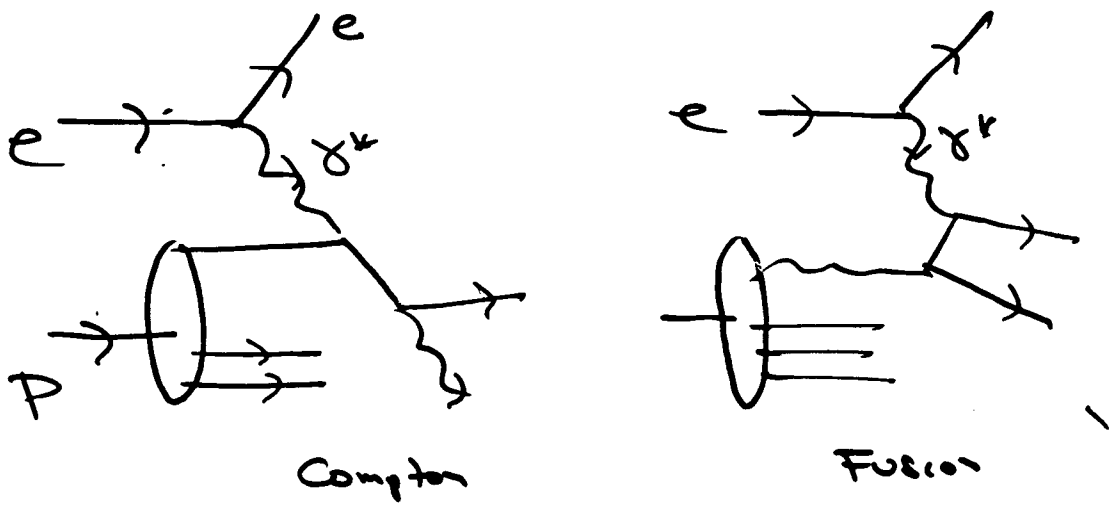
  $Q^* = \sqrt{ys}\; e^{-11/12}$

Lothaman & Ingelman:

Elimination of Renormalization

Scale uncertainty

in DIS Jet Cross Sections



rate of 2+1 jets to 1+1 jets

$$r = C(Q, x, P_T, \eta, \ldots)\, \alpha_s(\mu) + \ldots$$

choose scheme to define α_s e.g. $\overline{MS}$

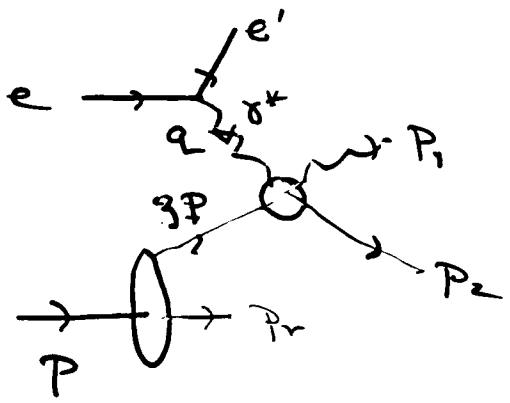
problem: what is μ

BLM: Separate Scales for Compton Fusion

Jets in Deep Inelastic Scattering

– A multi-scale problem

QCD Compton:



$$\xi = x\left[1 + \frac{(P_1+P_2)^2}{Q^2}\right] = \frac{x}{x_p}$$

$$z = \frac{P_1 \cdot P}{q \cdot P}$$

$$(P_i + P_j)^2 > y_{cut}\, W^2$$

QCD: Compton $\gamma^* q \to g q$ QCD: Fusion $\gamma^* g \to q\bar{q}$

$\overline{MS}$

$$\frac{d\sigma_{compton}}{dx\, dQ^2 \ldots} = A\,\frac{\alpha_s(\mu)}{\pi}\left[1 + (BN_f + C)\frac{\alpha_s}{\pi} + \ldots\right]$$

Graudenz
Brodkorb, Körner, Mirkes

$$= A\,\frac{\alpha_s(Q^*)}{\pi}\left[1 + \overset{\text{cont. coeff}}{D}\,\frac{\alpha_s}{\pi} + \ldots\right]$$

{Ingelman, Rathsman} BLM

$$* \quad Q^{*2} = \min\begin{cases} e^{-5/3}\, y_{cut}\, Q^2\, \frac{1-x}{x} & \text{Same for QE} \\ e^{-5/3}\,(1-z)\, Q^2\, \frac{1-x_p}{x} & \text{Comp} \end{cases}$$

$$A = A(x, Q^2, y_{cut}), \quad D = D(x, Q^2, y_{cut})$$

Ingelman + Rathsman

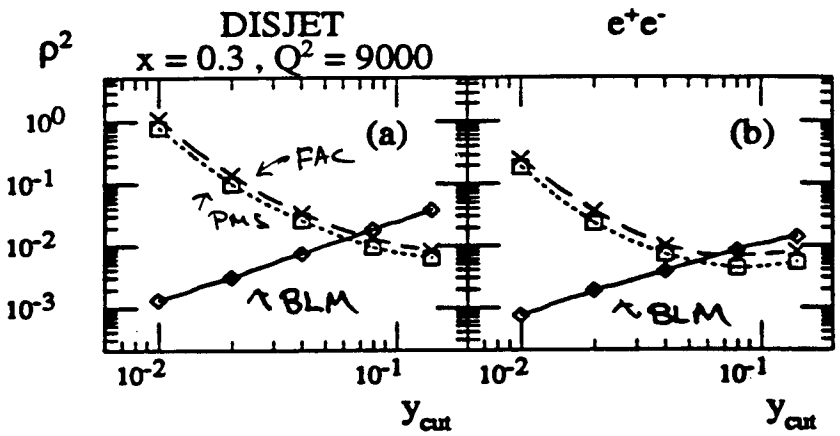
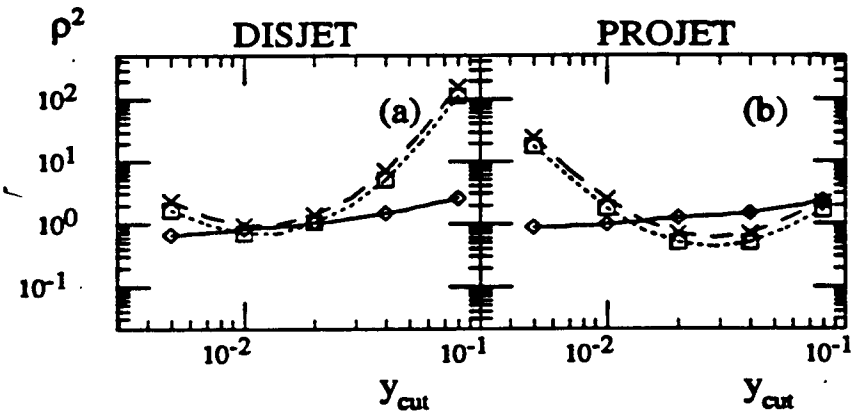


Figure 7: *Comparison of the scales obtained from the BLM (solid), FAC (dashed) and PMS (dotted) methods for (a) 2+1 jet production in DIS at $x = 0.3$, $Q^2 = 9000\ GeV^2$ and (b) 3-jet production in e^+e^- as given in ref. [22].*



$x = .001$

$Q^2 = 90\ GeV^2$

Figure 8: *Comparison of the scales obtained in $x = 0.001$, $Q^2 = 90 GeV^2$ according to the BLM (solid), FAC (dashed) and PMS (dotted) methods with (a) the complete α_s^2 cross-section (DISJET) and (b) the partial result (PROJET).*

$$\rho^2 = \frac{Q^{*2}}{Q^2}$$

from $\alpha_V \Rightarrow \alpha_{\overline{MS}}$

DIS: Find $\rho^2_{BLM} \cong \frac{\overline{W}^2}{Q^2} y_{cut}\ e^{-5/3}$

$W^2 = Q^2 \frac{1-x}{x} = (q+p)^2$

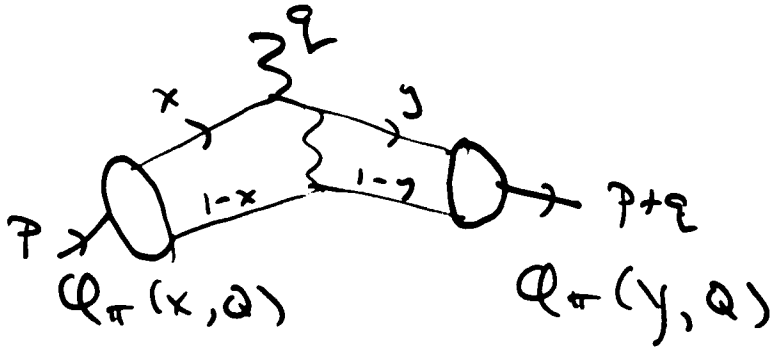
Application to Exclusive Processes

Form Factors

Elastic Scattering

Heavy Hadron Decays

e.g. $F_\pi(Q^2)$



$$\int_0^1 dx \int_0^1 dy \; Q_\pi(x,Q)\, Q_\pi(y,Q)\, T_H$$

di Pang Lu 830

$$T_H = \frac{\alpha_V\left((1-x)(1-y)Q^2\right)}{(1-x)(1-y)Q^2}$$

$\alpha_V(Q^{*2})$ $\qquad Q^{*2} \simeq e^{-3} Q^2$

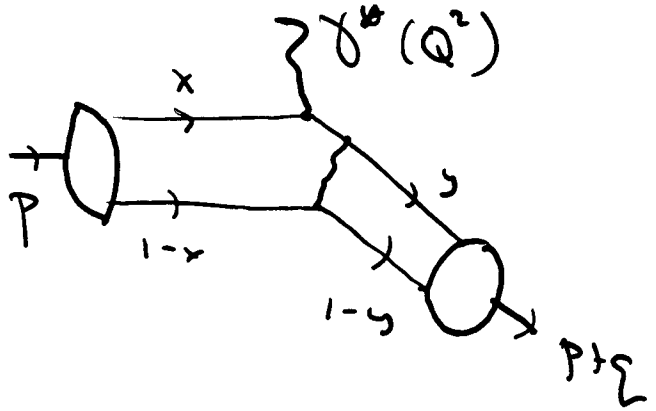
small scale even at large Q^2

$\alpha_V \sim$ const ? $\Rightarrow$ dim. counting

SJB
HLW
ERJ
A. Pang

Exclusive Amplitudes in QCD

$F_\pi(Q^2)$:



large Q^2 :

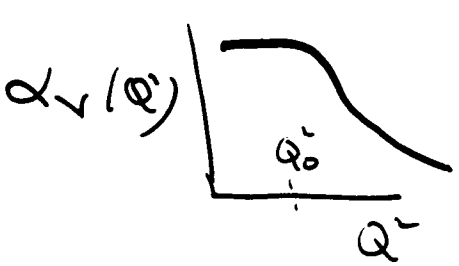
$$F_\pi(Q^2) = \int_0^1 dx \int_0^1 dy\ \phi_\pi(x,Q)\,\phi_\pi(y,Q)\,\frac{4\pi C_F\,\alpha_V\left[(1-x)(1-y)Q^2\right]}{(1-x)(1-y)Q^2}$$

For $\phi_\pi^{(x)} \propto x(1-x)$

$$F_\pi(Q^2) = C\,\frac{\alpha_V\left(e^{-3}Q^2\right)}{Q^2} + \text{L.O.}$$

← BLM scale

very low scale! $\sim \frac{1}{20}Q^2$



If $\alpha_V \sim$ const

$Q^2 F_\pi \sim C$

in $Q^2 < 20\,Q_0^2$.

$\alpha_V(Q^2)$ poorly determined at low Q^2

$$\alpha_V \sim \frac{4\pi}{\beta_0 \ln Q^2/\Lambda_V^2}$$ only valid at $Q^2 \gg \Lambda_V^2$

Program:

$$\bar{\alpha}_V^{(n)}(Q_0^2) = \int_0^{Q_0^2} dQ^2 \frac{(Q^2)^n}{(Q_0^2)^{n+1}} \alpha_V(Q^2)$$

Dokshitzer
Webber

physical parameters
constrain from expt.

BLM, CSR

Resum β terms into scale Q^*

Equivalent to n_f resummation

Large n_f resummation $\Rightarrow$ β_0 resummation

(Only n_f assoc. with renormalization)

Commensurate Scale Relations

* Relate Observable to Observable
* Convention - Independent Tests of QCD
* No scale ambiguity at any finite order
* Identifies underlying conformal relations
* Transitive : any intermediate scheme / scale
* Consistent with BLM, large n_f, naive non-abelian
* Consistent with Abelian ($N_C \to 0$)
* Expansion in finite effective charges
* I.R. Renormalon summed into scale
* Q^* expansion can have renormalon
* Extends $\overline{MS}$ to analytic scale fixed scheme
* α_V : special advantages

QCD Program

* Adopt Standard effective charge

e.g: $\alpha_V(Q^2)$ many advantages

* Use observables to set parameters, e.g.

$$\frac{1}{Q_0^2}\int_0^{Q_0^2} dQ^2\, \alpha_V(Q^2) = \overline{\alpha}_V(Q_0^2)$$

* Calculate from LGT, LCQ

* Commensurate Scale Relations:

Precise, convention-independent tests of QCD

Summary: PQCD / S.M.

* Connect Observable to Observable

⇒ no scale or scheme ambiguities

at any finite order

Commensurate Scale Relations

* "True" coefficients = conformal coefficients

$\mu = Q^*_{BLM}$ resums $\beta \neq 0$

$$Q^{*2}_{BLM} = \langle k_g^2 \rangle_V \qquad (MVT)$$

* $\mu \neq Q^*_{BLM}$: $\sum_n (\beta_0 \alpha_s)^n n!$

$g \Rightarrow 0 : Q^*_{FAC,PMS} \to \infty$

unphysical behavior

sums conformal physics

* Error estimate from BLM/FAC

* $\alpha_{\overline{MS}}(Q)$: physical definition ⇔ LGTh.

BLM Scale Fixing

* Physical Scales : test fermions probe gluon virtuality

* Multiple scales okay

* Commensurate scales:
  consistent treatment of quark thresholds

* Coefficients given by conformal-inv. theory

* Coefficients give meaningful criteria for convergence ; reflect physics

* Applications to all areas of Standard Model
  Uniform treatment of QED, EW, QCD

* BLM: resum β_0, β_1 into α_s

  → no criterion to minimize high order terms

Recent Developments CSR-BLM Scale Setting

* Ingelman Rathsman HERA 2+1 Jets

* Smith + Voloshin Sirlin Γ_{top}

* Zerwas, Spira, Graudenz Γ_{Higgs}

* Luke, Savage, Wise Γ_b

* Braun + Beneke Mueller Veubert UV Renormalons Q^*, Higher Twist

* SJB C-R Ji Exclusive Processes

* SJB Kai Wong $\frac{\partial}{\partial \ln Q^2} M_n(Q^2)$

* SJB G. Mirabelli Finite mass evolution $\alpha_s(Q)$

* SJB H.J. Lu A. Kataev Crewther Reln