Modern Physics (PHY 3305) Lecture Notes

Space and Time and Special Relativity

SteveSekula, 22 January 2010 (created 13 November 2009)

tags: lecture

CHAPTER 2.1-2.3

Review the salient parts of the previous lecture:

- Classical physics was a tremendous success but failed to explain measured properties of light (EM and Relativity) and blackbodies (Mechanics and Thermodynamics)
- the speed of light in vacuum is $c = 2.995 \times 10^8 \mathrm{m/s}$
- Michelson and Morley found that the speed of light did NOT depend on whether it was traveling with/against the earth's motion or perpendicular to it, though based on the ether hypothesis they did expect to measure such a difference.
- in our everyday lives, speeds vary from a few m/s to tens of m/s (highway speeds), sound is hundreds (10^2) of m/s, and in rocketry and satellites, speeds reach $10^3 - 10^4$m/s (25,000 mph) - all of these are far smaller than that of light
- Light is fast, but it is surprising how much we take for granted that light is instantaneous - it is not (consider that it takes ~75 years for light from the Big Dipper to reach us on Earth, and in cosmic terms the Big Dipper stars are fairly close!).

We will establish a notation that we will follow rigorously for the remainder of the course. We will denote quantities in a frame considered to be *at rest* using un-primed quantities (x,y,z,t,S). Those in a frame moving at constant velocity with respect to the frame considered stationary will be primed (x', y', z', t', S'). The velocity of one frame with respect to another will be written as $\vec{v}$. The velocity of objects IN a frame will be written as $\vec{u}$.

Avoid confusion: since we are dealing with relativity, and observers in each frame will claim they are stationary and the other is moving, we as scientists must enforce a choice (the choice does not matter) as to which we consider at rest. Then we make stick to that choice and do our work accordingly.

Consequences of the observation of no ether (needless ether) and the constancy of light

- the *Michelson-Morley* experiment helped to cast away the ether hypothesis
- electromagnetism explained the origin of light, predicted its constancy, but did not require a medium for its wave propagation
- the speed of light was observed to be unaffected by motion

Einstein's postulates:

- the laws of physics are invariant to observers in different inertial reference frames; that is, if you and I are conducting observations and experiments and moving relative to one another at constant velocity, the laws of physics governing the outcomes are not affected and the outcomes are the same. We will always agree that the mathematical framework governing the observation has not been affected by the motion, and we will agree on events themselves (occurrences in space and time).
- the speed of light is the same to all observers, regardless of their motion relative to the source of the light. If I am moving at half the speed of light relative to an LED, and I make a

measurement of how fast light is traveling from the LED to me, I will always observe light to travel at c.

Let us now explore qualitatively the consequences of these postulates using a story.

RELATIVE SIMULTANEITY

We are very accustomed to talking about things happening "at the same time" or "simultaneously". What does that ACTUALLY mean? Consider two people names Anna and Bob. They're old college friends, and Anna is passing through town on a train. She phones ahead to Bob and suggests it might be fun for him to wave to her, and her to him, as her train passes. She tells him to wave at the third car from the front, middle window on the car. Anna's train is moving east-to-west at high speed relative to Bob, who is standing on a platform next to the tracks. As Anna's train approaches, the weather turns sour. As the train passes, and just as Bob is lined up with Anna's window and they are waving, lightning strikes exactly one mile east and one mile west of where Bob is standing. Bob calls Anna on her mobile phone and says, "Wow, did you see that! I've never seen lightning strike simultaneously like that before - the two bright flashes hit just at the same instant!" Mathematically, we can translate his English to the following in Math:

- Let Event 1 be the east-side lightning strike and Event 2 be the west-side lightning strike. The strikes are at different positions in space when they happen - call them x_1 and x_2. But, to Bob they happen at the same time, so the times of Event 1 and Event 2 are simply related as $t_1 = t_2$.

Now, let's apply Einstein's postulates and see what happens. What does Anna see, since she is moving at high speed relative to Bob and the lightning?

- Postulate 2 consequence: this is the easier one to apply, so we'll do it first. The light from the bolts travels at c, both from the perspective of Anna and Bob. That means that if each of them was testing to see how fast light moved from each lightning strike as it traveled to their vantage point, BOTH would measure c. The speed of light is constant and independent of the state of motion of an observer.
- Postulate 1 consequence: Bob *observed* the lightning strikes simultaneously. That observation cannot be in question, according to postulate 1. Observers must agree on the events, even if they disagree on when and where they occur. Otherwise, there is a complete contradiction in the outcome of the experiment, which is not allowed by postulate 1. Events are not in question; when and where they occur are instead in question.
    - In math: When Anna talks to Bob on her phone after the train passes, Bob will report to her that in his frame although $x_1 \neq x_2$, he found $t_1 = t_2$.

So what is the conclusion? What does Anna see? If the events are not in question - that lightning struck in the east and west directions, and that Bob saw them happen simultaneously, there is only one conclusion: Anna argues that while the lightning strikes occurred, they did not happen *at the same time*. She agrees that the light from each reached Bob at the same time, but that's because she sees the western strike happen BEFORE the eastern strike. In her frame of reference, where Bob appears to be moving eastward at high speed and the speed of light is constant and c, this is the only way that he can be right. Once the light is emitted from the bolt, it travels at a fixed velocity toward Bob; in Anna's view, he is moving toward one wavefront of light and away from the other; thus, since the western bolt must travel further, it must have happened earlier.

However, do not be mis-led: the consequence of special relativity ARE NOT OPTICAL ILLUSIONS. It is quite directly the following: space and time are different for observers in relative motion. There are no optical illusions; there are fundamental changes to our notion of space and time. *Things simultaneous to one observer and not necessarily so for all observers.*

TIME DILATION

Einstein reported that one of his key revelations that led to the special theory of relativity happened on a streetcar ride in Bern. Hopping on the car as it pulled away, he looked back at the clocktower of Bern and realized that since he is moving away from the clock, light bouncing off the clock takes longer to reach his eyes. The ticks of the clock would appear to happen more slowly as he moves, than when he is standing still and watching the clock. This would become apparent at speeds comparable to that of light, as light itself would struggle to keep up with you. This is an optical illusion which helps you to appreciate the deeper truth about Einstein's postulates.

This consequence of the postulates deals with time alone and is called *time dilation*. It is another revealed property of the universe once you admit the postulates of relativity. Moving clocks run slower than stationary clocks, because time itself is slowing down. While two observers will agree that events occur, they will not agree on the time at which they occur. In the language of mathematics, $t_2 - t_1 \neq t_2' - t_1'$ when observations are made of events happening at the same place in space ($x_1 = x_2$ for instance).

As an example, consider that as Anna zooms by on the train she waves at Bob. Afterward, when talking on the phone, Bob asks, "Why did you wave at me so slowly, like a pageant winner?" Anna takes offense - she hates pageants - and contends she gave one wave and it happened very fast. They will agree that she waved, but Bob will think that she is doing to slowly while Anna doesn't notice. In Anna's frame, all time is running slowly and her perceptions are affected as well.

While it is comfortable to start from the "optical illusion" described by Einstein, that was but an inspiration for his idea. The idea that moving clocks run more slowly doesn't require physical ticks of a clock hand, viewed at some speed, to happen. We'll discuss at the end of a lecture a case of a fundamental subatomic particle, with no internal moving parts, which appears to respond to the slowing of time even though it has no physical clock. Time is a concept outside of only our ability to make a clock.

LENGTH CONTRACTION

The final consequence of the postulates goes hand-in-hand with the time dilation effect, and that is the contraction of moving objects along the direction of motion. As an example, let's pretend that the station where Bob is waiting for Anna to pass is famously known as *Hundred Meter Station*, because the architect insisted on making it that long even though the town wanted a shorter platform. As Anna passes, she times how long it takes for one end, then the next, to pass in front of her. Since the conductor of the train told them their cruising velocity, she computes the length of the station and finds it to be shorter than 100m. On her phone call with Bob, she asks whether the name is ironic. "Ironic?" he asks. "Yeah, that station is much shorter than 100m." Bob disagrees - he was on the team that built it and he knows it's 100m.

Why do they disagree? Since Anna's clocks are running slowly, so in the time it takes fo the train to pass by the station fewer seconds pass for Anna. Since distance is $L' = t' \times c$, Anna measures a shorter distance and finds the name to be ironic.

To Anna, Bob and everything outside the train appears shorter than it does to Bob. Likewise, to Bob the train and Anna seem shorter than they do to her. Were she to walk the length of a car or the train, and report her observation to Bob, he would disagree (he would measure the train by using the time difference between the front and back passing in front of him, at the same place x in his reference frame).

Mathematics of these scenarios

For each of the above scenarios, we can establish some useful mathematical relationships within each frame:

- The lightning strikes
  - For Bob, whom we choose to be at rest, $x_1 \neq x_2$ but $t_1 = t_2$. For Anna, $x'_1 \neq x'_2$ (and those are not necessarily equal to Bob's spatial coordinates), and $t'_1 \neq t'_2$
- The slow wave
  - For Bob, the time between her hand being on one side of the wave (event 1) and the other (event 2) seems longer than it does to Anna; thus $x_2 - x_2 \neq x'_2 - x'_1$
- The ironic station name
  - For Bob, the ends of the station are 100m apart; for instance, he can use two laser-rangers and stand in the middle of the platform, waiting for each pulse to bounce off something at each end and return to him. For him, $x_2 \neq x_1$ but the reception of the two pulses occur at the same time, $t_1 = t_2$. For Anna, who times the difference between one end and the other passing in front of her, $x'_2 = x'_1$ but $t'_2 \neq t'_1$, and she find the distance to be shorter.

Evidence for Relativistic Effects

The muon is a subatomic particle whose discovery shattered our understanding of the universe. Up until then, it was believed that all matter could be explained by atoms, each made from an arrangement of protons, neutrons, and electrons. But the discovery of the muons, something not of the atom, caused one physicist (Rabi) to exclaim, "Who ordered that?"

A great deal of study was done on the muon. They were produced in the upper atmosphere and rained down on the earth, which is how they were detected. Once captured, their lifetime (the time it takes for a muon to decay into other things) was measured and found to be $2.2\mu\mathrm{s}$. They were also found to be traveling downward at speeds approaching light. A classical calculation revealed that muons SHOULD NOT reach earth - their lifetimes are too short! Yet they do. Relativity explains this. The muons' clocks are ticking in their reference frame, which is moving relative to us at near c. That means their clock runs SLOW, so while in their frame a life lasts $2.2\mu\mathrm{s}$, in our frame it lasts much longer because to us their clock runs slowly. We'll explore this problem later, once we've established the mathematics.

Break

The Lorentz Transformation (or How I Learned to Stop Worrying and Love Relativity)

It's all fine and well that Anna and Bob agree on events but disagree on the space and time patterns of those events. But is there a way to relate what they see? Yes! The *Lorentz Transformation*. Let's figure this out by applying Einstein's postulates. We begin with the classical relationship between two frames of reference and then go from there.

Definitions Reminder

- (x, y, z, t) and S are the space-time coordinates in and notation for the stationary frame of reference, respectively

- (x', y', z', t') and S' are the space-time coordinates in and notation for the moving frame of reference, respectively

Classical Galilean Relativity

Consider Anna and Bob again. Anna is in a train moving west at speed ν. She throws a ball toward the front of the train at speed u'. The position of the ball at any time is given by $x' = u't'$. Bob sees the position of the ball in his frame as $x = u't + \nu t$, which can be written as $x = x' + \nu t$ to get the relationship between position in the two frames. The central relationship in classical physics is

$$x' = x - \nu t,\ t' = t$$

Space may be relative, related in different frames simply be knowing the speed of the relative motion, but time is assumed to march at the same pace in both frames. This equation is a TRANSFORMATION between one and the other frame. We can relate the speeds of the objects using this transformation:

$$u' = dx'/dt' = d/dt(x - \nu t) = u - \nu$$

But we get into trouble fast. If the object in motion in Anna's frame is light - say, a camera flash that is to reflect off a mirror at one end of the car - we know what experiments tell us. In Anna's frame, the light travels at c and the same is true in Bob's frame. But classical physics tells us that:

$$c' = c - \nu$$

which experimentally we know to be false. Why should light be special? Einstein chose to put light in its place as a normal thing and instead cast aside the assumption that space and time are special. Once you allow times to run slower and distances to contract for moving objects, suddenly the speed of light can be preserved while fundamentally altering the meaning of space and time.

The Lorentz Transformation

What is the transformation between frames S and S'? Let's consider two inertial frames; that is, there are no net forces in either frame and they move at constant velocity relative to one another. With no accelerations, the only relationship that can exist between the two frames is one in which space and time as measured in each frame are linearly related to one another. Thus, the most general relationship we can write relating the frames is

$$x' = Ax + Bt \ \text{ and } \ t' = Cx + Dt.$$

Let us now consider an object that can be in motion. The object can move at constant velocity u (u') in frame S (S'). In each frame, space, speed, and time for the object are related by $x = ut$ and $x' = u't'$. How do we use this situation to figure out the constant coefficients (A,B,C, and D) in our generalized transformation?

Since constants are constants, we can apply different scenarios and constrain the relationships. Whatever the relationships of constants in constrained situations, that's their relationship in any other situation!

1. For our first situation, let's consider the case where our object is pinned to the origin of S'. The mathematical expression for this condition is:

   $$u' = 0,\ x' = 0,\ u = \nu, \text{ and } x = \nu t.$$

   In this situation, the generalized relationship $x' = Ax + Bt$ becomes

   $$0 = A(\nu t) + Bt$$

   which yields the relationship

   $$B = -\nu A.$$

   So now we have B in terms of ν and A, and we've reduced the number of unknowns by one!
2. For our second situation, let's consider the case where our object is pinned to the origin of S. We can express this as

   $$u = 0\ x = 0\ u' = -\nu, \text{ and } x' = -\nu t'.$$

   Thus, we constrain the generalized relationship $t' = Cx + Dt$ to

   $$t' = Dt.$$

   We can then plug this into the other generalized equation, $x' = Ax + Bt$, and employ $x' = -\nu t'$, to find:

   $$-\nu t' = B(t'/D)$$

   $$-\nu D = B$$

   Substituting $B = -\nu A$ we arrive at

   $$D = A.$$

   We have now eliminated two of the unknown constants in favor of A.
3. For our third situation, let's consider that the object in motion is a beam of light. Thus,

   $$x' = ct',\ x = ct.$$

   Substituting into the general relationships, and using $D = A$ and $B = -\nu A$, we find

   $$x' = ct' = A(ct) - \nu At = At(c - \nu)$$

   and

   $$t' = C(ct) + At = (C \cdot c + A)t.$$

   Substituting for t' in the second equation using the first, we arrive at

   $$c(C \cdot c + A)t = At(c - \nu)$$

   , which we can rewrite to solve for C:

   $$C = (A(1 - \nu/c) - A)/c = A(-\nu/c^2).$$

   We have now eliminated all other constants in favor of A.
4. Let us write the general equations relating the two frames using A:

$$x' = A(x - \nu t) \text{ and } t' = A(-\nu/c^2 x + t).$$

In order to solve for A, let's use these two equations to solve for x and t. Doing this yields

$$x = \frac{1}{A(1-\nu^2/c^2)}(x' + \nu t') \text{ and } t = \frac{1}{A(1-\nu^2/c^2)}(\nu/c^2 x' + t').$$

To solve for A, we have to apply some physics: since the two frames are in constant relative motion with respect to one another, the ONLY difference between the frames is the direction each sees the other moving - that is, the sign of the relative velocity, $\pm\nu$, of the two frames. Applying that observation then forces us to set the coefficients equal to one another:

$$A = \frac{1}{A(1-\nu^2/c^2)}.$$

Solving for A yields:

$$A = \frac{1}{\sqrt{1-(\nu/c)^2}} \equiv \gamma_\nu$$

Thus we have arrived at the glorious Lorentz Transformation, first written down by H. A. Lorentz when he tried to understand the motion of matter in the ether and postulated that matter must contract along the direction of motion (thus causing the laws of electromagnetism to retain their form even in a moving frame):

$$\text{FRAME } S: \; x' = \gamma_\nu(x - \nu t), \; t' = \gamma_\nu\left(-\frac{\nu}{c^2}x + t\right)$$

$$\text{FRAME } S': \; x = \gamma_\nu(x' + \nu t'), \; t = \gamma_\nu\left(+\frac{\nu}{c^2}x' + t'\right)$$

Only properties ALONG the direction of motion are affected. We could also include motion in other directions (y,z) and solve it more generally, but since we can always define the x-axis as that which occurs along the direction of relative motion, it can be simplified to this form.

We also see a deep connection to the classical physics we already know. If the speeds in question are much smaller than light, $\nu << c$, then the above equations simple reduce to their classical counterparts! It's no wonder these effects remained undetected for so long, and thus seemed "outside experience."

A few other special cases are of note:

- When $\nu = 0$, $\gamma_\nu = 1$
- When $\nu > 0$, $\gamma_\nu > 1$
- When $\nu = c$, the gamma function is UNDEFINED (infinity). That is, the Lorentz Transformation cannot be applied to light or, conversely, light is the fastest anything can go and when you move at the speed of light time is infinitely dilated (time does not pass at all) and space is infinitely contracted (the universe appears to have no size at all).

Special Cases

Time:

A special case for time occurs for the frame where all events occur at the same location IN THAT FRAME. When that occurs, $x'_1 = x'_2$ even when $t_1 \neq t_2$. Time, as measured in this frame, is called *PROPER TIME*, and is usually given a special symbol: t_0. In that frame, $t_2 - t_1 = \gamma_\nu(t'_2 - t'_1) \to \Delta t = \frac{1}{\sqrt{1-(\nu/c)^2}}\Delta t_0$. Proper Time is the shortest time interval of any measured interval in any of the frames.

Space:

Again, there is a special case for a frame where all of the events occur SIMULTANEOUSLY, or at the same time. In that case, $t'_2 = t'_1$ and the Lorentz Transformation equation for distance in frame S is $\Delta x = \Delta x' / \frac{1}{\sqrt{1-(\nu/c)^2}} \to L = L_0 / \frac{1}{\sqrt{1-(\nu/c)^2}}$. The lengths measured in this special frame are called *PROPER LENGTHS* and are the longest lengths measured in any of the frames.

Revisiting the Consequences of Einstein's Postulates

We can now quantify the results reported in the example of Anna and Bob and the train. Keep in mind that we are now talking about the relationships between events, each marked by a particular place and time. Thus when we talk about the difference in space or time BETWEEN two events, we mean:

$$(x'_2 - x'_1) = \gamma_\nu[(x_2 - x_1) - \nu(t_2 - t_1)]$$

$$(t'_2 - t'_1) = \gamma_\nu \left[\left(-\frac{\nu}{c^2}(x_2 - x_1) + (t_2 - t_1) \right) \right]$$

- The lightning strikes: In Bob's frame, the lightning strikes are at different places along the direction of motion of the train but $t_2 = t_1$. What does Anna see? Using the above, we see that

$$(t'_2 - t'_1) = \gamma_\nu \left[\left(-\frac{\nu}{c^2}(x_2 - x_1) \right) \right]$$

If the lightning stikes are +1000km (west, event 2) and -1000km (east, event 1) from Bob and the train is moving at 0.87c, then $\gamma_\nu = 2.0$ and

$$(t'_2 - t'_1) = 2.0 \times \left[\left(-\frac{0.87}{c}(2000\text{m}) \right) \right] = -1.2 \times 10^{-5}$$

and Anna determines that the western lightning strike must have happened before (at an earlier time) the eastern one. This is what we concluded qualitatively, applying Einstein's postulates.

- The slow wave: To Anna, her waving basically appears to happen in the same place ($x'_2 = x'_1$) and takes about 1 second; but Bob sees Anna moving, so the start (event 1) of her wave is at a much different place then the end (event 2) of her wave. We can try to find out how much time Bob thinks the wave takes:

$$(t_2 - t_1) = \gamma_\nu \left[(t_2' - t_1')\right] = 2.0 \times [1\text{s}] = 2\text{s}$$

Bob concludes the wave taking a mocking 2 seconds to finish!

- The ironic station name: Bob knows the length of the station to be 100m, and that the time it will take Anna to pass from one end of the platform to the other will be $100\text{m}/\nu$. Anna will make the measurement by timing how long it takes to see the ends of the station pass her by; this will occur, in her frame, at the same place in space ($x_2' = x_1'$). What does Anna find when she uses her speed and the time between the ends of the station passing to measure the distance?

$$\nu \cdot (t_2' - t_1') \equiv L' = \gamma_\nu \left[\left(-\frac{\nu^2}{c^2}(x_2 - x_1) + (x_2 - x_1)\right)\right] = 100\text{m} \times \gamma_\nu \times 1/\gamma_\nu^2$$

Thus we determine that:

$$L' = L/\gamma_\nu$$

and in our example:

$$L' = 100\text{m}/2.0 = 50\text{m}$$

Anna thinks the station is half the length the name implies.

Some mental warm-ups

- *Living Longer by Commuting*: Can commuting to work/school extend your life? When you commute, time passes more slowly for you than those at rest with respect to the surface of the earth. Let's assume that you commute one hour every day, at 75.00 mph. Skipping weekends, and assuming you do this for 45 years, how much younger are you than you otherwise would be?
  - First, we compute the gamma factor (γ_ν) for commuting. The speeds here are so SMALL compared to the speed of light that blindly plugging the numbers into a computer (even in MATLAB, for instance) will return a gamma factor of 1.000 - computers are good at math but bad when it comes to knowing what YOU want. So we will use a neat trick for handling small numbers compared to 1: the binomial expansion! The Binomial Expansion of

    $$f(x) = (a+x)^n$$

    is

    $$f(x) = (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ...(x^2 < a^2).$$

    We'll also encounter the following a lot:

$$\frac{1}{(1-x^2)} = 1 + x^2 + x^4 + x^6 + ...$$

and

$$\sqrt{\frac{1}{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + ...$$

and finally

$$\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - ...$$

The gamma factor for our car at 75 mph is:

$$1/\sqrt{1-(33.53m/s)^2/(2.995\times 10^8 m/s)^2} = 1.000 + \frac{1}{2}(33.53m/s)^2/(2.995\times 10^8 m/s)^2 = 1.000 + 1.253\times 10^{-14}$$

.

- We can then apply time dilation to determine how much slower your biological clock is running compared to the clocks of people who live close to work and walk everyday. To do that, we know that time in frame S and time in frame S' are related by

  $$t = \frac{1}{\sqrt{1-(\nu/c)^2}}t' = t' \times \sqrt{1-\nu^2/c^2}.$$

  We then apply the Binomial Expansion again to the right-hand side and obtain

  $$t \approx t' \times \left(1.000 + \frac{1}{2}\nu^2/c^2\right)$$

  , which means that each day our high-speed commuter clocks fall behind earth-bound clocks by about $3600.s \times 1.253\times 10^{-14} = 4.512\times 10^{-11} s$ per day.
- What does this add up to, over a lifetime of commuting 5 days a week, 52 weeks a year, for 45 years? Well, we just multiply: $4.512\times 10^{-11}\, s/day \times 5\, days/week \times 52\, weeks/year \times 45\, years = 5.279\times 10^{-7} s$ or about 0.5 micro-seconds. Even over a lifetime of commutes, this is negligible! The risks of commuting far outweigh the relativistic benefits.

Next Time

- We'll review this material
- We'll do a rigorous application of Special Relativity and the Lorentz Transformation to a subatomic particle, the "muon," and see that time is something outside of clocks or human perception.
- We'll move onto discussions of alleged paradoxes in relativity
- We'll discuss motion in special relativity