Class_Problems_Solutions_Lecture007

February 11, 2019

1 Uniform Circular Motion

These are the notes that accompany the class period on uniform circular motion.

1.1 Instructor Problem: A Proton in the Large Hadron Collider

The Large Hadron Collider (LHC) is a one-of-a-kind particle accelerator straddling the France-Switzerland border. It is circular in shape, with a circumference of 17 miles (27km). It can accelerate protons to just below the speed of light (for this question, we'll treat the protons as traveling at exactly the speed of light, though in reality that's not possible). Answer the following questions:

- How much time does it take for a single proton to travel once around the accelerator?
- What is the "angular speed" of the proton in units of *radians per second*?
- What is the centripetal acceleration experienced by the proton?

1.1.1 SOLUTION

This is a problem about matter (a proton) going in a circle at constant speed (executing uniform circular motion). Therefore, we can combine notions of linear speed (v) with circular motion to arrive at answers to questions such as "what acceleration does the proton experience as it moves in a circle?"

Part 1 We are to find the time it takes to make 1 revolution about the LHC, a time T. This can be determined by relating the speed (speed of light) to the distance and the time it takes to travel that distance. The distance in question is 1 revolution, or 1 *circumference* of the LHC. We are given this distance... $C_{LHC} = 2.70 \times 10^4$m. Thus:

$$v_{proton} = \frac{C_{LHC}}{T} \longrightarrow T = \frac{C_{LHC}}{v_{proton}}$$

yielding $T = 9.01 \times 10^{-5}$s. (for $v_{proton} = c = 3.00 \times 10^8$m/s.

Part 2 What is the "angular speed" of the proton? This is the number of *radians* it travels around the circle in a certain amount of time. In our case, a proton goes 2π radians in 1 revolution, and we now know this takes $t = 9.01 \times 10^{-5}$s. Thus:

$$\omega_{proton} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T}$$

and we find that $\omega_{proton} = 6.98 \times 10^4$rad/s.

Part 3 We are finally to find the center-seeking, or *centripetal*, acceleration experienced by the proton as it remains in this huge circular trajectory. This is given by:

$$a_{proton} = \frac{v_{proton}^2}{R_{LHC}}$$

We need the LHC radius, which is related to the circumference by $C_{LHC} = 2\pi R_{LHC}$, and thus $R_{LHC} = C_{LHC}/(2\pi) = 4.30 \times 10^3$m.

Then $a_{proton} = 2.09 \times 10^{13}$ m/s^2.

Alternatively, we can employ the angular speed, relating the angular and the linear speed using the circumference. We can figure this out using some algebra, starting from:

$$v_{proton} = \frac{C_{LHC}}{T}$$

where T is the time for 1 revolution. Plugging in $C_{LHC} = 2\pi R_{LHC}$:

$$v_{proton} = \frac{2\pi R_{LHC}}{T}$$

We found above that the angular speed, determined from the total radians in 1 time of revolution, is $\omega_{proton} = 2\pi/T$. Substituting this into the equation,

$$v_{proton} = \omega_{proton} R_{LHC}$$

Then:

$$a_{proton} = \frac{\omega^2 R_{LHC}^2}{R_{LHC}} = \omega_{proton}^2 R_{LHC}$$

which yields $a_{proton}^{alternative} = 2.09 \times 10^{13}$ m/s^2. It's the same! Yay!

1.2 Student Problem: The International Space Station

The International Space Station orbits the Earth very quickly, completing one orbit every 92.7 minutes. It does so at an average altitude of 370km. If the orbit of the ISS is treated as uniform and circular, and the center of the orbit is the middle of the Earth (with a planetary average radius of 6371km)...

- What is the speed of the ISS? *(HINT: how far does it travel in each orbit?)*
- What centripetal acceleration does the ISS experience?
- In what direction does the centripetal acceleration point?

1.2.1 Solution to Part 1

We are to find the speed of the ISS. To do this, we need to know how for it goes in a certain amount of time. We are conveniently told that it goes 1 time around the Earth every 5562.0 seconds (I converted the minutes number to seconds already). This is one circle's worth of distance around the Earth. The circumference of that circle is the distance we seek!

The circumference of a circle is

$$C = 2\pi R \tag{1}$$

where R is the radius of the circle. The ISS is not located on the surface of the Earth, a distance $r_{earth} = 6371000$m from the center of the Earth, but rather a bit farther along a radial line... an additional h=370000m above the surface of the Earth. Thus

$$R = r_{earth} + h \tag{2}$$

which yields R=6741000m.

From this, we now know the radius and we had the time. Let's get the circumference, which is C=4.24 $\times$ 10^7m. This is the distance that the ISS travels in just one orbit. The speed is then $v = C/T$=7615.058m/s.

1.2.2 Solution to Part 2

This is about the centripetal acceleration felt by the ISS. It remains in orbit around the Earth (due to gravity's pull), and so it must experience a centripetal acceleration (a constant changing of its velocity direction that keeps it in a circle). Centripetal acceleration's magnitude is given by

$$a = \frac{v^2}{r} \tag{3}$$

In our case the speed, v, is known from part 1. The radius at which it orbits is also known from part 1. We can substitute numbers and find $a = 8.602\ m/s^2$.

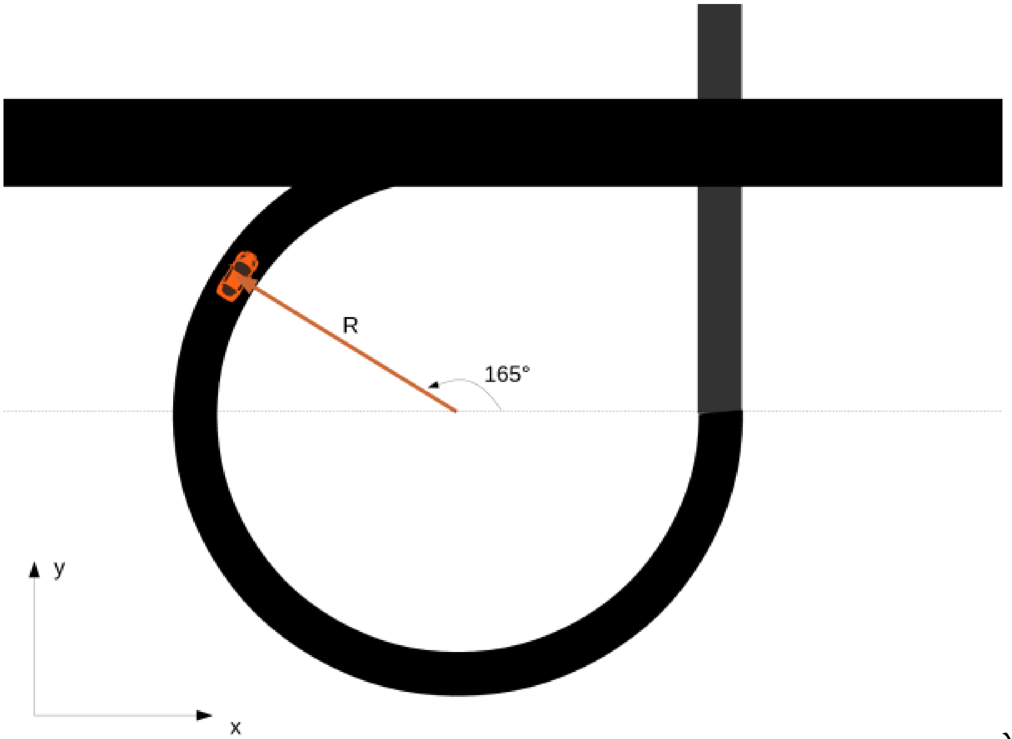
A Small Aside: Microgravity on the ISS Watch videos of astronauts on the space station. They float around in what appears to be near-zero-gravity conditions. This is because the ISS falls toward the Earth at the same rate that it is pulled by gravity, so while everyone on the space station (and the space station) are falling toward the Earth, pulled by gravity, the curved surface of the Earth remains at a constant distance from them as they fall, and they "miss" the Earth. There is an old joke here by the science fiction author, Douglas Adams, that the secret to flying is to throw yourself at the ground and miss. The ISS does this all the time. Gravity pulls it toward earth, but it moves forward perpendicular to the fall, and as a result it misses the surface of the Earth and maintains its orbit.

That said, the acceleration due to gravity at the ISS's height above the Earth is not $9.81 m/s^2$, but rather smaller: $g_{ISS} \approx 8.766\ m/s^2$. As you can see, the acceleration due to gravity and the centripetal acceleration are a near match... that's no accident, and it's why the ISS remains in orbit. The cancellation is very good, better than in the approximations I have been making here, and the ISS maintains a near-constant level of "microgravity" - near-zero-gravity conditions as a result.

1.2.3 Solution to Part 3

Centripetal acceleration *always points toward the center of the motion*, and so the vector of acceleration has the magnitude given in part 2 but points inward toward the center of the Earth.

1.3 Round the Bend



You are driving your car onto a highway. The on-ramp is a circular roadway of radius 250m. You are at the point in the circle indicated in the drawing at the right and have maintained a constant speed of 30mph during your time on the ramp. At the moment shown, what is your (a) velocity and (b) acceleration in both the x- and y-directions?

1.3.1 SOLUTION

This is, at first, a problem in trigonometry that ties into a problem of velocity and centripetal acceleration. Let's begin with the velocity.

PART A

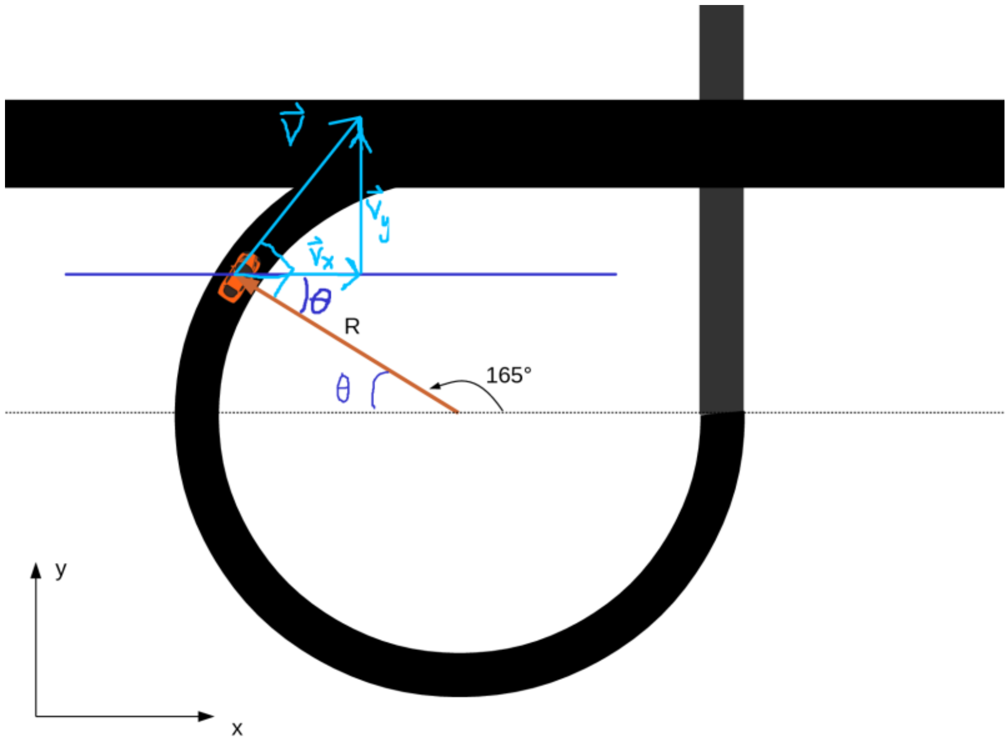
To find the velocity, we need to sketch the velocity vector of the car and think about how to decompose that vector into its components. First, there is speed. The speed is the length of the velocity vector. This is given in mph, but should be converted to MKS units: v=13.4112m/s.

Next, we sketch the vector arrow representing the velocity.

The car is, in the moment shown, driving up and to the right. Its velocity vector will be perpendicular to the radial line, because this is uniform circular motion. The velocity is tangent to the circle at the point shown. So it makes a 90-degree angle to the radial line. From this, I can mark similar angles in the picture, and we find that:

$$\theta = 180^\circ - 165^\circ = 15^\circ \tag{4}$$

The angle inside the velocity triangle, however, is $\phi = 90^\circ - \theta = 75^\circ$. and that:



This picture goes with the problem "Round the Bend" and shows how to setup the velocity vector.

$$v_x = v\cos\phi \quad (5)$$
$$v_y = v\sin\phi \quad (6)$$
$$(7)$$

We can then solve, plugging in numbers, and find $v_x = 3.47$ m/s and $v_y = 12.95$ m/s. The signs on these numbers are correct; both should be positive, according to the sketch.

PART B

Next, we want to repeat this for acceleration. But this is centripetal acceleration. The magnitude is given by:

$$a = \frac{v^2}{r} \quad (8)$$

and we were given the speed and the radius of the circular path. Thus $a = 0.72\ m/s^2$.

We need the acceleration point in toward the center of the motion, along the radial line... so this points down and to the right. The components are:

$$a_x = a\cos\theta \quad (9)$$
$$a_y = -a\sin\theta \quad (10)$$
$$(11)$$

and we obtain $a_x = 0.69\ m/s^2$ and $a_y =$ -0.19 m/s^2.

1.4 Challenge Problem: A Model of Hurricane Florence

Hurricane Florence is days from landfall on the eastern coast of the United States. A key parameter of any hurricane is the *radius of maximum winds*, $r = r_{v_m}$, the distance from the center of the hurricane ($r = 0$m) where the maximum wind speed (v_m) is found. For Florence, $v_m = 130$mph (as of 9/11/18), and its $r_{v_m} \approx 50$km. An older model for hurricane winds as a function of radius, called the Rankine combined vortex approximation, is the following:

$$v = v_m \left(\frac{r}{r_{v_m}} \right) \text{ for } \mathrm{r < r_{v_m}}$$

$$v = v_m \left(\frac{r_{v_m}}{r} \right)^{1/2} \text{ for } \mathrm{r \geq r_{v_m}}$$

(for more information, see Holland, Bellinger, and Fritz, "A Revised Model for Radial Profiles of Hurricane Winds," published in Monthly Weather Review (Volume 138 No. 12, December 2010). https://doi.org/10.1175/2010MWR3317.1)

- How does the centripetal acceleration experienced by, for instance, a pollen grain caught in the storm, or an air molecule that is part of the storm, vary as a function of radius?
- At half the radius of maximum winds in Florence, what is the centripetal acceleration?
- At twice the radius of maximum winds in Florence, what is the centripetal acceleration?

1.4.1 SOLUTIONS

Part 1 This is about using the centripetal acceleration formula,

$$a = v^2/r$$

in combination with the given model for wind speeds. For radii less than r_{v_m},

$$\begin{aligned} a_{r<r_{v_m}} &= \frac{v^2}{r} = \frac{\left[v_m \left(\frac{r}{r_{v_m}} \right) \right]^2}{r} \\ &= v_m^2 \frac{r}{r_{v_m}^2} \\ &= \frac{v_m^2}{r_{v_m}^2} r. \end{aligned}$$

For radii equal-to or greater-than r_{v_m},

$$\begin{aligned} a_{r \geq r_{v_m}} &= \frac{v^2}{r} = \frac{\left[v_m \left(\frac{r_{v_m}}{r} \right)^{1/2} \right]^2}{r} \\ &= v_m^2 \frac{r_{v_m}}{r^2}. \end{aligned}$$

So what we learn is that inside the RMW, the centripetal acceleration grows linearly with radius (and is zero at the center, as one would expect - this is supposed to be a point of zero rotation). At or beyond the RVM, the centripetal acceleration declines as the inverse-square of radius, so it falls off relatively fast.

Part 2 This is about applying the above centripetal acceleration formula at a specific radius, here $r = (1/2)r_{v_m}$. Plugging in that value to the correct equation:

$$a_{r<r_{v_m}} = \frac{v_m^2}{r_{v_m}^2} r$$

yields $a(\frac{1}{2} r_{v_m}) = 0.03377\ \mathrm{m/s^2}$.

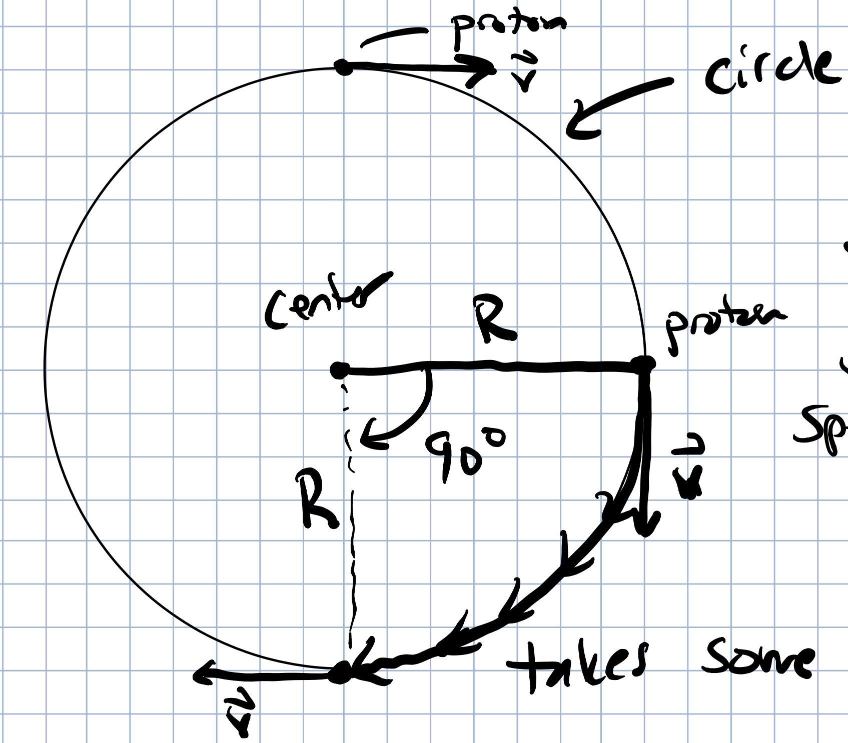
Part 3 Repeat the above, but for $r = 2r_{v_m}$. Using the apppropriate formula:

$$a_{r \geq r_{v_m}} = v_m^2 \frac{r_{v_m}}{r^2}$$

we find $a(2r_{v_m}) = 0.01689\ \mathrm{m/s^2}$.

Instructor Problem: The Large Hadron Collider (and a proton...)

circumference: 27 km



circle → LHC

$v \approx c = 3.0\times10^{8}\,m/s$

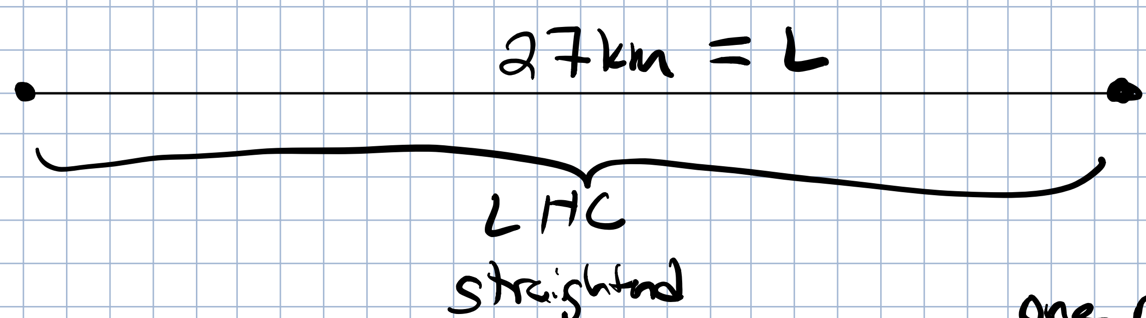
↓

speed of proton.

takes some time, Δt

a) time to make one lap ("revolution")?

imagine



$v = 3.0\times10^{8}\,m/s$

$$v = \frac{L}{t}$$

L ← one revolution

t ← time for one revolution

$$t_{1\,rev.} = \frac{L}{v} = 9\times10^{-5}\,s$$
$$= 90\,\mu s$$

b) what is the angular speed?

$$v = \text{linear speed} = \frac{\Delta x}{\Delta t}$$

$$\omega = \text{angular speed} = \frac{\Delta\theta}{\Delta t}$$ ← theta

↑ angular speed (lower-case "omega")

want angle → have distance

↳ I go a distance L in one revolution

(one revolution = $360^\circ = 2\pi$ radians)

$$L \longrightarrow \Delta\theta$$
(27 km) (2π radians)

$$\Delta t = 9\times10^{-5}\,s \text{ (1 revolution)}$$

$$\omega_{proton} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ radians}}{9\times10^{-5}\,s}$$

$$\approx 7 \times 10^4 \text{ rad./s}$$

c) proton's centripetal acceleration?

$$a_{centripetal} = \frac{v^2}{R}$$

v ← speed

R ← radius of circle

$$L = 2\pi R_{LHC}$$

$$R_{LHC} = \frac{27\text{km}}{2\pi} = \frac{\text{number}}{} \approx 4 \text{ km}$$

$$a_{proton} \cong \frac{(3.0\times10^8 \text{m/s})^2}{4\text{km}}$$

$$\approx 2\times10^{13} \text{ m/s}^2$$