Class_Problems_Solutions_Lecture014

March 7, 2019

1 The Centripetal Force

These are the notes that accompany the class period on the centripetal force.

1.1 Instructor Problem: Ball-on-a-String and Breaking Strength

In class, I have used a tennis ball tethered to a nylon string to demonstrate the ideas of uniform circular motion, centripetal acceleration, etc. Let's now consider this demonstration in lieu of centripetal force.

Consider me spinning the 0.057kg ball on the end of the nylon string (neglect the mass of the string). The radius of the circular motion path of the ball is about 0.75m. How fast would I have to whirl this ball to cause the string to break? The breaking strength of nylon string is about 4.0kN. Neglect the weight of the ball.

1.1.1 SOLUTION

We can use a free-body force diagram to analyze the motion. Since we are neglecting the weight of the ball (which is OK - the weight of the ball is a mere 0.559N, which is miniscule compared to the breaking strength of the string), the only forces acting on the ball and holding it in circular motion are the tension in the string. So we have, by Newton's Second Law:

$$\sum_i F_i = T = ma$$

What acceleration is the ball experiencing? At any constant speed, it's being held in a circular path, so it must be experiencing a centripetal acceleration. So, in this case, we know the form that the acceleration takes:

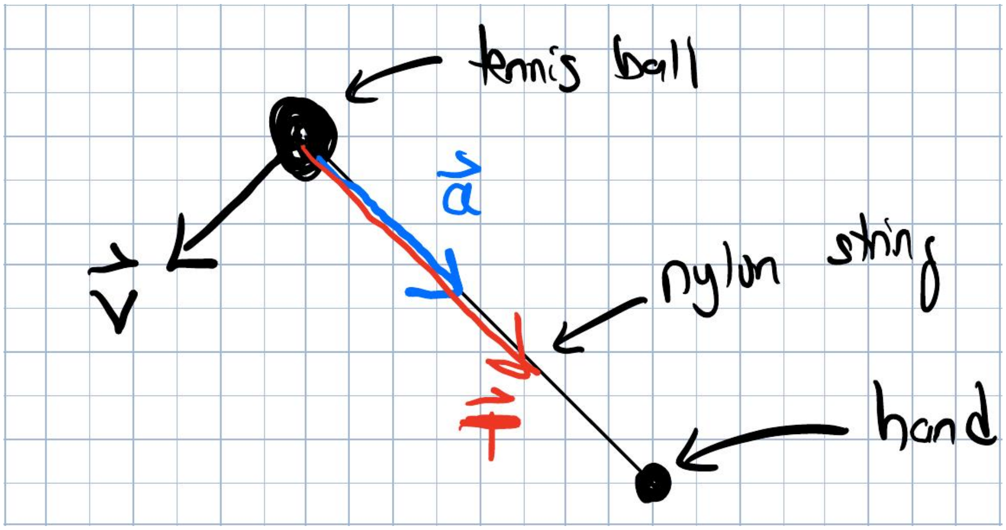
$$a = v^2/r$$

where v is the speed of the ball and r is the radius of the circular path. Putting this together:

$$T = m\frac{v^2}{r}$$

Now, we want to know how fast we can make the ball go to just reach the breaking strength of the string. That was given as $F_{break} = 4.0 \times 10^3$N. So we set the tension equal to this number and solve:

$$F_{break} = m\frac{v_{max}^2}{r} \longrightarrow v_{max} = \sqrt{\frac{rF_{break}}{m}}$$



This diagram accompanies the instructor problem on the whirling tennis ball

which yields $v_{max} = 229$m/s. For reference, the fastest that a tennis ball, once struck by a tennis player in a professional match, can move is about 80m/s.

Wow! That's fast. Let's put that in perspective. In class, I will have you compare the number of revolutions per second I can comfortably, maximally achieve with this setup. The above speed can be converted to revolutions per second as follows.

Each revolution requires traveling a distance equal to the circumference of the path, $C = 2\pi r = 4.7$m. We can find the number of revolutions traveled each second (the "frequency") by dividing the speed by the circumference of the path:

$$f_{max} = v_{max}/C$$

which yields $f_{max} = 49\text{s}^{-1}$.

There is no way I can comfortably achieve that... which is a good thing. We do not want anyone to get hit by a tennis ball moving at the above speed once it's released from its circular motion by the breaking of the string.

Some Bonus Investigations The student might investigate this breaking strength business farther. For instance, how might one determine the breaking strength of a material *in the first place*? One answer is to suspend the material vertically and then attach an increasing amount of mass to the bottom of the material. When the material fails, you have identified the threshold for it maximum tension.

How much mass is required to break the nylon rope used above? Well, if we imagine that the tension and the added weight (due to a mass, M, attached to the bottom of the string) balance each other just up to the point where the material fails (the breaking strength), then the Newton's Second Law equation yields:

$$T_{max} - w_{max} = 0$$

Just beyond w_{max}, the string fails and acceleration is no longer zero. So what maximum mass can we attach to the string?

$$T_{max} = w_{mag} = M_{max}g$$

Solving for M_{max}, we find $M_{max} = 4.08 \times 10^2$kg

How about another way to break this nylon rope? Instead of a tennis ball, I might attach one of our lead bricks to the end of the rope. These have mass of 11kg, far larger than the tennis ball. If I could spin the lead brick on the end of the nylon string, how fast would I have to spin that kind of mass to break the string? We setup the same Newton's 2nd Law Equation as earlier:

$$T_{max} = ma_{max} = m\frac{v_{max}^2}{r}$$

The radius of the string remains as before. We rearrange and solve for the speed of the uniform circular motion needed to achieve this:

$$v_{max} = \sqrt{\frac{rT_{max}}{m}}$$

We find that $v_{max} = 17$ m/s. How many rotations per second is this? Each rotation requires a distance equal to the circumference of the circular path, so the frequency (rotations per second) will be given by $f = v/(2\pi r) = 3.5$ per second

1.2 Student Problem: Merry-Go-Round

A "Merry-Go-Round" (or "Roundabout") is a spinning device often found on playgrounds (see picture). The Merry-Go-Round has a radius of 2.0m and is spinning, making 1 revolution every 6.0 seconds. If a child of mass 25kg is sitting on the edge of the Merry-Go-Round, try to answer the following questions:

- What is the magnitude of their *centripetal acceleration*?
- What is the magnitude of their centripetal force?
- What is the direction of the centripetal force they experience?
- What is the fastest speed the Merry-Go-Round can spin if the coefficient of static friction involved in preventing them from sliding off is $\mu_s = 0.30$?

1.2.1 SOLUTION

PART 1

This is a nice refresher on centripetal acceleration, and how to calculate it. Since $F = ma$, this will be a useful number to have for part 2.

First, recall that the magnitude of centripetal acceleration is given by:

$$a = \frac{v^2}{r} \tag{1}$$

Our first task is to figure out v (we already have r). We are told that the merry-go-round completes 1 rotation every 6 seconds. So we can say that it makes 1 revolution every 6 seconds, or equally that its *frequency of rotation*, f, is given by:

$$f \equiv \frac{rotations}{time} = \frac{1}{6s} \tag{2}$$

which yields $f = 0.17\,s^{-1}$. Each rotation means the edge of the merry-go-round travels through a length equal to *one circumference at that radius*. Thus the distance traveled in 6 seconds is $C = 2\pi r = 12.57$m.

So we can divide circumference by time, or equally validly multiply circumference by frequency (the answer is the exact same): $v = Cf = 2.09$m/s.

Now we have all we need to answer this first question. Using the definition of centripetal acceleration's magnitude, we plug in the speed and the radius and find $a = 2.19\ m/s^2$.

Part 2

We can now use Newton's Second Law to convert the centripetal acceleration to force. We need only multiply by the mass of the person sitting on the edge of the spinning merry-go-round. We find $F_{centripetal} = ma = 54.83$N.

Part 3

Which direction does this force point? Since centripetal acceleration always points inward to the center of the motion, so does centripetal force. Mass doesn't change anything about the direction.

Part 4

We are now to consider the maximum speed at which the person sitting on the merry-go-round could be spun and still be able to sit on the edge of the merry-go-round without overcoming the friction force holding them in place. It is the friction force (static friction, to be exact), that provides the *source* of the centripetal force. However, if the speed of the merry-go-round is too large, static friction cannot provide enough force to resist the changeover to kinetic friction, and the person would begin to slide. The transition occurs when the force required to prevent slipping exceeds the static friction force threshold:

$$F_{static} = \mu_s N = \mu_s mg \tag{3}$$

Since this is the threshold at which static friction will be overcome, we can see what speed this corresponds to in the centripetal force. We set centripetal force equal to this threshold, $F_{centripetal} = F_{static}$ and solve:

$$\begin{aligned}
F_{centripetal} &= F_{static} && (4)\\
m\frac{v_{max}^2}{r} &= \mu_s mg && (5)\\
v_{max}^2 &= \mu_s gr && (6)\\
v_{max} &= \sqrt{\mu_s gr} && (7)\\
& && (8)
\end{aligned}$$

We find that the speed is then $v_{max} = 2.43$m/s. This is not too much more speed than was given at the beginning of the problem, so this kid is somewhat close to sliding off the merry-go-round.

1.3 Student Problem: The Storm Horde of Jupiter

Jupiter is the fifth planet from the Sun. It is a "gas giant," composed primarily of Hydrogen (about 80%) and Helium (about 20%), though the smaller amounts of other elements (like Nitrogen, which forms Ammonia) are what provide the creamy whites you see in Jupiter's cloud bands. The reds are due to the hot hydrogen below the upper cloud layers. The "Great Red Spot" is red because you are looking through the eye of a storm into its deeper Hydrogen layers.

Jupiter has a radius about 11 times that of Earth (you can fit about 1300 earths inside Jupiter's volume!). The Great Red Spot, a storm that has raged for at least 300 years, is about Earth-sized. Jupiter's outer atmosphere makes one rotation every 10 hours (1 Jovian Day). Its south pole has been recently photographed by the Juno probe, which is studying Jupiter's magnetic field. What it has seen there is beyond earthly.

The south pole's jet stream creates a pentagonal swirl composed of 5 "stormlets". Each stormlet is approximately circular and roughly the width of the United States (giving each a radius of approximately 2150km). The outer edge of each stormlet rotates once every 6 hours. (Information from NASA, especially NASA Juno Findings - Jupiter's Jet-Streams Are Unearthly (March 7, 2018)).

- If a massive probe ($m = 2000kg$) could be placed in the outer edge of the storm to study a specific stormlet, what magnitude centripetal force would the probe experience?
- When trying to put the probe into the edge of the storm, it is knocked off course and instead ends up at a distance halfway between the edge and center of the storm (closer than intended to the center). Assuming the rotation rate of the storm is the same at this location, now what is the centripetal force on the probe?

1.3.1 SOLUTION

Part 1

We can again apply the centripetal force magnitude definition:

$$F_{centripetal} = \frac{mv^2}{r} \tag{9}$$

We have again been given a *frequency* of rotation (how long 1 revolutions takes, and thus the number of revolutions per unit time), and we can convert hours to seconds and obtain the frequency in MKS units: $f = 4.60 \times 10^{-5}\ s^{-1}$.

We can also then find the circumference corresponding to the distance traveled in one revolution: $C = 1.40 \times 10^7$m.

Finally, we can obtain the speed of the probe: $v = 6.30 \times 10^2$m/s. This allows us to then compute the centripetal force experienced by the probe, $F_{centripetal} = 3.60 \times 10^2$ N.

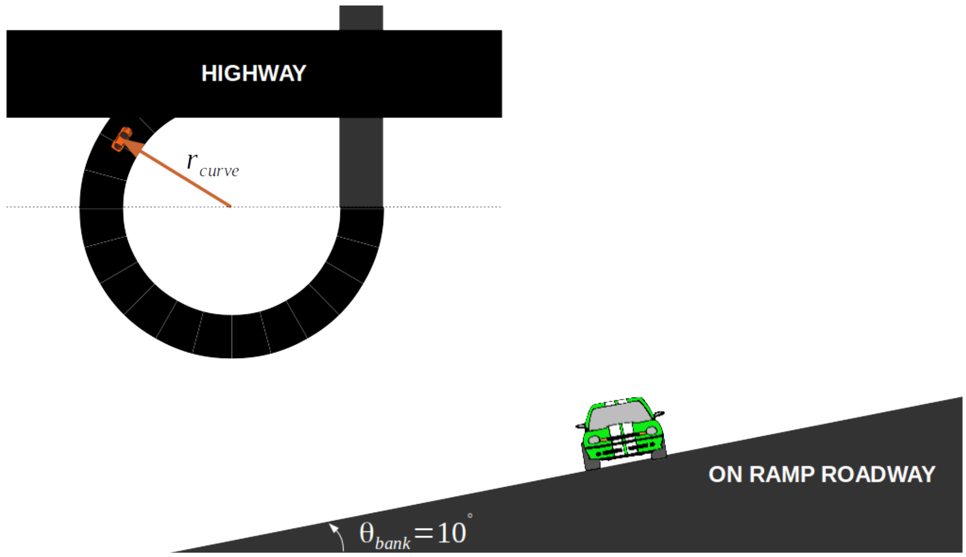
Part 2

For the second part, we can stop for a moment and think about what happens if all things remain the same - the frequency of rotation, the mass of the probe - except we change the radius of the orbit around the center of the storm. How does the radius ultimately enter into this problem?

It enters in two places: in the circumference of the orbit, and in the centripetal force equation. We know that $v = Cf = 2\pi rf$. We can insert that into the centripetal force equation:

$$F_{centripetal} = m\frac{v^2}{r} = m\frac{(2\pi f)^2 r^2}{r} = (2\pi f)^2 mr \tag{10}$$

We see that if frequency of orbit remains the same and mass remains the same, but the orbital radius is cut in half, we expect the centripetal force to be cut in half. Indeed, if an object moves right to the center of rotation, $r = 0m$, then it makes sense that all centripetal force ceases (for an extended object, it's impossible to all be at exactly r=0m, but if that were possible indeed the centripetal force would have to vanish). So we learn that $F^{new}_{centripetal} = 1.80 \times 10^2$N, half the original force.



This image goes with the Challenge Problem on the Maximum On-Ramp Speed

1.4 Challenge Problem: Maximum On-Ramp Speed

You work for an engineering firm that is designing a curved, banked on-ramp to a highway. The idea is that people will enter the highway from another road (with a lower speed limit), and the banked on-ramp will allow a typical car to enter the highway at a higher speed than that of the original roadway. The on-ramp will be circular in shape, with a radius of curvature of $r_{curve} = 0.65$km.

The roadway will have to bank at an angle of 10° (see picture). What is the maximum speed that a car will be allowed to drive on this on-ramp, assuming that the typical coefficient of static friction under dry conditions between the rubber tires and the roadway is 0.6? (HINT: the car must make the turn around the on-ramp without sliding up or down the inclined roadway)

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23.524477742797913
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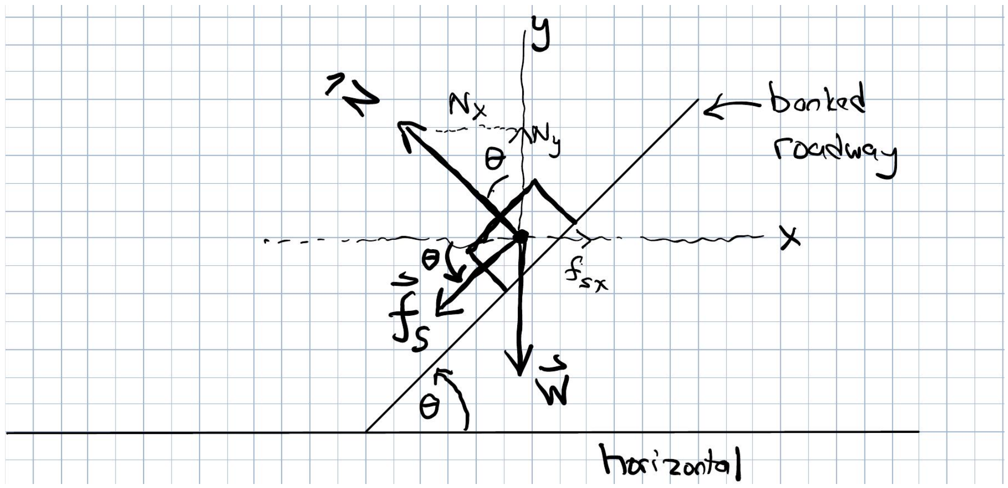
1.4.1 SOLUTION

This is a problem about an object that will enter circular motion when it travels through a circular path - the on-ramp. Since this is a car moving in a circular path at constant speed (whose maximum value we need to determine), the know that the net force in the horizontal direction must be a centripetal force. The car is neither sliding up the ramp, nor down the ramp, but at maximum possible speed is just on the cusp of sliding up the ramp (losing control). So at maximum speed, we must have just reached the maximum possible static friction force.

Let's draw a free-body force diagram illustrating the forces:

- In the x-direction (which is parallel to the earth, but not to the ramp surface) we have: a component of the friction force, a component of the normal force... and that's it. When combined, these forces together must equate to the net force, the total centripetal force required to maintain this circular path (seen from overhead).



This diagram accompanies the challenge problem on the banked roadway

- In the y-direction, we have a car that is neither moving up nor down - the net force in the vertical direction must be zero, by Newton's First Law. The forces are: a component of the normal force, the weight, and a component of the friction force.

Let's add the forces in the vertical direction, since those must sum to zero:

$$\sum_i F_{y,i} = 0$$

$$N\cos\theta - f_{s,max}\sin\theta - mg = 0$$

$$N\cos\theta - \mu_{s,max}N\sin\theta - mg = 0$$

$$N\left(\cos\theta - \mu_{s,max}\sin\theta\right) - mg = 0$$

$$N = \frac{mg}{\left(\cos\theta - \mu_{s,max}\sin\theta\right)}$$

Now, let's consider the x-direction:

$$\sum_i F_{x,i} = ma_{centripetal} = -m\frac{v^2}{r}$$

$$-f_{s,max}\cos\theta - N\sin\theta = -m\frac{v_{max}^2}{r}$$

$$\mu_{s,max}N\cos\theta + N\sin\theta = m\frac{v_{max}^2}{r}$$

$$N\left(\mu_{s,max}\cos\theta + \sin\theta\right) = m\frac{v_{max}^2}{r}$$

$$\frac{mg}{\left(\cos\theta - \mu_{s,max}\sin\theta\right)}\left(\mu_{s,max}\cos\theta + \sin\theta\right) = m\frac{v_{max}^2}{r}$$

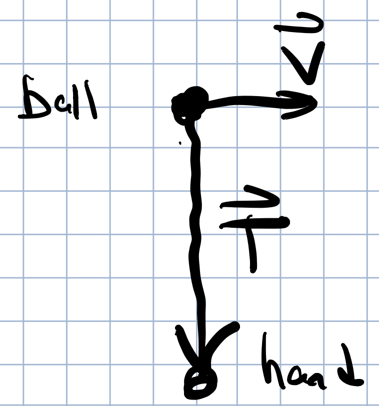
$$gr\frac{(\mu_{s,max}\cos\theta + \sin\theta)}{(\cos\theta - \mu_{s,max}\sin\theta)} = v_{max}^2$$

$$gr\frac{(\mu_{s,max} + \tan\theta)}{(1 - \mu_{s,max}\tan\theta)} = v_{max}^2$$

$$v_{max} = \sqrt{gr\frac{(\mu_{s,max} + \tan\theta)}{(1 - \mu_{s,max}\tan\theta)}}$$

If we now plug in the numbers ($\theta = 10^\circ$, $\mu_{s,max} = 0.60$, and $r = r_{curve} = 65$m), we find $v_{max} =$ 23.52m/s. This comes out to about 53mph, which is pretty reasonable for a typical, real-world on-ramp.

Instructor Problem: Ball on a String + Breaking Strength



$$F = ma$$

$$T = m\underbrace{\left(\frac{v^2}{r}\right)}_{\text{centripetal acceleration}}$$

want $T = F_{breaking}$ (4000N)

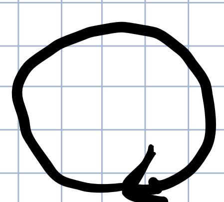
$r = 0.75\,m$

$$F_{breaking} = m\,\frac{v_{max}^2}{r}$$

$$v_{max} = \sqrt{\frac{r \cdot F_{breaking}}{m}}$$

$$= 229\ m/s$$

revolutions per unit time [second?]



1 revolution $\longrightarrow$ $C = 2\pi r$

$$\frac{2\pi r}{\text{revolution}}$$

$$v \rightarrow \frac{m}{s}$$

$$\frac{v}{(2\pi r/\text{revolution})} = 49 \ 1/s$$