Mechanics Laboratory 1105, Summer 2025

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https://www.physics.smu.edu/tneumann/110X_Summer2025/

Lab 12 — Gravity and Dark Matter Max. points: 55

How do objects move under gravity? In our Solar System, the planets orbit the Sun in a predictable way, described well by Newton's laws where almost all the mass is concentrated at the center. But when astronomers look at galaxies, they see something strange: stars and gas in the outer regions orbit much faster than expected based on the visible matter alone. This discrepancy, observed through galaxy rotation curves, points to one of the biggest mysteries in modern cosmology – the "missing mass" problem. Is there a vast amount of unseen "dark matter" providing the extra gravity, or do our theories of gravity need modification on galactic scales?

In this worksheet, you will explore this puzzle. First, you'll analyze the familiar Keplerian rotation of our Solar System as a baseline. Then, you'll examine real astronomical data from a spiral galaxy, plotting its rotation curve and calculating the total mass required to explain the observed speeds. Finally, you'll compare this required mass profile to the distribution of visible matter and investigate the implications, leading to the concepts of dark matter halos and alternative theories like MOND. By working through these steps, you'll gain insight into the evidence for dark matter and the ongoing quest to understand the fundamental workings of gravity and the composition of the universe.

Your preparation: Work through before coming to the lab

- Prepare for the lab by thoroughly reading and understanding the introductory text and the analysis procedures on this worksheet.
- Collect all your questions and ask your instructor at the beginning of the lab.
- Work through and review the following topics in your physics textbook (e.g., Halliday, Resnick, and Walker [1]):
  - **Newton's Law of Universal Gravitation:** Understand the force between two masses ($F = GMm/R^2$).
  - **Circular Motion and Centripetal Force:** Understand the relationship between velocity, radius, and the force required for circular motion ($F_c = mv^2/R$).
  - **Orbital Mechanics:** How gravitational force provides the centripetal force for orbits.
  - **The Doppler Effect (for light):** Understand how the observed frequency (or wavelength) of light changes when the source is moving relative to the observer (redshift and blueshift). How this relates to velocity.
  - **Astronomical Units:** Be familiar with units like Astronomical Units (AU), parsecs (pc), kiloparsecs (kpc), megaparsecs (Mpc), light-years, solar masses ($M_\odot$), and angular measurements (arcminutes, arcseconds). Understand basic conversions or how they relate

(e.g., using small angle approximation $d = D\theta$ for the physical size d of an object at a distance D given its angular size θ).

– **Newton's Shell Theorem (conceptual):** Understand the two main points regarding gravitational forces from spherical shells (no net force inside, acts as point mass outside).

Pre-lab: Upload to Canvas before coming to the lab

A reminder: Upload your answers as a text document (exported as PDF) to Canvas before the lab begins (Canvas uploads are no longer possible 60 minutes before the lab starts!).

Pre-lab 1 5 points

1. (1 point) Explain the Doppler effect for light. How can astronomers use it to determine if a part of a galaxy is moving towards us or away from us, and how fast?
2. (1 point) Consider an object of mass m orbiting a much larger central mass M in a circular path of radius R. By equating the gravitational force with the centripetal force, derive an expression for the orbital velocity v as a function of R. How does v depend on R?
3. (1 point) What is a galaxy rotation curve? Based on this worksheet and fig. 4, what is the main discrepancy between the observed rotation curves of spiral galaxies and the predictions based on their visible matter?
4. (1 point) Briefly describe the two main proposed explanations mentioned in the text for this discrepancy (Dark Matter and Modified Gravity).
5. (1 point) An object is observed at an angular radius of 1 arcminute from the center of a galaxy. If the galaxy is at a distance $D = 10\,\text{Mpc}$ from Earth, calculate the object's physical distance R from the galaxy's center in kiloparsecs (kpc). Use the small angle approximation ($\tan\theta \approx \theta$ for small θ in radians). Remember that 1 degree = 60 arcminutes, π radians = 180 degrees, 1 Mpc = 1000 kpc.

Analysis and Report: submission by end of class

A reminder: All calculations and plots must be documented. The final report must be uploaded to Canvas *by the end of the class* exported as PDF with plots and tables embedded as images.

Analysis 1 *Keplerian rotation curve* 18 points

Introduction: We first examine a familiar system where gravity dictates orbital motion: our own Solar System. By analyzing the relationship between orbital speed and distance from the central mass (the Sun), we can establish a baseline expectation based on Newton's Law of Universal Gravitation when most of the mass is concentrated at the center. This is often called Keplerian rotation.

a) (8 points) **Solar system rotation curve:** Using a reliable source (like Wikipedia or a NASA database)[1], find the average orbital speed (in km/s) and the semi-major axis (in AU) for the 8 planets in our solar system. Create a table of these values. Using Excel or similar software, create a plot of orbital speed (v, y-axis) versus semi-major axis (R, x-axis). Pay attention to the correct labeling of axes and data points.

b) (6 points) **Theoretical prediction:** Assume planets (mass m) orbit the Sun (mass $M_\odot$) due to gravity providing the centripetal force:

$$\frac{GM_\odot m}{R^2} = \frac{mv^2}{R} \tag{1}$$

where $G = 6.674 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$. Solve eq. (1) for v as a function of R. Look up the mass of the Sun ($M_\odot$). Calculate the predicted orbital speed $v(R)$ for several values of R spanning the range of the planets' orbits (or specifically at each planet's R). Add this theoretical curve to your plot from part (a). Make sure your units are consistent (km/s for v, AU for R - you'll need conversion factors: $1\,\mathrm{AU} \approx 1.496 \times 10^8\,\mathrm{km}$).

c) (4 points) **Discussion:** Discuss the agreement or disagreement between the theoretical curve and the planet data points. How well does the $v \propto 1/\sqrt{R}$ prediction fit the solar system? What are potential reasons for any minor deviations (e.g., non-circular orbits, gravitational influence of other planets)? Do you expect this same $1/\sqrt{R}$ relationship to hold for stars orbiting within a galaxy? Why or why not? (Consider how mass is distributed).

Analysis 2 *Galaxy rotation curve and enclosed mass* 20 points

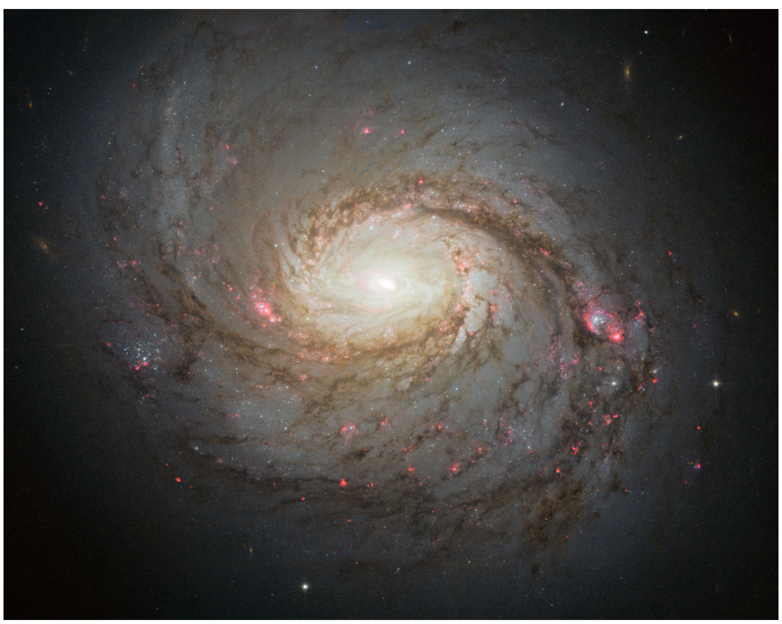

Figure 1: Messier 77 galaxy, also known as NGC 1068 [2].

[1] https://science.nasa.gov/solar-system/planets/, https://nssdc.gsfc.nasa.gov/planetary/factsheet/

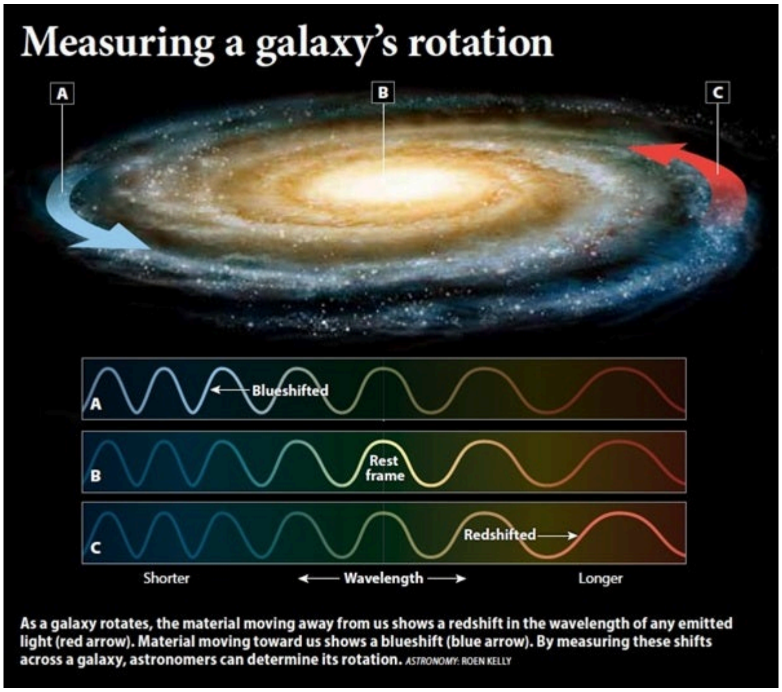

Figure 2: Using Doppler effect to measure a galaxy's rotation speed, from ref. [3].

Introduction: Most galaxies in our universe rotate, often exhibiting spiral structures like Messier 77 shown in fig. 1. Understanding how these galaxies form and evolve involves concepts like the gravitational collapse of interstellar gas and dust, combined with the conservation of angular momentum. The intricate spiral patterns themselves have been studied since the 1960s, with early explanations involving "density wave theory" [4]. Today, large-scale computer simulations based on Newton's laws of mechanics are crucial tools for modeling galactic evolution.

However, directly observing a galaxy complete a rotation is impossible due to the immense timescales involved – typically hundreds of millions of years. To study galactic rotation speeds, astronomers rely on the Doppler effect (fig. 2). Similar to how the pitch of an ambulance siren changes as it passes, the frequency (color) of light shifts based on the relative motion between the source and observer. Light from parts of a galaxy moving towards us is shifted to higher frequencies (blueshifted), while light from parts moving away is shifted to lower frequencies (redshifted). Measuring this shift allows us to determine the orbital speed (v) of stars or gas clouds at different distances (R) from the galactic center.

To accurately measure the Doppler shift, we need a light source with a precisely known original frequency. Ordinary starlight varies in color. Fortunately, neutral hydrogen gas, abundant in galaxies, emits radio waves at a very specific frequency of 1.42 GHz (corresponding to a wavelength of 21 cm) –– a well-understood quantum mechanical effect. This "21cm line" emission serves as a reliable standard. Furthermore, these radio waves have the advantage of penetrating the interstellar dust that often obscures visible light from distant parts of galaxies.

By measuring the Doppler shift of the 21cm line across a galaxy, astronomers can map out the orbital speed v as a function of radius R, creating what is known as a *rotation curve*, $v(R)$. Physicists compare these observed rotation curves to theoretical predictions based on Newton's law of gravity, considering the mass distribution inferred from the visible, luminous matter (stars and gas).

In this analysis, you will use observational data for a spiral galaxy (NGC 3198) to plot its rotation curve. You will then apply Newton's law of gravitation, specifically considering the total mass enclosed within a given radius ($M_{\text{enclosed}}(R)$), to calculate the mass profile required to explain the observed speeds. This calculation is fundamental to understanding the discrepancy between the mass we *see* and the mass required by gravity, a puzzle that points towards the existence of dark matter or the need for modified theories of gravity.

Table 2. Rotation Curve of NGC 3198

R (′)	V_C (km s^{-1})	ϕ (°)	i (°)	R (′)	V_C (km s^{-1})	ϕ (°)	i (°)
0.25	55±8	216.0	71.5	4.5	153±2	216.4	70.6
0.50	92±8	216.0	71.5	5.0	154±2	216.2	70.5
0.75	110±6	216.0	71.5	5.5	153±2	216.1	70.7
1.00	123±5	216.0	71.5	6.0	150±2	215.9	70.9
1.25	134±4	216.0	71.5	6.5	149±2	215.7	71.5
1.50	142±4	216.0	71.5	7.0	148±2	215.5	72.1
1.75	145±3	216.0	71.5	7.5	146±2	215.2	72.7
2.00	147±3	216.0	71.5	8.0	147±2	215.0	73.4
2.25	148±3	216.0	71.5	8.5	148±2	215.0	74.0
2.50	152±2	216.0	71.5	9.0	148±2	215.0	74.6
2.75	155±2	216.0	71.5	9.5	149±2	215.0	75.2
3.0	156±2	216.0	71.0	10.0	150±2	215.0	75.9
3.5	157±2	216.3	70.8	10.5	150±3	215.0	76.4
4.0	153±2	216.6	70.7	11.0	149±3	215.0	77.0

Figure 3: Rotation curve data for galaxy NGC3198 from ref. [5]. R is the angular radius in arcminutes (1/60 of a degree), V_C the rotation speed. The observation distance is 9.4 Mpc.

a) (6 points) **Plotting galaxy data:** The data table in Figure 3 shows rotation curve measurements for the spiral galaxy NGC 3198, obtained by studying the Doppler shift of the 21cm hydrogen line. The table provides the observed rotation speed V_C (in km/s) as a function of the angular radius R (in arcminutes) from the galaxy's center.

   First, convert the angular radius R (arcmin) for each data point into a physical radius in kiloparsecs (kpc). Use the given observation distance to the galaxy, $D = 9.4$ Mpc, and the small angle approximation $R_{\text{physical}} = D \times R_{\text{angular}}$ (remembering to convert R_{angular} from arcminutes to radians first).

   Create a table in your report listing the original R (arcmin), the calculated R (kpc), and the corresponding V_C (km/s) for all data points provided in Figure 3. Then, create a plot of the rotation speed V_C (y-axis) versus the physical radius R (kpc, x-axis). Ensure your plot axes are clearly labeled with units, and the data points are clearly shown.

b) (3 points) **Comparison:** How does the shape of this galaxy rotation curve differ from the solar system (Keplerian) rotation curve you plotted earlier, especially at large radii? Give possible explanations.

c) (6 points) **Calculating enclosed mass:** Unlike our Solar System where almost all the mass ($M_\odot$) is concentrated at the center, galaxies have their mass (stars, gas, dust, and potentially dark matter) distributed throughout their volume. This complicates the gravitational analysis. However, if we approximate the mass distribution as *spherically symmetric*, we can use *Newton's shell theorems* to understand its gravitational effects:

- **Theorem 1:** A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.
- **Theorem 2:** If the body is a spherically symmetric shell (i.e., a hollow ball), it exerts no net gravitational force on any object inside it.

Let's apply these theorems to an object (like a star or gas cloud) orbiting at radius R within the galaxy's assumed spherically symmetric mass distribution. We can conceptually divide the galaxy's mass into two parts:

(i) **Mass outside radius R:** Imagine all the mass located at radii greater than R. This mass can be thought of as a series of nested spherical shells, each with a radius larger than R. According to Theorem 2, each of these shells exerts zero net gravitational force on our object located inside them (at radius R). Therefore, the combined gravitational effect of *all* mass outside radius R on the object is zero.

(ii) **Mass inside radius R:** Now consider all the mass located at radii less than or equal to R. This collection of mass itself forms a spherically symmetric body. Our object at radius R is effectively *external* to this inner mass distribution. According to Theorem 1, this inner body exerts a gravitational force on the object as if its total mass, which we call the enclosed mass $M_{\text{enclosed}}(R)$, were concentrated at the very center of the galaxy.

Combining these two points, the total gravitational force experienced by the object at radius R depends *only* on the total mass enclosed within its orbit, $M_{\text{enclosed}}(R)$, acting as if it were a point mass at the center. The mass outside radius R does not affect its orbit. This allows us to adapt the equation used for Keplerian orbits, where the gravitational force due to the enclosed mass provides the necessary centripetal force for circular motion:

$$\frac{GM_{\text{enclosed}}(R)m}{R^2} = \frac{mv^2}{R} \quad \Longrightarrow \quad M_{\text{enclosed}}(R) = \frac{v(R)^2 R}{G} \tag{2}$$

Using your data points (R in kpc, V_C in km/s) from part (a), calculate the enclosed mass $M_{\text{enclosed}}(R)$ for each point using eq. (2). Be very careful with units! You will need to convert R from kpc to m and V_C from km/s to m/s before using G in standard SI units ($6.674 \times 10^{-11}\,\text{N} \cdot \text{m}^2/\text{kg}^2$). It is convenient to express the final mass in solar masses ($M_\odot \approx 1.989 \times 10^{30}$ kg). Add columns for M_{enclosed} (in $M_\odot$) to your table from part (a).

d) (4 points) **Plotting mass profile:** Create a plot of the enclosed mass M_{enclosed} (in $M_\odot$, y-axis) versus radius R (in kpc, x-axis).

e) (1 point) **Mass trend analysis:** Look at your plot for NGC 3198 at large radii ($R > 15$ kpc). The velocity $v(R)$ appeared to be roughly constant. What does your plot of $M_{\text{enclosed}}(R)$ show in this region? Does the enclosed mass appear to level off (become constant) or continue to increase with radius? Based on eq. (2), what functional form must $M_{\text{enclosed}}(R)$ have for $v(R)$ to be approximately constant? Does your calculated mass profile support this?

Analysis 3 *Interpreting the Mass Profile: Dark Matter and Alternatives* 12 points

Introduction: In the previous analysis, you calculated the total enclosed mass $M_{\text{enclosed}}(R)$ required by Newtonian gravity to explain the observed rotation curve $v(R)$ of galaxy NGC 3198. You likely found that to maintain the nearly constant velocities seen at large radii, the enclosed mass must continue to increase substantially, roughly linearly with radius ($M_{\text{enclosed}}(R) \propto R$). This result stands in stark contrast to the distribution of visible matter (stars and gas), which is mostly concentrated towards the galactic center, implying its contribution to the enclosed mass should level off at large distances. This discrepancy is the core of the "missing mass" problem, famously illustrated in fig. 4. In this section, you will delve into the interpretation of this required mass profile, comparing it explicitly to the visible matter distribution and exploring the two main proposed solutions: the existence of a vast dark matter halo and the possibility of modifying Newtonian dynamics (MOND).

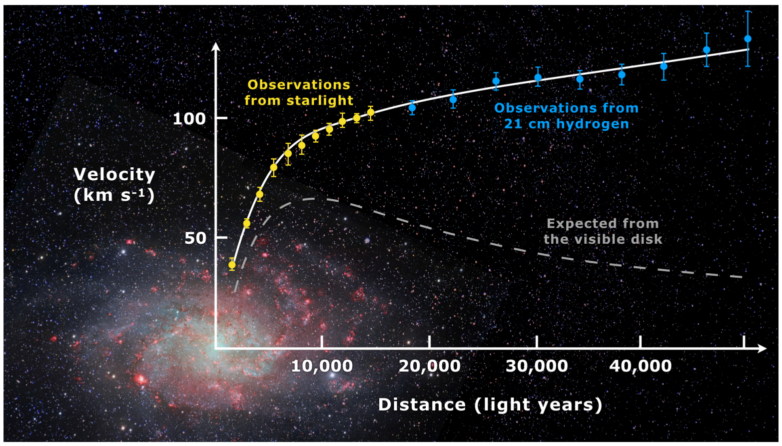

Figure 4: Rotation curve of the spiral galaxy Messier 33 (Triangulum). The "Expected from visible disk" curve falls far below the observed points at large radii, indicating missing mass. The "Dark matter halo" curve shows how adding a halo component can explain the observations [6].

a) (2 points) **Visible vs. Total mass:** Compare the trend of your calculated $M_{\text{enclosed}}(R)$ plot with the distribution of visible matter (stars and gas) in a typical spiral galaxy. Visible matter is highly concentrated towards the center, meaning its contribution to $M_{\text{enclosed}}(R)$ should level off at large radii (similar to how almost all the Solar System's mass is in the Sun). Does your calculated total $M_{\text{enclosed}}(R)$ level off? Discuss the discrepancy illustrated in fig. 4.

b) (4 points) **Dark Matter Halo Concept:** The discrepancy suggests the existence of "dark matter" – matter that does not emit or interact significantly with light but does exert gravitational force. This dark matter is thought to form a large, roughly spherical "halo" extending far beyond the visible disk of the galaxy. A simple model for the density profile of such a halo is the pseudo-isothermal sphere:

$$\rho(r) = \frac{\rho_0}{1 + (r/R_c)^2} \tag{3}$$

where ρ_0 is the central density and R_c is the core radius. At large radii ($r \gg R_c$), this density falls off as $\rho(r) \propto 1/r^2$. Show that if the density falls off as $1/r^2$, the enclosed mass $M_{\text{enclosed}}(R) = 4\pi \int_0^R \mathrm{d}r\, r^2 \rho(r)$ grows approximately linearly with R ($M_{\text{enclosed}}(R) \propto R$) at large R. Does this linear growth match the trend required for a flat rotation curve ($v \approx$ constant) that you deduced in the previous analysis section?

c) (3 points) **Alternative: MOND:** An alternative explanation, known as Modified Newtonian Dynamics (MOND), proposes that Newton's laws of gravity or motion might need adjustment at the very low accelerations experienced by stars in the outer parts of galaxies ($a \ll 10^{-10}\,\mathrm{m/s^2}$).

Instead of adding unseen matter, MOND modifies the relationship between force and acceleration (e.g., $F = m\mu(a/a_0)a$, where $\mu(x) \approx x$ for small x and $\mu(x) \approx 1$ for large x) or the gravitational force law itself. While less widely accepted than dark matter, MOND can also reproduce flat rotation curves without invoking dark matter. Based on the descriptions provided, articulate in your own words the fundamental conceptual difference between the dark matter hypothesis and the MOND hypothesis.

d) (3 points) **MOND's Force Law and Flat Rotation:** Let's explore qualitatively how MOND's modification to Newton's second law, $F = m\mu(a/a_0)a$, might explain flat rotation curves without dark matter. Consider the "deep MOND regime" relevant to the outer parts of galaxies where the acceleration a is much smaller than the characteristic acceleration a_0 ($a \ll a_0$). In this regime, the function $\mu(x) \approx x$ (where $x = a/a_0$), so the force law becomes approximately $F \approx m(a/a_0)a = ma^2/a_0$.

Assume a star of mass m is orbiting the galaxy's center at a large radius R with velocity v. The gravitational force is provided only by the enclosed *visible* mass M_{vis}, which we can approximate as being constant at these large radii ($F_g \approx GM_{\text{vis}}m/R^2$). The centripetal acceleration is $a = v^2/R$.

By equating the gravitational force F_g with the modified inertial force $F \approx ma^2/a_0$, substitute $a = v^2/R$ and show that this leads to a relationship where v^4 is proportional to M_{vis}. Explain why this result implies that the orbital velocity v becomes approximately constant at large radii (where M_{vis} is roughly constant), thus potentially explaining the observed flat rotation curves without invoking dark matter.

References

[1] D. Halliday, R. Resnick, and J. Walker. *Fundamentals of Physics*. Fundamentals of Physics. John Wiley & Sons.

[2] NASA, ESA & A. van der Hoeven. URL: http://www.spacetelescope.org/news/heic1305/.

[3] URL: https://www.astronomy.com/science/how-do-you-measure-the-rotational-speed-of-a-galaxy-taking-into-consideration-the-motion-of-our-galaxy-solar-system-planet-etc/.

[4] C. C. Lin and Frank H. Shu. "On the Spiral Structure of Disk Galaxies." In: *Astrophysical Journal* 140 (Aug. 1964), p. 646. DOI: 10.1086/147955.

[5] Kornelis Begeman. "HI rotation curves of spiral galaxies". PhD thesis. University of Groningen, 1987.

[6] Mario De Leo. URL: https://commons.wikimedia.org/w/index.php?curid=74398525.