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Adjoint QCD_{1+1} in Light-cone Gauge, Quantized at Equal Time ⋆

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Abstract

$SU(2)$ gauge theory coupled to massless fermions in the adjoint representation is quantized in light-cone gauge by imposing the equal-time canonical algebra. The theory is defined on a space-time cylinder with "twisted" boundary conditions, periodic for one color component (the diagonal 3- component) and antiperiodic for the other two. The focus of the study is on the non-trivial vacuum structure and the fermion condensate. It is shown that the indefinite-metric quantization of free gauge bosons is not compatible with the residual gauge symmetry of the interacting theory. A suitable quantization of the unphysical modes of the gauge field is necessary in order to guarantee the consistency of the subsidiary condition and allow the quantum representation of the residual gauge symmetry of the classical Lagrangian: the 3-color component of the gauge field must be quantized in a space with an indefinite metric while the other two components require a positive-definite metric. The contribution of the latter to the free Hamiltonian becomes highly pathological in this representation, but a larger portion of the interacting Hamiltonian can be diagonalized, thus allowing perturbative calculations to be performed. The vacuum is evaluated through second order in perturbation theory and this result is used for an approximate determination of the fermion condensate.

Key words: two dimensional gauge theory, light-cone gauge, chiral symmetry breaking, condensate
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1 Introduction

In this paper we study SU(2) gauge theory with the quarks in the adjoint representation. The model has been of interest since it possesses multiple ground states and a chiral condensate[1][2][3][4] [5] [6]. We shall quantize at equal time with periodicity conditions on the space interval $-L \leq \mathrm{x} \leq L$. We shall use "twisted" boundary conditions [7] [8], periodic for one color component (the diagonal 3-component) and antiperiodic for the other two. We note that these boundary conditions break the gauge symmetry but they lead to worthwhile technical simplifications and have been used in past studies; our work here will find most in common with [5], where the authors used equal-time quantization and reduced to gauge independent degrees of freedom by hand,and [6], where the authors used the light-cone gauge but quantized on the light-cone.

There are several reasons for wanting to study the light-cone gauge case using equal-time quantization. In the case of the Schwinger model the light-cone gauge solution quantized at equal-time with periodicity conditions has more in common with the continuum solution than the solution quantized on the light-cone with periodicity conditions[9] [10]. In particular, the chiral condensate goes to zero at large L in the case of light-cone quantization whereas it goes to the continuum value at large L in the equal-time case. Therefore, the authors of [6] were not able to make an estimate of the physical value of the condensate; we shall make such an estimate in the present paper. Also, the quantization at equal-time requires the use of Lagrange multiplier fields and an indefinite metric representation space, whereas, such fields are not required (or permitted) in the case of light-cone quantization with periodicity conditions and the representation space includes only physical states. In the continuum case, whether quantized at equal-time or on the light-cone, the Lagrange multiplier fields are required[11] and the representation space must be of indefinite metric. The properties of the Lagrange multiplier fields in the equal-time case with periodicity conditions are much like those of the continuum case We expect these qualitative differences between the equal-time periodic case and the light-cone periodic case to hold for the nonabelian case as well and the results presented below will, to some extent, justify that expectation.

In the present paper we shall quantize in light-cone gauge at equal time with twisted boundary conditions through the use of Lagrange multiplier fields. We explicitly construct the physical subspace and demonstrate that it is stable under time evolution. We also construct the algebra of the Lagrange multiplier fields; that algebra is likely to be the same as the continuum case and it may be of use in attempts to construct a continuum solution, particularly if quantizing on the light-cone. In setting up the quantization we encounter an unexpected difficulty: the system possesses a residual gauge symmetry at the classical level where the gauge matrix is given by

$$U(x) = e^{iN\frac{\pi}{L}(t+\mathrm{x})\tau^3} \tag{1}$$

It is this residual gauge symmetry that leads to the multiple vacua which is something we want to study. If this symmetry is not implemented at the quantum level the multiple vacua are not present. Since we have introduced unphysical degrees of freedom we would expect to have to quantize the components of the gauge field in indefinite metric in order to be able to consistently remove the unphysical states. That is what is necessary in the case of the Schwinger model[10]. But here we find that if we quantize all the components of the gauge field in indefinite metric we cannot implement the residual gauge symmetry at the quantum level. On the other hand, if we quantize all the components of the gauge field in positive metric, we cannot consistently remove the unphysical states. the solution, for the present case, is to quantize the periodic component of the gauge field in indefinite metric and the antiperiodic components of the gauge field in positive metric. We show that that procedure allows us to implement the residual gauge symmetry and to consistently remove the unphysical states. With the use of the mixed quantization scheme we then find that there are two possible vacua, in agreement with the findings in [1][3][5][6]. The success of the mixed quantization procedure depends on the breaking of the gauge symmetry through the use of the twisted boundary conditions. It is an open question as to how the quantization should be done in the continuum or in a case where the same periodicity conditions are imposed on all components of the gauge field. The question may have some importance since the same issue arises in the light-cone quantization of standard QCD.

Once the quantization is set up we write the Hamiltonian in a form suitable for perturbative calculations. The existence of the condensate is a nonperturbative effect and neither the vacuum nor any other state can be calculated by a perturbative calculation in which the interaction is the perturbing operator. But we find that we can find an "unperturbed" operator consisting of the kinetic energies plus a small part of the interaction which we can diagonalize in closed form. The eigenstates of this "unperturbed" operator contain all the singularities in the coupling constant and we can then perform standard perturbation calculations using the rest of the interaction as the perturbing operator.

We use the perturbative formalism to calculate the vacuum through second order. We then use the vacuum to calculate the condensate through the same order. We are able to find the exact dependence of the condensate on the parameters but have a constant (a pure number) for which we have only an expansion. We use Padé approximants to estimate the value of this number in the limit $L \to \infty$. It would be interesting to compare our estimate with estimates obtained by other means but we currently know of no such calculations.

Our light-cone conventions are as follows:

$$\mathrm{x}^- = \frac{t-\mathrm{x}}{\sqrt{2}} \quad , \quad \mathrm{x}^+ = \frac{t+\mathrm{x}}{\sqrt{2}}$$

$$\partial_- = \frac{\partial}{\partial \mathrm{x}^-} = \frac{\partial_0 - \partial_1}{\sqrt{2}} \quad , \quad \partial_+ = \frac{\partial}{\partial \mathrm{x}^+} = \frac{\partial_0 + \partial_1}{\sqrt{2}} \; .$$

2 SU(2) Gauge Theory Coupled to Adjoint Fermions

2.1 Basics

The lagrangian density for the theory is [1]

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}) + \mathrm{Tr}(\bar{\Psi}\gamma^\mu \mathrm{D}_\mu \Psi)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \quad , \quad D_\mu = \partial_\mu + ig[A_\mu, \quad] \; .$$

A_μ and Ψ are matrices in the adjoint representation of $SU(2)$:

$$A_\mu = A_\mu^a \tau^a \quad , \quad \Psi = \Psi^a \tau^a \qquad a = 1, 2, 3$$

where $\tau^a = \frac{\sigma^a}{2}$ and σ^a are the Pauli matrices, so that

$$\left[\tau^a, \tau^b\right] = i\epsilon^{abc} \qquad , \qquad \mathrm{Tr}\left(\tau^a \tau^b\right) = \tfrac{1}{2}\delta^{ab} \; .$$

γ^0 and γ^1 are 2×2 matrices satisfying the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \; .$$

We shall use the following representation

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; .$$

[1] The notation used here is similar to that of ref.[6].

Ψ^a is a 2-component Dirac field :

$$\Psi^a = \begin{pmatrix} \Psi^a_R \\ \Psi^a_L \end{pmatrix}$$

The lagrangian is invariant under the gauge transformation

$$\Psi'_{R/L} = U \Psi_{R/L} U^{-1} ,$$

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} \partial_\mu U U^{-1}$$

where U is a spacetime-dependent element of SU(2).
Note that $F_{\mu\nu}$ and D_μ transform covariantly under gauge transformations:

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \qquad , \qquad D'_\mu = U D_\mu U^{-1} .$$

The equations of motion for the gauge fields are

$$D_\mu F^{\mu\nu} = g J^\nu \tag{2}$$

where the fermion current J^μ is defined as

$$J^\mu \equiv i \bar{\Psi}^b \gamma^\mu \Psi^a \left[\tau^a, \tau^b \right] . \tag{3}$$

The conservation law associated with the gauge invariance is

$$\partial_\nu \left(J^\nu - i \left[A_\mu, F^{\mu\nu} \right] \right) = 0 ,$$

the fermion current being conserved in the covariant sense:

$$D_\mu J^\mu = 0 .$$

We shall work in the light-cone gauge. The light-cone gauge condition

$$n^\mu A^a_\mu = 0 \qquad \text{with} \quad n = \frac{1}{\sqrt{2}}(1, -1) \tag{4}$$

or, equivalently,

$$A^a_- \equiv \frac{1}{\sqrt{2}}(A^a_0 - A^a_1) = 0 , \tag{5}$$

can be enforced by means of a Lagrange multiplier $\lambda(x) = \lambda^a(x)\tau^a$, by adding the gauge-fixing term [12]

$$\mathcal{L}_{gf} = 2\mathrm{Tr}\,(\lambda\, n^\mu A_\mu)$$

to the Lagrangian. The theory defined by the Lagrangian

$$\mathcal{L}' = \mathcal{L} + \mathcal{L}_{gf}$$

can be consistently quantized by means of Dirac's procedure[13]. The gauge conditions (4) can be obtained as the Euler-Lagrange equations associated to the fields $\lambda^a(x)$.

The quantum commutators corresponding to the classical Dirac's brackets are

$$[A_0^a(t,\mathrm{x}), (\pi^1)^b(t,\mathrm{y})] = [A_1^a(t,\mathrm{x}), (\pi^1)^b(t,\mathrm{y})] = \delta_{ab}(\mathrm{x}-\mathrm{y})$$

where $(\pi^1)^b = F_{01}^b$. We can see that the gauge constraint $A_-^a = 0$ can be imposed in a strong sense while A_+^a satisfies

$$[A_+^a(t,\mathrm{x}), F_{01}^b(t,\mathrm{y})] = \sqrt{2}\delta_{ab}(\mathrm{x}-\mathrm{y})\ .$$

This procedure introduces spurious degrees of freedom into the theory. The Euler-Lagrange equations associated to the gauge fields

$$D_\mu F^{\mu\nu} + \lambda n^\nu = gJ^\nu \tag{6}$$

are not equivalent to eqs.(2) owing to the presence of the Lagrange multiplier. Equivalence with the original theory can be recovered by imposing the subsidiary condition $\lambda = 0$. However, since the commutators of λ with the other fields are not zero, such condition is incompatible with the quantization of the theory and cannot be imposed in a strong sense. As in the standard Gupta-Bleuler quantization of QED in the Feynman gauge, the subsidiary condition will have to be imposed as a weak condition selecting the physical subspace $\mathcal{V}_{phys}$ of the theory:

$$|phys\rangle \in \mathcal{V}_{phys} \Leftrightarrow \langle phys|\lambda|phys\rangle = 0 \tag{7}$$

The stability of the physical subspace under time evolution is guaranteed by the fact that, as we shall see, λ satisfies a free-field equation of motion and has, therefore, a well defined decomposition into positive and negative frequency parts, so that one can equivalently state the subsidiary condition as

$$|phys\rangle \in \mathcal{V}_{phys} \Leftrightarrow \lambda^{(+)}|phys\rangle = 0 \tag{8}$$

where $\lambda^{(+)}$ denotes the annihilation, or positive frequency, component of the field $\lambda(x)$.

It is convenient to introduce the helicity basis [6]

$$\tau^{+} = \frac{\tau^1 + i\,\tau^2}{\sqrt{2}} \qquad , \qquad \tau^{-} = \frac{\tau^1 - i\,\tau^2}{\sqrt{2}}$$

These satisfy

$$[\tau^+, \tau^-] = \tau^3 \qquad , \qquad [\tau^3, \tau^\pm] = \pm\tau^\pm \tag{9}$$

and

$$\mathrm{Tr}(\tau^+\tau^-) = \mathrm{Tr}(\tau^3)^2 = \frac{1}{2} \quad , \quad \mathrm{Tr}(\tau^\pm)^2 = \mathrm{Tr}(\tau^3\tau^\pm) = 0\;. \tag{10}$$

With respect to this basis A_μ and Ψ are decomposed as

$$A_\mu = A_\mu^3\tau^3 + A_\mu^-\tau^+ + A_\mu^+\tau^- \tag{11}$$

where $A_\mu^\pm \equiv \frac{1}{\sqrt{2}}(A_\mu^1 \pm A_\mu^2)$,

$$\Psi_{R/L} = \phi_{R/L}\tau^3 + \psi_{R/L}\tau^+ + \psi_{R/L}^\dagger\tau^- \tag{12}$$

where $\phi_{R/L} \equiv \Psi_{R/L}^3$ and $\psi_{R/L} \equiv \frac{1}{\sqrt{2}}\left(\Psi_{R/L}^1 - i\Psi_{R/L}^2\right)$, $\psi_{R/L}^\dagger \equiv \frac{1}{\sqrt{2}}(\Psi_{R/L}^1 + i\Psi_{R/L}^2)$.

We shall restrict the space variable to the interval $-L \leq \mathrm{x} \leq L$ and impose "twisted" boundary conditions: the fields ψ_R and ψ_L will be taken to be antiperiodic; it will be convenient, however, to take ϕ_R and ϕ_L to be periodic. For consistency, then, $A_\mu^\pm$ must be taken to be antiperiodic while A_μ^3 is periodic.

With the above definitions the Lagrangian density can be written as

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{2}(F_{01}^3)^2 - \tfrac{1}{2}F_{01}^+F_{01}^- - \tfrac{1}{2}F_{01}^-F_{01}^+ + \frac{i}{\sqrt{2}}\left[\phi_R\partial_+\phi_R + \phi_L\partial_-\phi_L\right] \\ &+\frac{i}{\sqrt{2}}\left[\psi_R^\dagger\partial_+\psi_R + \psi_R\partial_+\psi_R^\dagger + \psi_L^\dagger\partial_-\psi_L + \psi_L\partial_-\psi_L^\dagger\right]\end{aligned} \tag{13}$$

$$-\frac{g}{\sqrt{2}}\left[A_+^3 J_R^3 + A_+^- J_R^+ + A_+^+ J_R^- + A_-^3 J_L^3 + A_-^- J_L^+ + A_-^+ J_L^-\right]$$

$$+\lambda^3 A_-^3 + \lambda^+ A_-^- + \lambda^- A_-^+$$

where

$$A_-^{3,\pm} \equiv \frac{1}{\sqrt{2}}\left(A_0^{3,\pm} - A_1^{3,\pm}\right) \quad , \quad A_+^{3,\pm} \equiv \frac{1}{\sqrt{2}}\left(A_0^{3,\pm} + A_1^{3,\pm}\right)$$

and

$$J_L^{3,\pm} = \frac{1}{\sqrt{2}}\left(J_0^{3,\pm} + J_1^{3,\pm}\right) \quad , \quad J_R^{3,\pm} = \frac{1}{\sqrt{2}}\left(J_0^{3,\pm} - J_1^{3,\pm}\right)$$

The equations of motion for the gauge fields take the form

$$\partial_- F^3 + J_R^3 = 0 \tag{14}$$
$$\partial_+ F^3 + ig(F^+A^- - F^-A^+) - J_L^3 + \lambda^3 = 0 \tag{15}$$
$$\partial_- F^- + J_R^- = 0 \tag{16}$$
$$\partial_+ F^- + ig(F^-A^3 - F^3A^-) - J_L^- + \lambda^- = 0 \tag{17}$$
$$\partial_- F^+ + J_R^+ = 0 \tag{18}$$
$$\partial_+ F^+ + ig(F^3A^+ - F^+A^3) - J_L^+ + \lambda^+ = 0 \tag{19}$$
$$A_- = 0 \tag{20}$$

where $A^{3,\pm} \equiv A_+^{3,\pm}$ and $F^{3,\pm} \equiv F_{01}^{3,\pm}$, which is the only non-vanishing component of the antisymmetric tensor $F_{\mu\nu}^{3,\pm}$ in two dimensions. The condition $A_- = 0$ implies

$$F^{3,\pm} = \partial_0 A_1^{3,\pm} - \partial_1 A_0^{3,\pm} = \partial_- A^{3,\pm}$$

From the expression of the energy-momentum tensor

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)} \partial^\nu \varphi_\alpha - \mathcal{L} g^{\mu\nu}$$

one obtains the canonical Hamiltonian

$$P^0 \equiv H = \int_{-L}^{L} dx\, \Theta^{00}(x).$$

where

$$\Theta^{00} = F^3\partial_0 A_1^3 + F^-\partial_0 A_1^+ + F^+\partial_0 A_1^- + \frac{i}{2}\left(\phi_R\partial_0\phi_R + \psi_R^\dagger\partial_0\psi_R + \psi_R\partial_0\psi_R^\dagger\right)$$
$$+\frac{i}{2}\left(\phi_L\partial_0\phi_L + \psi_L^\dagger\partial_0\psi_L + \psi_L\partial_0\psi_L^\dagger\right) - \mathcal{L}$$

With some manipulations and using the constraint $A_- = 0$ one gets

$$H = \int_{-L}^{L} d\mathrm{x}\left\{\tfrac{1}{2}\left(F^3\right)^2 + F^+F^- - \frac{1}{\sqrt{2}}\left(\partial_1 F^3 A^3 + \partial_1 F^+ A^- + \partial_1 F^- A^+\right)\right.$$
$$+\frac{i}{2}\left(\phi_L\partial_1\phi_L + \psi_L^\dagger\partial_1\psi_L + \psi_L\partial_1\psi_L^\dagger - \phi_R\partial_1\phi_R - \psi_R^\dagger\partial_1\psi_R - \psi_R\partial_1\psi_R^\dagger\right)$$
$$\left.+g\left(A^3J_R^3 + A^-J_R^+ + A^+J_R^-\right)\right\}.$$

2.2 Quantization of the Fermi field

Our treatment of the fermi field is standard. The Fock representation for the fermionic degrees of freedom at $t = 0$ is obtained by Fourier expanding $\Psi_{R/L}(0, \mathrm{x})$. We have

$$\phi_R(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{N=1}^{\infty}\left(r_N e^{ik_N\mathrm{x}} + r_N^\dagger e^{-ik_N\mathrm{x}}\right) + \overset{o}{\phi}_R \tag{21}$$
$$\psi_R(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left(b_n e^{ik_n\mathrm{x}} + d_n^\dagger e^{-ik_n\mathrm{x}}\right) \tag{22}$$
$$\phi_L(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{N=1}^{\infty}\left(\rho_N e^{-ik_N\mathrm{x}} + \rho_N^\dagger e^{ik_N\mathrm{x}}\right) + \overset{o}{\phi}_L \tag{23}$$
$$\psi_L(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^{\infty}\left(\beta_n e^{-ik_n\mathrm{x}} + \delta_n^\dagger e^{ik_n\mathrm{x}}\right)\ , \tag{24}$$

where $\overset{o}{\phi}_{R/L}$ are the zero modes of $\phi_{R/L}$. The lower-case (upper-case) indices run over positive half-odd integers (integers) and $k_n = n\pi/L$, $k_N = N\pi/L$.

The canonical anti-commutation relations for the Fermi fields are

$$\{\phi_R(0,\mathrm{x}), \phi_R(0,y)\} = \delta_P(x-y)$$
$$\{\phi_L(0,\mathrm{x}), \phi_L(0,y)\} = \delta_P(x-y)\ ,$$

where δ_P denotes the periodic delta function, which can be expanded in the interval $[-L,+L]$ as

$$\delta_P(\mathrm{x}-\mathrm{y}) = \frac{1}{2L}\sum_{N=-\infty}^{\infty} e^{i\frac{\pi}{L}N(\mathrm{x}-\mathrm{y})} \ ,$$

and

$$\left\{\psi_R(0,\mathrm{x}),\psi_R^\dagger(0,y)\right\} = \delta_A(x-y) \tag{25}$$

$$\left\{\psi_L(0,\mathrm{x}),\psi_L^\dagger(0,y)\right\} = \delta_A(x-y) \ , \tag{26}$$

where δ_A denotes the anti-periodic delta function

$$\delta_A(\mathrm{x}-\mathrm{y}) = \frac{1}{2L}\sum_{n=-\infty}^{+\infty} e^{i\frac{\pi}{L}n(\mathrm{x}-\mathrm{y})} \ .$$

All the other anti-commutators vanish.
These induce the following algebra for the Fourier modes:

$$\{\rho_N^\dagger,\rho_M\} = \{r_N^\dagger, r_M\} = \delta_{N,M} \tag{27}$$

$$\{b_n^\dagger, b_m\} = \{d_n^\dagger, d_m\} = \{\beta_n^\dagger,\beta_m\} = \{\delta_n^\dagger,\delta_m\} = \delta_{n,m} \tag{28}$$

$$\{\overset{o}{\phi}_R,\overset{o}{\phi}_R\} = \{\overset{o}{\phi}_L,\overset{o}{\phi}_L\} = \frac{1}{2L} \ , \tag{29}$$

all other anti-commutators vanishing.
The fermionic Fock space is generated in the usual way by the action of the creation operators on a vacuum state $|0\rangle$.

We must define the currents with a regularization procedure that is consistent with gauge invariance. We shall use the gauge invariant point splitting procedure [6] and define the currents as

$$\hat{J}_R(0,\mathrm{x}) \equiv -\lim_{\epsilon\to 0}\Psi_R^a(0,\mathrm{x}+\epsilon)\Psi_R^b(0,\mathrm{x})\,[e^{ig\int_{\mathrm{x}}^{\mathrm{x}+\epsilon}A_1\cdot d\mathrm{x}}\,\tau^a\,e^{-ig\int_{\mathrm{x}}^{\mathrm{x}+\epsilon}A_1\cdot d\mathrm{x}},\tau^b]$$

This definition gives the expected result that

$$\begin{aligned}\hat{J}_R^3 &= \tilde{J}_R^3 + \frac{g}{\sqrt{2\pi}}A_1^3\\ \hat{J}_R^+ &= \tilde{J}_R^+ + \frac{g}{\sqrt{2\pi}}A_1^+\\ \hat{J}_R^- &= \tilde{J}_R^- + \frac{g}{\sqrt{2\pi}}A_1^- \ .\end{aligned} \tag{30}$$

and

$$
\begin{aligned}
\hat{J}_L^3 &= \tilde{J}_L^3 - \frac{g}{\sqrt{2}\pi} A_1^3 \\
\hat{J}_L^+ &= \tilde{J}_L^+ - \frac{g}{\sqrt{2}\pi} A_1^+ \\
\hat{J}_L^- &= \tilde{J}_L^- - \frac{g}{\sqrt{2}\pi} A_1^- \ .
\end{aligned}
\tag{31}
$$

Here, the tilde operators, $\tilde{J}$, are the normal ordered, but un-gauge-corrected, currents:

$$
\tilde{J}_R^3(0,\mathrm{x}) = \sqrt{2}\,\boldsymbol{:}\psi_R^\dagger(0,\mathrm{x})\psi_R(0,\mathrm{x})\boldsymbol{:} = \frac{1}{\sqrt{2}L}\sum_{m,n=\frac{1}{2}}^{\infty}\Big(d_n b_m e^{i(k_n+k_m)\mathrm{x}} - d_m^\dagger d_n e^{i(k_n-k_m)\mathrm{x}}
$$
$$
+ b_n^\dagger b_m e^{-i(k_n-k_m)\mathrm{x}} + b_n^\dagger d_m^\dagger e^{-i(k_n+k_m)\mathrm{x}}\Big) \ .
$$

$$
\tilde{J}_R^+(0,\mathrm{x}) = \frac{1}{2L}\sum_{N=0}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\Big(r_N d_n e^{i(N+n)\frac{\pi}{L}} - b_n^\dagger r_N e^{i(N-n)\frac{\pi}{L}}
$$
$$
+ r_N^\dagger d_n e^{-i(N-n)\frac{\pi}{L}} + r_N^\dagger b_n^\dagger e^{-i(N+n)\frac{\pi}{L}}\Big)
$$

$$
\tilde{J}_R^-(0,\mathrm{x}) = \frac{1}{2L}\sum_{N=0}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\Big(b_n r_N e^{i(N+n)\frac{\pi}{L}\mathrm{x}} - r_N^\dagger b_n e^{-i(N-n)\frac{\pi}{L}\mathrm{x}}
$$
$$
+ d_n^\dagger r_N e^{i(N-n)\frac{\pi}{L}\mathrm{x}} + d_n^\dagger r_N^\dagger e^{-i(N+n)\frac{\pi}{L}\mathrm{x}}\Big)
$$

where we have set

$$
\overset{o}{\phi}_R = \frac{r_0 + r_0^\dagger}{\sqrt{2L}} \ . \tag{32}
$$

With similar expressions for the $\tilde{J}_L$ currents.

We must also regularize the kinetic energy in a gauge invariant manner and again we shall use gauge invariant point splitting. We define

$$
[\mathrm{Tr}(\mathrm{i}\Psi_\mathrm{R}(0,\mathrm{x})\partial_1\Psi_\mathrm{R}(0,\mathrm{x})]_{reg} \equiv
$$
$$
\equiv \lim_{\epsilon\to 0}\Big\{\mathrm{Tr}\Big(\mathrm{i}e^{\mathrm{ig}\int_\mathrm{x}^{\mathrm{x}+\epsilon}\mathrm{A}_1(0,\mathrm{y})\mathrm{dy}}\Psi_\mathrm{R}(0,\mathrm{x}+\epsilon)e^{-\mathrm{ig}\int_\mathrm{x}^{\mathrm{x}+\epsilon}\mathrm{A}_1(0,\mathrm{y})\mathrm{dy}}\partial_1\Psi_\mathrm{R}(0,\mathrm{x})\Big) - \mathrm{v.e.v.}\Big\}
$$

Which leads to

$$[\mathrm{Tr}(\mathrm{i}\Psi_{\mathrm{R}}(0,\mathrm{x})\partial_1\Psi_{\mathrm{R}}(0,\mathrm{x})]_{reg} = \frac{i}{2} : \psi_R^\dagger \overset{\leftrightarrow}{\partial_1} \psi_R : + \frac{i}{2} : \phi_R\partial_1\phi_R : + \frac{g^2}{4\pi}\left(2A_1^+A_1^- + (A_1^3)^2\right) . (33)$$

Analogously,

$$[\mathrm{Tr}(\mathrm{i}\Psi_{\mathrm{L}}(0,\mathrm{x})\partial_1\Psi_{\mathrm{L}}(0,\mathrm{x})]_{reg} = \frac{i}{2} : \psi_L^\dagger \overset{\leftrightarrow}{\partial_1} \psi_L : + \frac{i}{2} : \phi_R\partial_1\phi_L : - \frac{g^2}{4\pi}\left(2A_1^+A_1^- + (A_1^3)^2\right) . (34)$$

It will be convenient to perform our analysis in the bosonized basis. To that end we shall need the Fourier components of the tilde currents. We write

$$\tilde{J}_R^3(0,\mathrm{x}) = \frac{1}{\sqrt{2}L}\sum_{N=1}^{\infty}\left(C_N^3 e^{ik_N\mathrm{x}} + {C_N^3}^\dagger e^{-ik_N\mathrm{x}}\right) + \frac{C_0^3}{\sqrt{2}L} \tag{35}$$

where

$$C_N^3 = \sum_{n=\frac{1}{2}}^{\infty}\left(b_n^\dagger b_{N+n} - d_n^\dagger d_{N+n}\right) - \sum_{n=\frac{1}{2}}^{N-\frac{1}{2}} b_n d_{N-n}$$

and

$$C_0^3 = \sum_n (b_n^\dagger b_n - d_n^\dagger d_n) . \tag{36}$$

Also

$$\tilde{J}_L^3(0,\mathrm{x}) = \frac{1}{\sqrt{2}L}\sum_{N=1}^{\infty}\left(D_N^3 e^{-ik_N\mathrm{x}} + {D_N^3}^\dagger e^{ik_N\mathrm{x}}\right) + \frac{D_0^3}{\sqrt{2}L} \tag{37}$$

where

$$D_N^3 = \sum_{n=\frac{1}{2}}^{\infty}\left(\beta_n^\dagger \beta_{N+n} - \delta_n^\dagger \delta_{N+n}\right) - \sum_{n=\frac{1}{2}}^{N-\frac{1}{2}} \beta_n \delta_{N-n}$$

and

$$D_0^3 = \sum_n (\beta_n^\dagger \beta_n - \delta_n^\dagger \delta_n) . \tag{38}$$

For the $\pm$ currents we have

$$\tilde{J}^{\pm}_R(0,\mathrm{x}) = \frac{1}{\sqrt{2}L}\sum_{n=\frac{1}{2}}^{\infty}\left(C^{\pm}_n e^{ik_n\mathrm{x}} + C^{\mp\dagger}_n e^{-ik_n\mathrm{x}}\right) \tag{39}$$

where

$$C^{+}_n = \sum_{M=0}^{\infty} r^{\dagger}_M d_{n+M} - \sum_{m=\frac{1}{2}}^{\infty} b^{\dagger}_m r_{n+m} - \sum_{m=\frac{1}{2}}^{n} d_m r_{n-m} \tag{40}$$

$$C^{-}_n = \sum_{m=\frac{1}{2}}^{\infty} d^{\dagger}_m r_{n+m} - \sum_{M=0}^{\infty} r^{\dagger}_M b_{M+n} - \sum_{m=\frac{1}{2}}^{n} r_{n-m} b_m \ . \tag{41}$$

Similar expressions can be found for the operators $D^{\pm}_n$ such that

$$\tilde{J}^{\pm}_L(0,\mathrm{x}) = \frac{1}{\sqrt{2}L}\sum_{n=\frac{1}{2}}^{\infty}\left(D^{\pm}_n e^{-ik_n\mathrm{x}} + D^{\mp\dagger}_n e^{ik_n\mathrm{x}}\right) \tag{42}$$

Using the fundamental anti-commutators (27), (28) and (29) one can verify that these operators satisfy the commutation relations [2] [14]

$$\left[C^3_N, C^3_M\right] = N\delta_{N,-M} \tag{43}$$

$$\left[C^{\pm}_n, C^{\pm}_m\right] = 0 \tag{44}$$

$$\left[C^3_N, C^{\pm}_m\right] = \pm C^{\pm}_{N+m} \tag{45}$$

$$\left[C^{+}_n, C^{-}_m\right] = C^3_{n+m} + n\delta_{n,-m} \ . \tag{46}$$

where we have defined

$$C^3_{-N} \equiv (C^3_N)^{\dagger} \ , \qquad C^{\pm}_{-n} \equiv (C^{\mp}_n)^{\dagger}$$

The algebra satisfied by the Ds is of course identical.

Following [15] we define

$$\varphi^{(+)}_R(0,\mathrm{x}) = -\sum_{N=1}^{\infty}\frac{1}{N}C^3_N e^{ik_N\mathrm{x}}$$

[2] Note that these relations do not hold in this form if the zero mode of $\phi_{R/L}$ is discarded, as in ref. [6]

$$\varphi_R^{(-)}(0,\mathrm{x}) = \sum_{N=1}^{\infty} \frac{1}{N} (C_N^3)^\dagger e^{-ik_N \mathrm{x}}$$
$$\varphi_L^{(+)}(0,\mathrm{x}) = -\sum_{N=1}^{\infty} \frac{1}{N} D_N^3 e^{-ik_N \mathrm{x}}$$
$$\varphi_L^{(-)}(0,\mathrm{x}) = \sum_{N=1}^{\infty} \frac{1}{N} (D_N^3)^\dagger e^{ik_N \mathrm{x}}$$

and

$$\sigma_{R/L}(0,\mathrm{x}) = \sqrt{2L} e^{\varphi_{R/L}^{(-)}(0,\mathrm{x})} \psi_{R/L}(0,\mathrm{x}) e^{\varphi_{R/L}^{(+)}(0,\mathrm{x})} \ . \tag{47}$$

The following relations hold [15]

$$\sigma_R^+(0,\mathrm{x})\sigma_R(0,\mathrm{x}) = \sigma_R(0,\mathrm{x})\sigma_R^+(0,\mathrm{x}) = 1 \ , \tag{48}$$

$$\sigma_L^+(0,\mathrm{x})\sigma_L(0,\mathrm{x}) = \sigma_L(0,\mathrm{x})\sigma_L^+(0,\mathrm{x}) = 1 \ , \tag{49}$$

$$\{\sigma_R(x), \sigma_L(y)\} = \{\sigma_R(x), \sigma_L^+(y)\} = 0 \ , \tag{50}$$

$$[C_0^3 \, , \, \sigma_L] = [D_0^3 \, , \, \sigma_R] = 0 \tag{51}$$

$$[C_0^3 \, , \, \sigma_R] = -\sigma_R \quad , \qquad [D_0^3 \, , \, \sigma_L] = -\sigma_L \tag{52}$$

$$[C_N^3, \sigma_{R/L}] = [D_N^3, \sigma_{R/L}] = 0 \ . \tag{53}$$

The action of the spurions, $\sigma_{R/L} \equiv \sigma_{R/L}(0,0)$, on the vacuum is given as follows:

$$|M, N\rangle = \sigma_L^M \sigma_R^N |0\rangle \ ,$$

where

$$\sigma_{R/L}^{-N} \equiv (\sigma_{R/L}^\dagger)^N \ .$$

and for $M, N > 0$,

$$|M, N\rangle = \delta^\dagger_{M-\frac{1}{2}} \cdots \delta^\dagger_{\frac{1}{2}} \, d^\dagger_{N-\frac{1}{2}} \cdots d^\dagger_{\frac{1}{2}} |0\rangle$$
$$|-M, N\rangle = \beta^\dagger_{M-\frac{1}{2}} \cdots \beta^\dagger_{\frac{1}{2}} \, d^\dagger_{N-\frac{1}{2}} \cdots d^\dagger_{\frac{1}{2}} |0\rangle$$
$$|M, -N\rangle = \delta^\dagger_{M-\frac{1}{2}} \cdots \delta^\dagger_{\frac{1}{2}} \, b^\dagger_{N-\frac{1}{2}} \cdots b^\dagger_{\frac{1}{2}} |0\rangle$$
$$|-M, -N\rangle = \beta^\dagger_{M-\frac{1}{2}} \cdots \beta^\dagger_{\frac{1}{2}} \, b^\dagger_{N-\frac{1}{2}} \cdots b^\dagger_{\frac{1}{2}} |0\rangle \ .$$

It is easy to see that the states $|M,N\rangle$ are eigenstates of C_0^3 and D_0^3 :

$$C_0^3|M,N\rangle = -N|M,N\rangle$$
$$D_0^3|M,N\rangle = -M|M,N\rangle \ .$$

One can verify that, for any $P>0$

$$C_P^3|M,N\rangle = 0 \quad , \qquad D_P^3|M,N\rangle = 0$$

and since

$$[C_0^3, C_P^{3\,\dagger}] = [D_0^3, C_P^{3\,\dagger}] = 0$$

$$[C_0^3, D_P^{3\,\dagger}] = [D_0^3, D_P^{3\,\dagger}] = 0$$

the action of $C_P^{3\,\dagger}$ and $D_P^{3\,\dagger}$ does not modify the eigenvalues of C_0^3 and D_0^3.
It can be shown [16] that the fermion Fock space $\mathcal{F}$, generated by the action of the creation operators $b_n^\dagger$, $d_n^\dagger$, $\beta_n^\dagger$, $\delta_n^\dagger$ on the vacuum $|0\rangle$, can be decomposed as an infinite direct sum of irreducible representations of the bosonic algebra satisfied by the operators C_P^3 and D_P^3 ($P\neq 0$), each representation corresponding to an eigenspace of C_0^3 and D_0^3 . More explicitly we have

$$\mathcal{F} = \oplus_{M,N}\mathcal{F}_{MN} \qquad M,N = 0,\pm1,\pm2,\dots$$

where $\mathcal{F}_{MN}$ is the Fock space generated by applying products of the operators $C_P^{3\,\dagger}$ and $D_P^{3\,\dagger}$ to the vacuum $|M,N\rangle$ and

$$\forall\,|\Phi_{MN}\rangle\in\mathcal{F}_{MN} \ : \quad C_0^3|\Phi_{MN}\rangle = -N|\Phi_{MN}\rangle \quad , \qquad D_0^3|\Phi_{MN}\rangle = -M|\Phi_{MN}\rangle \ .$$

The free fermion hamiltonian

$$\begin{aligned} H_\psi &= \frac{i}{2}\int_{-L}^{L} d\mathrm{x}\left(\psi_L^\dagger(0,\mathrm{x})\overleftrightarrow{\partial}_1\,\psi_L(0,\mathrm{x}) - \psi_R^\dagger(0,\mathrm{x})\overleftrightarrow{\partial}_1\,\psi_R(0,\mathrm{x})\right) \\ &= \sum_{n=\frac{1}{2}}^{\infty} k_n(\beta_n^\dagger\beta_n + \delta_n^\dagger\delta_n + b_n^\dagger b_n + d_n^\dagger d_n) \end{aligned} \tag{54}$$

and the momentum operator

$$P_\psi = \frac{i}{2}\int_{-L}^{L} d\mathrm{x}\left(\psi_L^\dagger(0,\mathrm{x})\overleftrightarrow{\partial}_1\,\psi_L(0,\mathrm{x}) + \psi_R^\dagger(0,\mathrm{x})\overleftrightarrow{\partial}_1\,\psi_R(0,\mathrm{x})\right)$$

$$= \sum_{n=\frac{1}{2}}^{\infty} k_n(\beta_n^\dagger \beta_n + \delta_n^\dagger \delta_n - b_n^\dagger b_n - d_n^\dagger d_n) \tag{55}$$

can be expressed in terms of the boson operators by means of the Kronig identities:

$$H = \frac{\pi}{2L}\Big((C_0^3)^2 + (D_0^3)^2\Big) + \frac{\pi}{L}\sum_{N=1}^{\infty}\Big({C_N^3}^\dagger C_N^3 + {D_N^3}^\dagger D_N^3\Big) \tag{56}$$

$$P_1 = \frac{\pi}{2L}\Big((D_0^3)^2 - (C_0^3)^2\Big) + \frac{\pi}{L}\sum_{N=1}^{\infty}\Big({D_N^3}^\dagger D_N^3 - {C_N^3}^\dagger C_N^3\Big)\ . \tag{57}$$

Finally, using

$$\sigma_{R/L}(0,\mathrm{x}) = e^{-iP_\psi \mathrm{x}}\sigma_{R/L}e^{P_\psi \mathrm{x}}$$

it is easy to see that

$$\sigma_R(0,\mathrm{x}) = e^{\frac{i\pi}{2L}C_0^3\mathrm{x}}\sigma_R e^{\frac{i\pi}{2L}C_0^3\mathrm{x}}$$
$$\sigma_L(0,\mathrm{x}) = e^{-\frac{i\pi}{2L}D_0^3\mathrm{x}}\sigma_R e^{-\frac{i\pi}{2L}D_0^3\mathrm{x}}$$

and the operators $\psi_{R/L}$ at $t = 0$ can then be written in terms of bosonic operators as

$$\psi_R(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\, e^{-\varphi_R^{(-)}(0,\mathrm{x})} e^{\frac{i\pi}{2L}C_0^3\mathrm{x}}\sigma_R e^{\frac{i\pi}{2L}C_0^3\mathrm{x}} e^{-\varphi_R^{(+)}(0,\mathrm{x})} \tag{58}$$

$$\psi_L(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\, e^{-\varphi_L^{(-)}(0,\mathrm{x})} e^{-\frac{i\pi}{2L}D_0^3\mathrm{x}}\sigma_L e^{-\frac{i\pi}{2L}D_0^3\mathrm{x}} e^{-\varphi_L^{(+)}(0,\mathrm{x})}\ . \tag{59}$$

2.3 the Hamiltonian and the Subsidiary Condition

In this subsection we shall give the regularized quantum Hamiltonian and shall discuss the subsidiary condition. We shall show that the physical subspace is dynamically stable and shall discuss some properties of the physical states.

Using the regularized expressions (30) and (33) for the currents and the fermion kinetic terms we obtain the regularized quantum hamiltonian

$$\hat{H} = \int_{-L}^{L} d\mathrm{x}\Big\{\tfrac{1}{2}\Big(F^3\Big)^2 + F^+F^- - \frac{1}{\sqrt{2}}\Big(\partial_1 F^3 A^3 + \partial_1 F^+ A^- + \partial_1 F^- A^+\Big)$$

$$+\frac{i}{2}\left(:\phi_L\partial_1\phi_L: + :\psi_L^\dagger \overleftrightarrow{\partial}_1 \psi_L: + -:\phi_R\partial_1\phi_R: - :\psi_R^\dagger \overleftrightarrow{\partial}_1 \psi_R:\right)$$
$$+g\left(A^3J_R^3 + A^-J_R^+ + A^+J_R^-\right) + \frac{g^2}{4\pi}\left[(A^3)^2 + 2A^+A^-\right]\Bigg\} \quad (60)$$

where the products of gauge fields will also have to be defined.

Starting from the Fourier expansions of $A_1^a = \frac{1}{\sqrt{2}}A^a$ and F^a at $t = 0$ in the space interval $[-L, L]$ with the chosen boundary conditions :

$$\begin{aligned} A_1^3(0,\mathrm{x}) &= \frac{1}{\sqrt{2L}}\sum_N a_N^3 e^{-ik_N\mathrm{x}} \\ A_1^{1,2}(0,\mathrm{x}) &= \frac{1}{\sqrt{2L}}\sum_n a_n^{1,2} e^{-ik_n\mathrm{x}} \\ F^3(0,\mathrm{x}) &= \frac{1}{\sqrt{2L}}\sum_N b_N^3 e^{-ik_N\mathrm{x}} \\ F^{1,2}(0,\mathrm{x}) &= \frac{1}{\sqrt{2L}}\sum_n b_n^{1,2} e^{-ik_n\mathrm{x}} \end{aligned}$$

and using $a_{-N}^3 = a_N^{3\,\dagger}$, $b_{-N}^3 = b_N^{3\,\dagger}$, $a_{-n}^\pm = a_n^{1\,\dagger} \pm ia_n^{2\,\dagger} = a_n^{\mp\dagger}$, $b_{-n}^\pm = b_n^{\mp\dagger}$, we can write

$$A^3(0,\mathrm{x}) = \frac{1}{\sqrt{L}}\sum_{N=1}^\infty\left(a_N^3 e^{-ik_N\mathrm{x}} + a_N^{3\,\dagger} e^{ik_N\mathrm{x}}\right) + \frac{1}{\sqrt{L}}\,a_0^3 \quad (61)$$
$$A^\pm(0,\mathrm{x}) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^\infty a_n^\pm e^{-ik_n\frac{\pi}{L}\mathrm{x}} = \frac{1}{\sqrt{L}}\sum_{n=\frac{1}{2}}^\infty\left(a_N^\pm e^{-ik_n\mathrm{x}} + a_n^{\mp\dagger} e^{ik_n\mathrm{x}}\right) \quad (62)$$
$$F^3(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{N=1}^\infty\left(b_N^3 e^{-ik_N\mathrm{x}} + b_N^{3\,\dagger} e^{ik_N\mathrm{x}}\right) + \frac{1}{\sqrt{2L}}\,b_0^3 \quad (63)$$
$$F^\pm(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{n=-\infty}^\infty b_n^\pm e^{-ik_n\mathrm{x}} = \frac{1}{\sqrt{2L}}\sum_{n=\frac{1}{2}}^\infty\left(b_n^\pm e^{-ik_n\mathrm{x}} + b_n^{\mp\dagger} e^{ik_n\mathrm{x}}\right)\ . \quad (64)$$

From the commutation relations

$$[A^a(0,\mathrm{x}), F^b(0,\mathrm{y})] = i\sqrt{2}\delta^{ab}\delta(\mathrm{x}-\mathrm{y})$$

we see that we must have

$$\begin{aligned} &[A^\pm(0,\mathrm{x}), F^\mp(0,\mathrm{y})] = i\sqrt{2}\delta(\mathrm{x}-\mathrm{y}) \\ &[A^3(0,\mathrm{x}), F^3(0,\mathrm{y})] = i\sqrt{2}\delta(\mathrm{x}-\mathrm{y}) \end{aligned}$$

and

$$[a^3_M, {b^3_N}^\dagger] = i\delta_{MN} \quad , \quad [a^\pm_m, {b^\pm_n}^\dagger] = i\delta_{mn} \quad , \tag{65}$$

all the other commutators vanishing. We shall postpone the discussion of implementing this algebra in a representation space until after we have discussed the subsidiary condition. For that discussion we need the regularized hamiltonian $\hat{H}$

$$\begin{aligned}
\hat{H} = & H^0_F + \tfrac{1}{2}\left(b^3_0\right)^2 + \frac{g^2}{2\pi}\left(a^3_0\right)^2 + g\sqrt{\frac{2}{L}}C^3_0 a^3_0 \\
& + \sum_{N=1}^{\infty}\left\{{b^3_N}^\dagger b^3_N + ik_N {a^3_N}^\dagger b^3_N - ik_N {b^3_N}^\dagger a^3_N + \frac{g^2}{\pi}{a^3_N}^\dagger a^3_N + g\sqrt{\frac{2}{L}}\left(C^3_N a^3_N + {C^3_N}^\dagger {a^3_N}^\dagger\right)\right\} \\
& + \sum_{n=\frac{1}{2}}^{\infty}\left\{{b^+_n}^\dagger b^+_n + {b^-_n}^\dagger b^-_n + ik_n {a^+_n}^\dagger b^+_n - ik_n {b^-_n}^\dagger a^-_n + ik_n {a^-_n}^\dagger b^-_n - ik_n {b^+_n}^\dagger a^+_n\right. \\
& \left. + \frac{g^2}{\pi}\left({a^+_n}^\dagger a^+_n + {a^-_n}^\dagger a^-_n\right) + g\sqrt{\frac{2}{L}}\left(C^+_n a^-_n + C^-_n a^+_n + {C^+_n}^\dagger {a^-_n}^\dagger + {C^-_n}^\dagger {a^+_n}^\dagger\right)\right\}
\end{aligned} \tag{66}$$

where H^0_F is the free fermion Hamiltonian

$$H^0_F = \sum_{n=\frac{1}{2}}^{\infty} k_n\left(\beta^\dagger_n\beta_n + \delta^\dagger_n\delta_n + b^\dagger_n b_n + d^\dagger_n d_n\right) + \sum_{N=0}^{\infty}\left(\rho^\dagger_N\rho_N + r^\dagger_N r_N\right) \tag{67}$$

which, as we have seen, can also be expressed as

$$H^0_F = \frac{\pi}{2L}\Big((C^3_0)^2 + (D^3_0)^2\Big) + \frac{\pi}{L}\sum_{N=1}^{\infty}\left({C^3_N}^\dagger C^3_N + {D^3_N}^\dagger D^3_N\right) + \sum_{N=0}^{\infty}\left(\rho^\dagger_N\rho_N + r^\dagger_N r_N\right). \tag{68}$$

The Lagrange multipliers λ^3, $\lambda^\pm$ are given in terms of the other fields by

$$\begin{aligned}
\lambda^3 &= -\sqrt{2}\partial_1 F^3 - igF^+A^- + igF^-A^+ + \sqrt{2}g\tilde{J}^3_0 \\
\lambda^+ &= -\sqrt{2}\partial_1 F^+ - igF^3A^+ + igF^+A^3 + \sqrt{2}g\tilde{J}^+_0 \\
\lambda^- &= -\sqrt{2}\partial_1 F^- - igF^-A^3 + igF^3A^- + \sqrt{2}g\tilde{J}^-_0 \ .
\end{aligned}$$

Consider the Fourier expansion

$$\lambda^3(0,\mathrm{x}) = \frac{1}{\sqrt{2L}}\sum_{N=-\infty}^{\infty}\lambda^3_N e^{-ik_N\mathrm{x}} \ ,$$

$$\lambda^{\pm}(0,\mathrm{x}) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} \lambda_n^{\pm} e^{-ik_n \mathrm{x}} \,.$$

Note that from $\lambda^3 = (\lambda^3)^\dagger$ and $\lambda^\pm = (\lambda^\mp)^\dagger$ it follows that $\lambda^3_{-N} = (\lambda^3_N)^\dagger$ and $\lambda^\pm_{-n} = (\lambda^\mp_n)^\dagger$. We want to show that the time evolution of the Lagrange multipliers in the Heisenberg picture is that of free fields satisfying the simple equation $\partial_- \lambda = 0$. In order to see this let us evaluate their commutators with the Hamiltonian.

$[\hat{H}, \lambda^3(0,\mathrm{y})]$ consists of the following terms:

$$1.\, ig \int_{-L}^{L} d\mathrm{x} [F^+(0,\mathrm{x}) F^-(0,\mathrm{x})\,,\; F^-(0,\mathrm{y}) A^+(0,\mathrm{y}) - F^+(0,\mathrm{x}) A^-(0,\mathrm{y})]$$

$$2.\, -\frac{ig}{\sqrt{2}} \int_{-L}^{L} d\mathrm{x}\, [\, \partial_1 F^+(0,\mathrm{x}) A^-(0,\mathrm{x}) + \partial_1 F^-(0,\mathrm{x}) A^+(0,\mathrm{x})\,,$$
$$F^-(0,\mathrm{y}) A^+(0,\mathrm{y}) - F^+(0,\mathrm{y}) A^-(0,\mathrm{y})\,]$$

$$3.\, \int_{-L}^{L} d\mathrm{x}\, [\, \partial_1 F^3(0,\mathrm{x}) A^3(0,\mathrm{x})\,,\; \partial_1 F^3(0,\mathrm{y})\,]$$

$$4.\, g\frac{\pi}{L} \sum_{N=1}^{\infty} \Big[\, {C^3_N}^\dagger C^3_N + {D^3_N}^\dagger D^3_N\,,\; \tilde{J}^3_R(0,\mathrm{y}) + \tilde{J}^3_L(0,\mathrm{y}) \,\Big]$$
$$-g\sqrt{2} \int_{-L}^{L} d\mathrm{x}\, \Big[\, A^3(0,\mathrm{x}) \tilde{J}^3_R(0,\mathrm{x})\,,\; \partial_1 F^3(0,\mathrm{y}) \,\Big]$$

$$5.\, g^2 \int_{-L}^{L} d\mathrm{x} \Big[A^3(0,\mathrm{x}) \tilde{J}^3_R(0,\mathrm{x})\,,\, \tilde{J}^3_R(0,\mathrm{y}) \Big] - \frac{g^2}{2\sqrt{2}\pi} \int_{-L}^{L} d\mathrm{x} \Big[(A^3)^2(0,\mathrm{x}), \partial_1 F^3(0,\mathrm{y}) \Big]$$

$$6.\, g^2 \int_{-L}^{L} d\mathrm{x}\, \Big[\, A^-(0,\mathrm{x}) \tilde{J}^+_R(0,\mathrm{x})\,,\; -iF^+(0,\mathrm{y}) A^-(0,\mathrm{y}) + \tilde{J}^3_R(0,\mathrm{y}) \,\Big]$$

$$7.\, g^2 \int_{-L}^{L} d\mathrm{x}\, \Big[\, A^+(0,\mathrm{x}) \tilde{J}^-_R(0,\mathrm{x})\,,\; iF^-(0,\mathrm{y}) A^+(0,\mathrm{y}) + \tilde{J}^3_R(0,\mathrm{y}) \,\Big]$$

$$8.\, \frac{ig^3}{2\pi} \int_{-L}^{L} d\mathrm{x}\, \Big[\, A^+(0,\mathrm{x}) A^-(0,\mathrm{x}), F^-(0,\mathrm{y}) A^+(0,\mathrm{y}) - F^+(0,\mathrm{y}) A^-(0,\mathrm{y}) \,\Big]$$

We have

$$1. = g\sqrt{2} \int_{-L}^{L} d\mathrm{x} \Big(-F^+(0,\mathrm{y}) F^-(0,\mathrm{x}) \delta_A(\mathrm{x}-\mathrm{y}) + F^+(0,\mathrm{x}) F^-(0,\mathrm{y}) \delta(\mathrm{x}-\mathrm{y}) \Big) = 0$$

$$2. = g \int_{-L}^{L} d\mathrm{x}\, \Big(-\partial_\mathrm{x} F^+(0,\mathrm{x}) A^-(0,\mathrm{y}) \delta_A(\mathrm{x}-\mathrm{y}) + F^+(0,\mathrm{y}) A^-(0,\mathrm{x}) \partial_\mathrm{x} \delta_A(\mathrm{x}-\mathrm{y}) \Big)$$

$$+g\int_{-L}^{L} d\mathrm{x}\,\Big(\partial_{\mathrm{x}}F^{-}(0,\mathrm{x})A^{+}(0,\mathrm{y})\delta(\mathrm{x}-\mathrm{y})-F^{-}(0,\mathrm{y})A^{+}(0,\mathrm{x})\partial_{\mathrm{x}}\delta(\mathrm{x}-\mathrm{y})\Big)$$

$$=g\,\partial_{\mathrm{y}}\Big(F^{-}(0,\mathrm{y})A^{+}(0,\mathrm{y})-F^{+}(0,\mathrm{y})A^{-}(0,\mathrm{y})\Big)$$

$$3. = i\sqrt{2}\int_{-L}^{L} d\mathrm{x}\,\partial_{\mathrm{x}}F^{3}(0,\mathrm{x})\partial_{\mathrm{y}}\delta_{P}(\mathrm{x}-\mathrm{y}) = i\sqrt{2}\,\partial_{y}^{2}F^{3}(0,\mathrm{y})$$

$$4. = \frac{g}{2L}\sum_{N=1}^{\infty}\left(-k_N C_N^3 e^{ik_N\mathrm{x}} + k_N C_N^{3\,\dagger} e^{-ik_N\mathrm{x}} - k_N D_N^3 e^{-ik_N\mathrm{x}} + k_N D_N^{3\,\dagger} e^{ik_N\mathrm{x}}\right) +$$

$$-2ig\int_{-L}^{L} d\mathrm{x}\tilde{J}^{3}(0,\mathrm{x})\partial_{\mathrm{y}}\delta_{P}(\mathrm{x}-\mathrm{y})$$

$$= ig\partial_{\mathrm{y}}\tilde{J}_{R}^{3}(0,\mathrm{y}) - ig\partial_{\mathrm{y}}\tilde{J}_{L}^{3}(0,\mathrm{y}) - 2ig\partial_{\mathrm{y}}\tilde{J}_{R}^{3}(0,\mathrm{y}) = -ig\,\partial_{\mathrm{y}}\Big(\tilde{J}_{R}^{3}(0,\mathrm{y})+\tilde{J}_{L}^{3}(0,\mathrm{y})\Big)$$

$$5. = \int_{-L}^{L} d\mathrm{x}\left[\frac{g^2}{2L^2}A^3(0,\mathrm{x})\sum_{N=1}^{\infty} N\left(e^{ik_N(\mathrm{x}-\mathrm{y})} - e^{-ik_N(\mathrm{x}-\mathrm{y})}\right) - \frac{ig^2}{\pi}A^3(0,\mathrm{x})\partial_{\mathrm{y}}\delta_P(\mathrm{x}-\mathrm{y})\right]$$

$$= \int_{-L}^{L} d\mathrm{x}\left[\frac{ig^2}{\pi}A^3(0,\mathrm{x})\partial_{\mathrm{y}}\delta_P(\mathrm{x}-\mathrm{y}) - \frac{ig^2}{\pi}A^3(0,\mathrm{x})\partial_{\mathrm{y}}\delta_P(\mathrm{x}-\mathrm{y})\right] = 0$$

$$6. = \sqrt{2}g^2\tilde{J}_R^{+}(0,\mathrm{y})A^{-}(0,\mathrm{y}) - \frac{g^2}{2L^2}\int_{-L}^{L} d\mathrm{x}A^{-}(0,\mathrm{x})\sum_{n=-\infty}^{\infty}\sum_{N=-\infty}^{\infty} C_{n+N}^{+}e^{ik_n\mathrm{x}}e^{ik_N\mathrm{y}}$$

$$= \sqrt{2}g^2\tilde{J}_R^{+}(0,\mathrm{y})A^{-}(0,\mathrm{y}) - \frac{g^2}{2L^2}\int_{-L}^{L} d\mathrm{x}A^{-}(0,\mathrm{x})\sum_{n=-\infty}^{\infty} C_{n}^{+}e^{ik_n\mathrm{x}}\sum_{N=-\infty}^{\infty} e^{-ik_N(\mathrm{x}-\mathrm{y})}$$

$$= \sqrt{2}g^2\tilde{J}_R^{+}(0,\mathrm{y})A^{-}(0,\mathrm{y}) - g^2\sqrt{2}\int_{-L}^{L} d\mathrm{x}A^{-}(0,\mathrm{x})\tilde{J}_R^{+}(0,\mathrm{x})\delta_P(\mathrm{x}-\mathrm{y}) = 0$$

$$7. = \sqrt{2}g^2\tilde{J}_R^{-}(0,\mathrm{y})A^{+}(0,\mathrm{y}) - g^2\sqrt{2}\int_{-L}^{L} d\mathrm{x}A^{+}(0,\mathrm{x})\tilde{J}_R^{-}(0,\mathrm{x})\delta_P(\mathrm{x}-\mathrm{y}) = 0$$

$$8. = \frac{g^2}{\sqrt{2}\pi}\int_{-L}^{L} d\mathrm{x}\Big(-A^{-}(0,\mathrm{x})A^{+}(0,\mathrm{y})\delta_A(\mathrm{x}-\mathrm{y}) + A^{+}(0,\mathrm{x})A^{-}(0,\mathrm{y})\delta_A(\mathrm{x}-\mathrm{y})\Big) = 0$$

so that

$$[\hat{H},\lambda^3(0,\mathrm{y})] = -i\partial_{\mathrm{y}}\lambda^3(0,\mathrm{y})$$

and

$$[\hat{H}, \lambda_N^3] = \frac{1}{\sqrt{2L}} \int_{-L}^{L} d\mathrm{y}\, e^{ik_N \mathrm{y}} [\hat{H}, \lambda^3(0,\mathrm{y})] = -k_N \lambda_N^3 \ .$$

The time evolution of λ^3 is given by

$$e^{i\hat{H}t} \lambda^3(0,\mathrm{x}) e^{-i\hat{H}t} = \frac{1}{\sqrt{2L}} \sum_{N=1}^{\infty} \Big(\lambda_N^3 e^{-ik_N(t+\mathrm{x})} + {\lambda_N^3}^\dagger e^{ik_N(t+\mathrm{x})} + \lambda_0^3 \Big)$$

Let us consider $[\hat{H}, \lambda^+(0,\mathrm{y})]$. A similar calculation gives

$$\begin{aligned}
&\int_{-L}^{L} d\mathrm{x} \ \Big[\ \tfrac{1}{2} \Big(F^3(x) \Big)^2 + F^+(x)F^-(x) - \frac{1}{\sqrt{2}} \Big\{ \partial_1 F^3(x) A^3(x) + \partial_1 F^+(x) A^-(x) \\
&\qquad +\partial_1 F^-(x) A^+(x) \Big\} + \frac{g^2}{4\pi} \Big\{ (A^3)^2(x) + 2A^+(x)A^-(x) \Big\} \ , -\sqrt{2}\partial_1 F^+(y) \\
&\qquad -igF^3(y)A^+(y) + igF^+(y)A^3(y) \ \Big]_{x_0=y_0=0} = \\
&\quad = -i\partial_\mathrm{y} \Big\{ -\sqrt{2}\partial_1 F^+(0,\mathrm{y}) - igF^3(0,\mathrm{y})A^+(0,\mathrm{y}) + igF^+(0,\mathrm{y})A^3(0,\mathrm{y}) \Big\} \\
&\qquad -\frac{ig^2}{\pi} \partial_\mathrm{y} A^+(0,\mathrm{y})
\end{aligned}$$

and

$$g^2 \int_{-L}^{L} d\mathrm{x} \Big\{ [A^3(0,\mathrm{x}) \tilde{J}_R^3(0,\mathrm{x}) \ , \ \tilde{J}_R^+(0,\mathrm{y})] + i[A^-(0,\mathrm{x}) \tilde{J}_R^+(0,\mathrm{x}) \ , \ A^3(0,\mathrm{y}) F^+(0,\mathrm{y})] \Big\} = 0$$

while

$$\begin{aligned}
&g^2 \int_{-L}^{L} d\mathrm{x} \Big\{ A^+(0,\mathrm{x}) \Big[\tilde{J}_R^-(0,\mathrm{x}) \ , \ \tilde{J}_R^+(0,\mathrm{y}) \Big] - i\tilde{J}_R^3(0,\mathrm{x}) [A^3(0,\mathrm{x}) \ , \ F^3(0,\mathrm{y})] A^+(0,\mathrm{y}) \Big\} \\
&= g^2 \int_{-L}^{L} d\mathrm{x} \Big\{ \frac{A^+(0,\mathrm{x})}{2L^2} \sum_{m,n=-\infty}^{\infty} e^{ik_n \mathrm{x}} e^{ik_m \mathrm{y}} [C_n^- \ , \ C_m^+] + \sqrt{2} \tilde{J}_R^3(0,\mathrm{x}) A^+(0,\mathrm{y}) \delta(\mathrm{x}-\mathrm{y}) \Big\} \\
&= -\frac{g^2}{2L^2} \int_{-L}^{L} d\mathrm{x} A^+(0,\mathrm{x}) \sum_{m,n=-\infty}^{\infty} e^{ik_n \mathrm{x}} e^{ik_m \mathrm{y}} \Big(C_{m+n}^3 + m\delta_{m,-n} \Big) + g^2 \sqrt{2} \tilde{J}_R^3(0,\mathrm{y}) A^+(0,\mathrm{y})
\end{aligned}$$

$$= -\frac{g^2}{L}\int_{-L}^{L} d\mathrm{x} A^+(0,\mathrm{x})\Big\{\sum_{N=-\infty}^{\infty} C_N^3 e^{ik_N \mathrm{x}}\delta(\mathrm{x}-\mathrm{y}) + \frac{iL}{\pi}\partial_{\mathrm{x}}\delta(\mathrm{x}-\mathrm{y})\Big\}$$
$$+g^2\sqrt{2}\tilde{J}_R^3(0,\mathrm{y})A^+(0,\mathrm{y})$$
$$= \frac{ig^2}{\pi}\partial_{\mathrm{y}} A^+(0,\mathrm{y})$$

and

$$-g\sqrt{2}\int_{-L}^{L} d\mathrm{x}\tilde{J}_R^+(0,\mathrm{x})[A^-(0,\mathrm{x})\,,\,\partial_{\mathrm{y}}F^+(0,\mathrm{y})] = -2ig\partial_{\mathrm{y}}\tilde{J}_R^+(0,\mathrm{y})$$

so that

$$[\hat{H},\lambda^+(0,\mathrm{y})] = -i\partial_{\mathrm{y}}\Big\{-\sqrt{2}\partial_1 F^+(0,\mathrm{y}) - igF^3(0,\mathrm{y})A^+(0,\mathrm{y}) + igF^+(0,\mathrm{y})A^3(0,\mathrm{y})\Big\}$$
$$-2ig\partial_{\mathrm{y}}\tilde{J}_R^+(0,\mathrm{y}) + g\left[H_F\,,\,\tilde{J}_R^+(0,\mathrm{y}) + \tilde{J}_L^+(0,\mathrm{y})\right]\ .$$

Let us evaluate $\left[H_F\,,\,\tilde{J}_R^+(0,\mathrm{y})\right]$ and $\left[H_F\,,\,\tilde{J}_L^+(0,\mathrm{y})\right]$. We have

$$\left[H_F\,,\,C_n^+\right] = \Big[\sum_{m=\frac{1}{2}}^{\infty} k_m(b_m^\dagger b_m + d_m^\dagger d_m) + \sum_{M=1}^{\infty} k_M r_M^\dagger r_M\ ,\ \sum_{N=0}^{\infty} r_N^\dagger d_{n+N}$$
$$-\sum_{j=\frac{1}{2}}^{\infty} b_j^\dagger r_{n+j} - \sum_{j=\frac{1}{2}}^{n} d_j r_{n-j}\Big]$$

and, using the relation

$$[AB\,,\,CD] = A\{B\,,C\}D - AC\{B\,,D\} + \{A\,,C\}DB - C\{A\,,D\}B$$

we get, for positive n,

$$\left[H_F\,,\,C_n^+\right] = -\sum_{m,j=\frac{1}{2}}^{\infty} k_m b_m^\dagger r_{n+j}\delta_{mj} - \sum_{m=\frac{1}{2}}^{\infty}\sum_{N=0}^{\infty} k_m r_N^\dagger d_m \delta_{m,\,n+N}$$
$$-\sum_{m=\frac{1}{2}}^{\infty}\sum_{j=\frac{1}{2}}^{n} k_m r_{n-j} d_m \delta_{mj} + \sum_{M=1}^{\infty}\sum_{N=0}^{\infty} k_M r_M^\dagger d_{n+N}\delta_{MN}$$

$$
\begin{aligned}
&+\sum_{M=1}^{\infty}\sum_{j=\frac{1}{2}}^{\infty} k_M b_j^\dagger r_M \delta_{M,n+j} + \sum_{M=1}^{\infty}\sum_{j=\frac{1}{2}}^{n} k_M d_j r_M \delta_{M,n-j} \\
=&-\sum_{j=\frac{1}{2}}^{\infty}(k_j - k_{n+j}) b_j^\dagger r_{n+j} - \sum_{N=0}^{\infty}(k_{n+N} - k_N) r_N^\dagger d_{n+N} \\
&-\sum_{j=\frac{1}{2}}^{n}(k_j + k_{n-j}) d_j r_{n-j} \\
=&-k_n C_n^+ \ .
\end{aligned}
$$

Analogously one obtains

$$
\begin{aligned}
&\left[H_F\,,\,(C_n^-)^\dagger\right] = k_n (C_n^-)^\dagger \\
&\left[H_F\,,\,(D_n^+)\right] = -k_n D_n^+ \\
&\left[H_F\,,\,(D_n^-)^\dagger\right] = k_n (D_n^-)^\dagger \ .
\end{aligned}
$$

Therefore we have

$$
\left[H_F\,,\,\tilde{J}_R^+(0,\mathrm{y})\right] = \frac{1}{\sqrt{2}L}\sum_{n=\frac{1}{2}}^{\infty}\left(-k_n C_n^+ e^{ik_n \mathrm{y}} + k_n C_n^{-\dagger} e^{-ik_n \mathrm{y}}\right) = i\partial_{\mathrm{y}} \tilde{J}_R^+(0,\mathrm{y})
$$

and

$$
\left[H_F\,,\,\tilde{J}_L^+(0,\mathrm{y})\right] = \frac{1}{\sqrt{2}L}\sum_{n=\frac{1}{2}}^{\infty}\left(-k_n D_n^+ e^{-ik_n \mathrm{x}} + k_n (D_n^-)^\dagger e^{ik_n \mathrm{x}}\right) = -i\partial_{\mathrm{y}} \tilde{J}_L^+(0,\mathrm{y}) \ .
$$

Finally we can write

$$
\begin{aligned}
[\hat{H}, \lambda^+(0,\mathrm{y})] =& -i\partial_{\mathrm{y}}\Big\{ -\sqrt{2}\partial_1 F^+(0,\mathrm{y}) - igF^3(0,\mathrm{y})A^+(0,\mathrm{y}) + igF^+(0,\mathrm{y})A^3(0,\mathrm{y})\Big\} \\
& -ig\partial_{\mathrm{y}}\left(\tilde{J}_R^+(0,\mathrm{y}) + \tilde{J}_L^+(0,\mathrm{y})\right) \\
=& -i\partial_{\mathrm{y}}\lambda^+(0,\mathrm{y})
\end{aligned}
$$

and, obviously,

$$
[\hat{H}, \lambda^-(0,\mathrm{y})] = -i\partial_{\mathrm{y}}\lambda^-(0,\mathrm{y})
$$

so that

$$[\hat{H}, \lambda_n^{\pm}] = \frac{1}{\sqrt{2L}} \int_{-L}^{L} d\mathrm{y} e^{ik_n \mathrm{y}} [\hat{H}, \lambda^{\pm}(0,\mathrm{y})] = -k_n \lambda_n^{\pm} \; .$$

As a consequence we have

$$\lambda^{\pm}(t,\mathrm{x}) = e^{i\hat{H}t} \lambda^{\pm}(0,\mathrm{x}) e^{-i\hat{H}t} = \frac{1}{\sqrt{2L}} \sum_{n=\frac{1}{2}}^{\infty} \left(\lambda_n^{\pm} e^{-ik_n(t+x)} + (\lambda_n^{\pm})^{\dagger} e^{ik_n(t+x)} \right)$$

We have thus shown that the Heisenberg field $\lambda(t,\mathrm{x})$ has a free-field decomposition into positive and negative frequency components, which is fundamental for a consistent quantization of the theory. This result guarantees that the decomposition of λ into Fock creation and annihilation operators and the definition of the physical subspace by means of the subsidiary condition are stable under time evolution.

Another important result is

$$[\hat{H}, \lambda_0^3] = 0 \, .$$

Indeed, the zero mode of λ^3 is a conserved charge. In order to satisfy the subsidiary condition, we shall require that its physical eigenstates have zero eigenvalue.

To further investigate the structure of the physical subspace let us consider the algebra of the Lagrange multipliers. Using the canonical commutation relations we get

$$\begin{aligned} [\lambda^3(0,\mathrm{x}), \lambda^+(0,\mathrm{y})] = & -2gF^+(0,\mathrm{y})\partial_{\mathrm{x}}\delta_P(x-y) - 2gF^+(0,\mathrm{x})\partial_{\mathrm{y}}\delta_A(\mathrm{x}-\mathrm{y}) \\ & + i\sqrt{2}g^2 F^+(0,\mathrm{x}) A^3(0,\mathrm{y}) \delta_A(x-y) \\ & - i\sqrt{2}g^2 F^3(0,\mathrm{y}) A^+(0,\mathrm{x}) \delta_A(x-y) \end{aligned}$$

$$+g^2[\tilde{J}^3_R(0,\mathrm{x})+\tilde{J}^3_L(0,\mathrm{x})\,,\,\tilde{J}^+_R(0,\mathrm{y})+\tilde{J}^+_L(0,\mathrm{y})]$$

and

$$\begin{aligned}
[\lambda^3_N,\lambda^+_m] &= \frac{1}{2L}\int\limits_{-L}^{L} d\mathrm{x}\, e^{ik_N\mathrm{x}}\int\limits_{-L}^{L} d\mathrm{y}\, e^{ik_m\mathrm{y}}\ [\lambda^3(0,\mathrm{x}),\lambda^+(0,\mathrm{y})] \\
&= \frac{g}{\sqrt{2}L}\int\limits_{-L}^{L} d\mathrm{x}\, e^{i(k_N+k_m)\mathrm{x}}\Big[i(k_N+k_m)\sqrt{2}F^+(0,\mathrm{x})+igF^+(0,\mathrm{x})A^3(0,\mathrm{x}) \\
&\qquad -igF^3(0,\mathrm{x})A^+(0,\mathrm{x})\Big]+\frac{g^2}{L}[C^3_{-N}+D^3_N,C^+_{-m}+D^+_m] \\
&= \frac{g}{\sqrt{2}L}\int\limits_{-L}^{L} d\mathrm{x}\, e^{i(k_N+k_m)\mathrm{x}}\Big[-\sqrt{2}\partial_1F^+(0,\mathrm{x})+igF^+(0,\mathrm{x})A^3(0,\mathrm{x}) \\
&\qquad -igF^3(0,\mathrm{x})A^+(0,\mathrm{x})\Big]+\frac{g^2}{L}\left(C^+_{-N-m}+D^+_{N+m}\right)
\end{aligned}$$

where (35), (37), (39), (42) and (45) were used. Finally one has

$$[\lambda^3_N,\lambda^+_m]=\frac{g}{\sqrt{L}}\ \lambda^+_{N+m}$$

and, analogously,

$$[\lambda^3_N,\lambda^-_m]=-\frac{g}{\sqrt{L}}\ \lambda^-_{N+m}\ .$$

In particular, for $N=0$

$$[\lambda^3_0,\lambda^\pm_m]=\pm\frac{g}{\sqrt{L}}\ \lambda^\pm_m\ .$$

which shows that $\lambda^\pm$ are charged fields. This result has the important consequence that the subsidiary conditions involving $\lambda^\pm$ are identically satisfied for states with zero eigenvalue of the charge λ^3_0 :

$$\langle phys|\lambda^\pm_n|phys\rangle=\pm\frac{g}{\sqrt{L}}\langle phys|[\lambda^3_0\,,\,\lambda^\pm_n]|phys\rangle=0$$

as long as

$$\lambda_0^3|phys\rangle = 0 \ .$$

Therefore we only need to require that physical states satisfy the conditions

$$\begin{aligned}&\lambda_N^3|phys\rangle = 0 \quad \text{for N} > 0 \ ,\\ &\lambda_0^3\,|phys\rangle = 0 \ .\end{aligned}$$

One can also show that

$$\begin{aligned}[\lambda_n^+, \lambda_m^-] = \frac{g}{\sqrt{2}L}\int_{-L}^{L} d\mathrm{x}\, e^{i(k_n+k_m)\mathrm{x}}\Big[-\sqrt{2}\partial_1 F^3(0,\mathrm{x}) + igF^-(0,\mathrm{x})A^+(0,\mathrm{x})\\ -igF^+(0,\mathrm{x})A^-(0,\mathrm{x})\Big] + \frac{g^2}{L}[C_{-n}^+ + D_n^+, C_{-m}^- + D_m^-]\end{aligned}$$

which, using $\left[C_{-n}^+, C_{-m}^-\right] = C_{-n-m}^3 - n\delta_{n,-m}$ and $[D_n^+, D_m^-] = D_{n+m}^3 + n\delta_{n,-m}$, gives

$$[\lambda_n^+, \lambda_m^-] = \frac{g}{\sqrt{L}}\ \lambda_{n+m}^3$$

and

$$\begin{aligned}[\lambda^3(0,\mathrm{x})\,,\ \lambda^3(0,\mathrm{y})] &= g^2[\tilde{J}_R^3(0,\mathrm{x})\,,\ \tilde{J}_R^3(0,\mathrm{y})] + g^2[\tilde{J}_L^3(0,\mathrm{x})\,,\ \tilde{J}_L^3(0,\mathrm{y})]\\ &= \frac{g^2}{2L^2}\sum_{N,M=-\infty}^{\infty}\Big([C_N^3\,,\ C_M^3]e^{ik_N\mathrm{x}}e^{ik_M\mathrm{y}} + [D_N^3\,,\ D_M^3]e^{-ik_N\mathrm{x}}e^{-ik_M\mathrm{y}}\Big)\\ &= \frac{g^2}{2L^2}\sum_{N=-\infty}^{\infty}(Ne^{ik_N(\mathrm{x}-\mathrm{y})} + Ne^{-ik_N(\mathrm{x}-\mathrm{y})}) = 0\end{aligned}$$

These relations imply that the Lagrange multipliers generate zero norm states when applied to physical states. This is consistent with the expectation that modes of the Lagrange multipliers can be found in zero norm physical states, in analogy with the Gupta-Bleuler quantization of QED, where zero norm combinations of the unphysical scalar and longitudinal photons are present in the physical subspace.

2.4 Quantization of the Bose field

Our quantization of the Fermi Field follows standard methods. The quantization of the gauge field involves more complex issues and is more delicate. In the case of the Schwinger modes the Bose field had to be quantized in indefinite metric [10]. For the free case that will also work here. Let us consider the part of the unperturbed Hamiltonian which involves the non-zero unphysical modes of the gauge field:

$$H_G = \sum_{n=\frac{1}{2}}^{\infty} \left\{ b_n^{+\dagger} b_n^+ + b_n^{-\dagger} b_n^- + ik_n a_n^{+\dagger} b_n^+ - ik_n b_n^{-\dagger} a_n^- + ik_n a_n^{-\dagger} b_n^- - ik_n b_n^{+\dagger} a_n^+ \right\} + \sum_{N=1}^{\infty} \left\{ b_N^{3\,\dagger} b_N^3 + ik_N a_N^{3\,\dagger} b_N^3 - ik_N b_N^{3\,\dagger} a_N^3 \right\} . \quad (69)$$

We are naturally led to a Fock representation with a vacuum state defined as the state $|0\rangle$ such that $a_N^3|0\rangle = b_N^3|0\rangle = 0$ and $a_n^{\pm}|0\rangle = b_n^{\pm}|0\rangle = 0$ for $n, N > 0$. To implement that idea, following [10], we define, for $n, N > 0$:

$$A_N^3 \equiv \frac{1}{\sqrt{2L}}\, a_N^3 + i\sqrt{\frac{L}{2}}\, b_N^3 \quad (70)$$

$$A_{-N}^3 \equiv \frac{1}{\sqrt{2L}}\, a_N^3 - i\sqrt{\frac{L}{2}}\, b_N^3 \quad (71)$$

$$A_n^{\pm} \equiv \frac{1}{\sqrt{2L}}\, a_n^{\pm} + i\sqrt{\frac{L}{2}}\, b_n^{\pm} \quad (72)$$

$$A_{-n}^{\pm} \equiv \frac{1}{\sqrt{2L}}\, a_n^{\mp} - i\sqrt{\frac{L}{2}}\, b_n^{\mp} \quad (73)$$

so that

$$a_N^3 = \sqrt{\frac{L}{2}} \left(A_N^3 + A_{-N}^3 \right) \quad , \qquad b_N^3 = \frac{A_N^3 - A_{-N}^3}{i\sqrt{2L}}$$
$$a_n^\pm = \sqrt{\frac{L}{2}} \left(A_n^\pm + A_{-n}^\mp \right) \quad , \qquad b_n^\pm = \frac{A_n^\pm - A_{-n}^\mp}{i\sqrt{2L}}$$

The commutation relations

$$[A_M^3, (A_N^3)^\dagger] = \delta_{MN} \quad , \qquad [A_{-M}^3, (A_{-N}^3)^\dagger] = -\delta_{MN}$$
$$[A_m^\pm, (A_n^\pm)^\dagger] = \delta_{mn} \quad , \qquad [A_{-m}^\pm, (A_{-n}^\pm)^\dagger] = -\delta_{mn}$$

can be represented in a Fock space endowed with an indefinite metric where the daggered operators are creation operators and the undaggered operators are destruction operators. As a consequence of the unphysical nature of the degrees of freedom we are considering, the presence of an indefinite metric is not surprising and we know that it can be dealt with consistently provided that its restriction to the physical subspace is positive semidefinite. Note that (69) is not diagonal in this representation, nor can it be diagonalized. The vacuum and the states created out of it by repeated action of the operators $(b_N^3)^\dagger$ and $(b_n^\pm)^\dagger$ provide an incomplete set of eigenstates. This anomalous situation is related to the fact that the metric is not positive definite. A similar situation occurs in the Schwinger model where it can, nonetheless, be shown that the *complete* set of one-particle eigenstates of the full Hamiltonian, as given by the known solution of the model, can be obtained perturbatively starting from the *incomplete* set of unperturbed one-particle eigenstates [17].

The above quantization of the unphysical non-zero modes of the gauge field is required for the non-interacting gauge theory, where the subsidiary conditions can be expressed as $b^3_N|phys\rangle = 0$, $b^{\pm}_n|phys\rangle = 0$ (as can easily be seen by setting $g = 0$ in the expressions for the Lagrange multipliers). The physical subspace can be defined by the 3 independent conditions

$$\left(A^3_N - A^3_{-N}\right)|phys\rangle = \left(A^{\pm}_n - A^{\pm}_{-n}\right)|phys\rangle = 0\,,$$

expressed in terms of annihilation operators. It is then possible to follow the Gupta-Bleuler procedure and show that the physical subspace has a positive semi-definite metric, with zero-norm states being the ones containing ghost-like modes, and

$\langle phys|H_G|phys\rangle = 0$, so that unphysical modes do not contribute to the energy spectrum.

As is characteristic of two-dimensional pure Yang-Mills theories in light-cone gauge, the Hamiltonian (69) has no interaction terms and coincides with that of free gauge bosons. The interaction is carried by the Lagrange multipliers and has the effect of modifying the subsidiary condition and the physical subspace. We expect more restrictive conditions as a consequence of the g-dependent terms in λ. The colour components of λ do not commute with one another and the subsidiary conditions are not independent. As a matter of fact, the Lagrange multipliers satisfy the same algebra as in the previously considered case where fermions are present. The conditions that need to be imposed are $\widetilde{\lambda}^3_0|phys\rangle = 0$ and $\widetilde{\lambda}^3_N|phys\rangle = 0$, for $N > 0$ where

$$\widetilde{\lambda}^3_0 = \frac{ig}{\sqrt{2}L} \sum_{m=\frac{1}{2}}^{\infty} \Big((a^-_m)^\dagger b^-_m + (b^+_m)^\dagger a^+_m - (a^+_m)^\dagger b^+_m - (b^-_m)^\dagger a^-_m \Big)$$

$$\widetilde{\lambda}^3_N = \frac{i}{\sqrt{L}} k_N b^3_N + \frac{ig}{\sqrt{2}L} \sum_{m=\frac{1}{2}}^{N-\frac{1}{2}} \Big(b^-_m a^+_{N-m} - b^+_m a^-_{N-m} \Big)$$
$$+ \frac{ig}{\sqrt{2}L} \sum_{m=\frac{1}{2}}^{\infty} \Big((a^-_m)^\dagger b^-_{m+N} + (b^+_m)^\dagger a^+_{m+N} - (a^+_m)^\dagger b^+_{m+N} - (b^-_m)^\dagger a^-_{m+N} \Big) .$$

We can see that the eigenstates of (69) generated by the action of $b^\pm_n$ are no longer physical states as in the free case. We need to require the more restrictive condition that no modes of $A^\pm$ be present in the physical subspace. Only physical states with modes of A^3 are now zero-norm states and again one has $\langle phys|H_G|phys\rangle = 0$.

This indefinite metric representation of the gauge field, suggested by the free nature of the Hamiltonian associated with it, turns out to be unsuitable for the quantization of the full non-abelian gauge theory, on account of its residual gauge invariance. To see this let us consider the operator

$$U(x) = e^{iN\frac{\pi}{L}(t+\mathrm{x})\tau^3} \tag{74}$$

It satisfies the condition $U(t,-L) = U(t,L)$ and $A'_- = UA_-U^\dagger + \frac{i}{g}\partial_- UU^\dagger = 0$. It leaves the gauge-fixing condition invariant and it preserves the boundary conditions. It is, therefore, a residual gauge symmetry of the classical theory.

Let us see what happens when we try to implement this symmetry in the quantized theory in the representation space where the gauge fields are quantized in indefinite metric. To calculate the action of u on A we use $[\tau^+, \tau^-] = \tau^3$ and $[\tau^3, \tau^\pm] = \pm\tau^\pm$ to get:

$$A' = UAU^\dagger + \frac{i}{g}\partial_+ U U^\dagger = e^{iN\frac{\pi}{L}(t+\mathrm{x})} A^- \tau^+ + e^{-iN\frac{\pi}{L}(t+\mathrm{x})} A^+ \tau^- - \frac{N\pi\sqrt{2}}{gL}\tau^3$$

or

$$A^{-\prime} = e^{iN\frac{\pi}{L}(t+\mathrm{x})} A^- \quad , \quad A^{+\prime} = e^{-iN\frac{\pi}{L}(t+\mathrm{x})} A^+ \quad , \quad A^{3\prime} = A^3 - \frac{N\pi\sqrt{2}}{gL}$$

and from $F' = UFU^\dagger$ we have

$$F^{-\prime} = e^{iN\frac{\pi}{L}(t+\mathrm{x})} F^- \quad , \quad F^{+\prime} = e^{-iN\frac{\pi}{L}(t+\mathrm{x})} F^+ \quad , \quad F^{3\prime} = F^3 \; .$$

Let us concentrate on the transformation properties of the + and − colour components. A quantum operator T^N representing this symmetry in the space of states must be such that the quantum fields represented at $t = 0$ as in (62) and (64) have the following transformation properties:

$$T^N A^-(0,\mathrm{x})(T^N)^\dagger = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} a_n^- e^{-in\frac{\pi}{L}\mathrm{x}} e^{iN\frac{\pi}{L}\mathrm{x}} = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} a_{n+N}^- e^{-in\frac{\pi}{L}\mathrm{x}}$$
$$T^N A^+(0,\mathrm{x})(T^N)^\dagger = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} a_n^+ e^{-in\frac{\pi}{L}\mathrm{x}} e^{-iN\frac{\pi}{L}\mathrm{x}} = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} a_{n-N}^+ e^{-in\frac{\pi}{L}\mathrm{x}}$$
$$T^N F^-(0,\mathrm{x})(T^N)^\dagger = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} b_n^- e^{-in\frac{\pi}{L}\mathrm{x}} e^{iN\frac{\pi}{L}\mathrm{x}} = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} b_{n+N}^- e^{-in\frac{\pi}{L}\mathrm{x}}$$
$$T^N F^+(0,\mathrm{x})(T^N)^\dagger = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} b_n^+ e^{-in\frac{\pi}{L}\mathrm{x}} e^{-iN\frac{\pi}{L}\mathrm{x}} = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} b_{n-N}^+ e^{-in\frac{\pi}{L}\mathrm{x}} \; .$$

We see that must have

$$T^N a_n^+ (T^N)^\dagger = a_{n-N}^+ \quad , \qquad T^N b_n^+ (T^N)^\dagger = b_{n-N}^+$$
$$T^N a_n^- (T^N)^\dagger = a_{n+N}^- \quad , \qquad T^N b_n^- (T^N)^\dagger = b_{n+N}^-$$

or, in terms of the Fock creation and annihilation operators defined in (72–73):

$$T^N A_n^+ (T^N)^\dagger = A_{n-N}^+ \qquad \text{for } n \geq N + \tfrac{1}{2} \tag{75}$$
$$T^N A_n^+ (T^N)^\dagger = (A_{n-N}^+)^\dagger \qquad \text{for } \tfrac{1}{2} \leq n \leq N - \tfrac{1}{2} \tag{76}$$
$$T^N A_{-n}^+ (T^N)^\dagger = A_{-n-N}^+ \qquad \text{for } n \geq \tfrac{1}{2} \tag{77}$$
$$T^N A_{-n}^- (T^N)^\dagger = A_{-n+N}^- \qquad \text{for } n \geq N + \tfrac{1}{2} \tag{78}$$
$$T^N A_{-n}^- (T^N)^\dagger = (A_{N-n}^-)^\dagger \qquad \text{for } \tfrac{1}{2} \leq n \leq N - \tfrac{1}{2} \tag{79}$$
$$T^N A_n^- (T^N)^\dagger = A_{n+N}^- \qquad \text{for } n \geq \tfrac{1}{2} \tag{80}$$

We can see from (76) and (79) that T must turn annihilation operators into creation operators. This does not allow the vacuum to be invariant. The transformed vacuum $T|0\rangle$ must be such that $(A^+_{-\frac{1}{2}})^\dagger T|0\rangle = 0$, a condition which cannot be satisfied by a state in the Fock space we are considering. The symmetry of the theory under index-shifting at a classical level suggests that in a Fock quantization the creation or annihilation nature of the Bose operators must be preserved under index-shifting (that need not be true for Fermi operators where a relation such as $(\delta_{-\frac{1}{2}})^\dagger T|0\rangle = 0$ is easily satisfied). Interpreting $(A^\pm_{-n})^\dagger$ as creation operators generating negative-norm states when acting on the vacuum appears to be inconsistent with this symmetry transformation. As we shall see, implementing this symmetry as a unitary operator in the Hilbert space will be necessary to obtain the non-trivial vacuum structure which is characteristic of this theory when coupled to fermions. One possibility is that the relation $(A^+_{-\frac{1}{2}})^\dagger T|0\rangle = 0$ must be implemented weakly in the physical subspace so that the relation holds in the factor space that forms the physical Hilbert space. We do not know whether or not that idea can be realized and we will not pursue it further in this paper. Here we shall solve the problem by modifying the quantization of the fields $A^\pm$.

In the quantization of $A^\pm$ in the free gauge theory, as well as of A^3 in both the free and the interacting case, the indefinite metric is necessary to express the subsidiary condition in terms of annihilation operators and it allows to get rid of ghost-like modes, which are present in zero-norm physical states, by constructing a Hilbert space with positive definite metric as a quotient space. But in the quantization of the interacting $A^\pm$ the indefinite metric does not seem to play a crucial role. As a matter of fact, a standard definition of creation and annihilation operators with canonical commutators:

$$A^\pm_n \equiv \frac{1}{\sqrt{2L}}\, a^\pm_n + i\sqrt{\frac{L}{2}}\, b^\pm_n \ , \qquad [A^\pm_m, (A^\pm_n)^\dagger] = \delta_{mn} \ ,$$

for both positive and negative n, leads to the following expressions for $\widetilde{\lambda}^3_0$ and $\widetilde{\lambda}^3_N$:

$$\begin{aligned}\widetilde{\lambda}^3_0 &= \frac{g}{\sqrt{2L}} \sum_{m=-\infty}^{\infty} \Big((A^-_m)^\dagger A^-_m - (A^+_m)^\dagger A^+_m \Big) \\ \widetilde{\lambda}^3_N &= \frac{i}{\sqrt{L}}\, k_N b^3_N + \frac{g}{\sqrt{2L}} \sum_{m=-\infty}^{\infty} \Big((A^-_{m-N})^\dagger A^-_m - (A^+_{m-N})^\dagger A^+_m \Big) \ .\end{aligned}$$

The subsidiary conditions can be satisfied by requiring that

$$
\begin{aligned}
&(A^3_N - A^3_{-N})|phys\rangle = 0 \ , \forall N > 0 \\
&A^\pm_n |phys\rangle = 0
\end{aligned}
$$

and (69) can be written as

$$
\begin{aligned}
H_G = &\sum_{n=-\infty}^{\infty} \left\{ \left(k_n + \frac{1}{2L} \right) \left((A^+_n)^\dagger A^+_n + (A^-_n)^\dagger A^-_n \right) - (A^+_n)^\dagger (A^-_{-n})^\dagger - A^+_n A^-_{-n} \right\} \\
&+ \sum_{N=1}^{\infty} \left\{ {b^3_N}^\dagger b^3_N + ik_N {a^3_N}^\dagger b^3_N - ik_N {b^3_N}^\dagger a^3_N \right\} .
\end{aligned}
$$

The vacuum is the only physical state in the positive metric Fock representation of A^+ and A^- and, although it is not an eigenstate of H_G, we still have $\langle 0|H_G|0\rangle = 0$. The transformation U can now be represented by an operator T^N such that:

$$
T^N A^+_n (T^N)^\dagger = A^+_{n-N} \quad , \qquad T^N (A^+_n)^\dagger (T^N)^\dagger = (A^+_{n-N})^\dagger \tag{81}
$$

$$
T^N A^-_n (T^N)^\dagger = A^-_{n+N} \quad , \qquad T^N (A^+_n)^\dagger (T^N)^\dagger = (A^+_{n-N})^\dagger \tag{82}
$$

for any positive or negative n.

One can check that (81–82) are satisfied for $N = 1$ by

$$
\begin{aligned}
\widetilde{T} = \cdots e^{\frac{\pi}{2}\left(A^+_n(A^+_{n-1})^\dagger - (A^+_n)^\dagger A^+_{n-1} + A^-_{-n}(A^-_{-n+1})^\dagger - (A^-_{-n})^\dagger A^-_{-n+1}\right)} \cdots \\
\cdots e^{\frac{\pi}{2}\left(A^+_{3/2}(A^+_{1/2})^\dagger - (A^+_{3/2})^\dagger A^+_{1/2} + A^-_{-3/2}(A^-_{-1/2})^\dagger - (A^-_{-3/2})^\dagger A^-_{-1/2}\right)} \\
e^{\frac{\pi}{2}\left(A^+_{1/2}(A^+_{-1/2})^\dagger - (A^+_{1/2})^\dagger A^+_{-1/2} + A^-_{-1/2}(A^-_{1/2})^\dagger - (A^-_{-1/2})^\dagger A^-_{1/2}\right)} \\
e^{\frac{\pi}{2}\left(A^+_{-1/2}(A^+_{-3/2})^\dagger - (A^+_{-1/2})^\dagger A^+_{-3/2} + A^-_{1/2}(A^-_{3/2})^\dagger - (A^-_{1/2})^\dagger A^-_{3/2}\right)} \cdots \\
\cdots e^{\frac{\pi}{2}\left(A^+_{-n}(A^+_{-n-1})^\dagger - (A^+_{-n})^\dagger A^+_{-n-1} + A^-_{n}(A^-_{n+1})^\dagger - (A^-_{n})^\dagger A^-_{n+1}\right)} \cdots
\end{aligned}
$$

2.5 A Hamiltonian Suitable for Perturbative Calculations

Although unphysical, the modes of A^+ and A^- interact with fermions and can no longer be eliminated from the theory when coupling to fermions is considered. The fact that the gauge-invariant vacuum state associated with a positive-definite Fock representation of A^+ and A^- is not an eigenstate of the unperturbed Hamiltonian prevents us from performing a standard perturbative calculation. On the other hand, as a result of the gauge-invariant renormalization of the fermion products, the term $\frac{g^2}{2\pi}A^+A^-$ has been introduced into the theory. Being quadratic in the gauge field it has the well known form of a mass term, the "mass" being $m \equiv \frac{g}{\sqrt{\pi}}$. This suggests treating it as part of the unperturbed Hamiltonian, in spite of its dependence on g^2, leaving only the order-g terms $g(A^+\widetilde{J}^-_R + A^-\widetilde{J}^+_R)$, which couple the gauge fields to the fermion currents, in the perturbation Hamiltonian. By doing so we can diagonalize the "unperturbed" hamiltonian related to the + and − gauge fields. We can write

$$
H_G = \sum_{n=-\infty}^{\infty}(k_n + m)\left((A^+_n)^\dagger A^+_n + (A^-_n)^\dagger A^-_n\right) + \sum_{N=1}^{\infty}\left\{ {b^3_N}^\dagger b^3_N + ik_N {a^3_N}^\dagger b^3_N - ik_N {b^3_N}^\dagger a^3_N \right\}
$$

where now the Fock operators $A_n^{\pm}$ are defined as

$$A_n^{\pm} \equiv \sqrt{\frac{m}{2}}\, a_n^{\pm} + \frac{i}{\sqrt{2m}}\, b_n^{\pm}$$

for both positive and negative n, with $m = \frac{g}{\sqrt{\pi}}$.

Inverting these relations we obtain

$$a_n^{\pm} = \frac{A_n^{\pm} + (A_{-n}^{\mp})^{\dagger}}{\sqrt{2m}} \quad , \qquad b_n^{\pm} = \sqrt{\frac{m}{2}}\, \frac{A_n^{\pm} - (A_{-n}^{\mp})^{\dagger}}{i} \ .$$

From (65) we see that these operators satisfy the Fock algebra

$$[A_m^{\pm}, A_n^{\pm\dagger}] = \delta_{mn}$$

all the other commutators vanishing.

Analogously, we quantize the zero mode of A^3 by defining

$$A_0^3 \equiv \sqrt{\frac{m}{2}}\, a_0^3 + \frac{i}{\sqrt{2m}}\, b_0^3$$

so that we can write

$$\tfrac{1}{2}\left(b_0^3\right)^2 + \frac{g^2}{2\pi}\left(a_0^3\right)^2 = m A_0^{3\dagger} A_0^3 \ .$$

The Hamiltonian can now be written as

$$\hat{H} = H_0 + H_I$$

where

$$H_0 = H_F^0 + \sum_{N=1}^{\infty} \left\{ {b_N^3}^\dagger b_N^3 + ik_N {a_N^3}^\dagger b_N^3 - ik_N {b_N^3}^\dagger a_N^3 \right\} + m(A_0^3)^\dagger A_0^3$$
$$+ \sum_{n=-\infty}^{\infty} \Big(k_n + m\Big)\Big((A_n^+)^\dagger A_n^+ + (A_n^-)^\dagger A_n^-\Big)$$

and

$$H_I = \sum_{N=1}^{\infty} \left\{ g\sqrt{\frac{2}{L}} \left(C_N^3 a_N^3 + {C_N^3}^\dagger {a_N^3}^\dagger \right) + \frac{g^2}{\pi} {a_N^3}^\dagger a_N^3 \right\}\Bigg\} + \frac{g}{\sqrt{mL}} C_0^3 \Big((A_0^3)^\dagger + A_0^3\Big)$$
$$+ \frac{g}{\sqrt{mL}} \Bigg\{ \sum_{n=-\infty}^{\infty} \Big(A_n^- C_n^+ + A_n^+ C_n^- + (A_n^-)^\dagger (C_n^+)^\dagger + (A_n^+)^\dagger (C_n^-)^\dagger \Big)$$

H_F^0 is the free fermion Hamiltonian (67– 68).

3 Perturbative Calculations

3.1 The Vacuum

Let

$$\begin{aligned} |\Omega\rangle &= |0\rangle + |\Omega^{(1)}\rangle + |\Omega^{(2)}\rangle + \dots \\ E &= E_0 + E_1 + E_2 + \dots \end{aligned}$$

We shall determine the corrections to the vacuum state, $|\Omega^{(1)}\rangle$ and $|\Omega^{(2)}\rangle$, and to the vacuum energy, E_1 and E_2, by requiring that

$$H_0|0\rangle = E_0|0\rangle \tag{83}$$
$$H_0|\Omega^{(1)}\rangle + H_1|0\rangle = E_0|\Omega^{(1)}\rangle + E_1|0\rangle \tag{84}$$
$$H_0|\Omega^{(2)}\rangle + H_1|\Omega^{(1)}\rangle = E_0|\Omega^{(2)}\rangle + E_1|\Omega^{(1)}\rangle + E_2|0\rangle \ . \tag{85}$$

H_0 is normal-ordered in such a way that

$$H_0|0\rangle = 0 \quad \text{and} \quad E_0 = 0$$

while higher order corrections to the energy are not necessarily zero. We have

$$H_I|0\rangle = \sum_{N=1}^{\infty} g\sqrt{\frac{2}{L}}(C^3_N)^\dagger {a^3_N}^\dagger|0\rangle + \frac{g}{\sqrt{mL}}\sum_{n=-\infty}^{\infty}\Big((A^-_n)^\dagger(C^+_n)^\dagger|0\rangle + (A^+_n)^\dagger(C^-_n)^\dagger|0\rangle\Big)\,.$$

One can immediately see that

$$E_1 = \langle 0|H_I|0\rangle = 0\,.$$

As we have seen, $[H_F\,,\,C^+_n] = -k_n C^+_n$, so that we have

$$[H_0\,,\,C^+_n] = -k_n C^+_n \quad \text{and} \quad [H_0\,,\,(C^+_n)^\dagger] = k_n (C^+_n)^\dagger\,.$$

Analogously one can prove that

$$[H_0\,,\,C^-_n] = -k_n C^-_n \quad \text{and} \quad [H_0\,,\,(C^-_n)^\dagger] = k_n (C^-_n)^\dagger\,.$$

These relations, together with $H_0|0\rangle = 0$, tell us that

$$H_0(C^\pm_n)^\dagger|0\rangle = k_n(C^\pm_n)^\dagger|0\rangle\,,$$

while from the expressions of $C^\pm_n$ (40–41) we see that

$$C^\pm_n|0\rangle = (C^\mp_{-n})^\dagger|0\rangle = 0\,.$$

We also have $C^3_N|0\rangle = (C^3_{-N})^\dagger|0\rangle = 0$ and, from (68),

$$H_0(C^3_N)^\dagger|0\rangle = k_N(C^3_N)^\dagger|0\rangle\,.$$

It is now easy to verify that the state

$$
\begin{aligned}
|\Omega^{(1)}\rangle = &-\frac{g\sqrt{2}}{\sqrt{L}}\sum_{N=1}^{\infty}\left(\frac{a_N^3{}^\dagger}{2k_N}+\frac{ib_N^3{}^\dagger}{(2k_N)^2}\right)C_N^3{}^\dagger|0\rangle \\
&-\frac{g}{\sqrt{mL}}\sum_{n=\frac{1}{2}}^{\infty}\frac{1}{2k_n+m}\Big((A_n^-)^\dagger(C_n^+)^\dagger|0\rangle+(A_n^+)^\dagger(C_n^-)^\dagger|0\rangle\Big)
\end{aligned}
$$

satisfies (84) with $E_0 = E_1 = 0$.

In order to evaluate E_2 and $|\Omega^{(2)}\rangle$ we need $H_I|\Omega^{(1)}\rangle$. Using (43), (46) we get

$$
\begin{aligned}
H_I|\Omega^{(1)}\rangle = &\frac{2g^2}{L}\sum_{N=1}^{\infty}\frac{N}{(2k_N)^2}|0\rangle-\frac{g^2}{mL}\sum_{n=\frac{1}{2}}^{\infty}\frac{2n}{2k_n+m}|0\rangle \\
&\frac{2g^2}{L}\sum_{M,N=1}^{\infty}a_M^3{}^\dagger C_M^3{}^\dagger\left(\frac{a_N^3{}^\dagger}{2k_N}+\frac{ib_N^3{}^\dagger}{(2k_N)^2}\right)C_N^3{}^\dagger|0\rangle \\
&\frac{g^2\sqrt{2}}{\sqrt{mL}}\sum_{M=1}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\frac{a_M^3{}^\dagger(C_M^3)^\dagger\Big((A_n^-)^\dagger(C_n^+)^\dagger+(A_n^+)^\dagger(C_n^-)^\dagger\Big)}{2k_n+m}|0\rangle \\
&\frac{g^2\sqrt{2}}{\sqrt{mL}}\sum_{m=-\infty}^{\infty}\sum_{N=1}^{\infty}\Big((A_m^-)^\dagger(C_m^+)^\dagger+(A_m^+)^\dagger(C_m^-)^\dagger\Big)\left(\frac{a_N^3{}^\dagger}{2k_N}+\frac{ib_N^3{}^\dagger}{(2k_N)^2}\right)C_N^3{}^\dagger|0\rangle \\
&\frac{g^2}{mL}\sum_{n=\frac{1}{2}}^{\infty}\frac{(A_0^3)^\dagger C_0^3\Big((A_n^-)^\dagger(C_n^+)^\dagger+(A_n^+)^\dagger(C_n^-)^\dagger\Big)}{2k_n+m}|0\rangle \\
&\frac{g^2}{mL}\sum_{m=-\infty}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\frac{\Big((A_m^-)^\dagger(C_m^+)^\dagger+(A_m^+)^\dagger(C_m^-)^\dagger\Big)\Big((A_n^-)^\dagger(C_n^+)^\dagger+(A_n^+)^\dagger(C_n^-)^\dagger\Big)}{2k_n+m}|0\rangle\ .
\end{aligned}
$$

We therefore have

$$
E_2 = \frac{2g^2}{L}\sum_{N=1}^{\infty}\frac{N}{(2k_N)^2}-\frac{g^2}{mL}\sum_{n=\frac{1}{2}}^{\infty}\frac{2n}{2k_n+m}
$$

a diverging quantity that has to be subtracted from the Hamiltonian.

It is not hard to verify that (85) is satisfied if the state $|\Omega^{(2)}\rangle$ is given by

$$
\begin{aligned}
|\Omega^{(2)}\rangle = & \frac{g^2}{L}\sum_{M=1}^{\infty}\sum_{N=1}^{\infty}\left(\frac{a_M^{3\,\dagger}}{2k_M}+\frac{ib_M^{3\,\dagger}}{(2k_M)^2}\right)\left(\frac{a_N^{3\,\dagger}}{2k_N}+\frac{ib_N^{3\,\dagger}}{(2k_N)^2}\right)C_M^{3\,\dagger}C_N^{3\,\dagger}|0\rangle \\
& +\frac{g^2\sqrt{2}}{\sqrt{m}L}\sum_{M=1}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\Bigg(\frac{a_M^{3\,\dagger}}{2k_M+2k_n+m} \\
& \qquad +\frac{ib_M^{3\,\dagger}}{(2k_n+2k_M+m)^2}\Bigg)C_M^{3\,\dagger}\,\frac{(A_n^-)^\dagger(C_n^+)^\dagger+(A_n^+)^\dagger(C_n^-)^\dagger}{2k_n+m}|0\rangle \\
& +\frac{g^2\sqrt{2}}{\sqrt{m}L}\sum_{N=1}^{\infty}\sum_{p=-\infty}^{\infty}\Bigg\{\frac{1}{2k_p+2k_N+m}\left(\frac{a_N^{3\,\dagger}}{2k_N}+\frac{ib_N^{3\,\dagger}}{(2k_N)^2}\right) \\
& \qquad +\frac{ib_N^{3\,\dagger}}{2k_N(2k_p+2k_N+m)^2}\Bigg\}\Big((A_p^-)^\dagger(C_p^+)^\dagger+(A_p^+)^\dagger(C_p^-)^\dagger\Big)C_N^{3\,\dagger}|0\rangle \\
& +\frac{g^2}{mL}\sum_{n=\frac{1}{2}}^{\infty}\frac{(A_0^3)^\dagger\Big(-(A_n^-)^\dagger(C_n^+)^\dagger+(A_n^+)^\dagger(C_n^-)^\dagger\Big)}{2(k_n+m)(2k_n+m)}|0\rangle \\
& +\frac{g^2}{mL}\sum_{p=\frac{1}{2}}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\left[\frac{(A_p^+)^\dagger(A_n^+)^\dagger(C_p^-)^\dagger(C_n^-)^\dagger}{2(2k_p+m)(2k_n+m)}|0\rangle+\frac{(A_p^-)^\dagger(A_n^-)^\dagger(C_p^+)^\dagger(C_n^+)^\dagger}{2(2k_p+m)(2k_n+m)}|0\rangle\right] \\
& +\frac{g^2}{mL}\sum_{p=-\infty}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\left[\frac{(A_p^-)^\dagger(A_n^+)^\dagger(C_p^+)^\dagger(C_n^-)^\dagger}{2(k_n+k_p+m)(2k_n+m)}|0\rangle+\frac{(A_p^+)^\dagger(A_n^-)^\dagger(C_p^-)^\dagger(C_n^+)^\dagger}{2(k_n+k_p+m)(2k_n+m)}|0\rangle\right].
\end{aligned}
$$

3.1.1 *The subsidiary condition*

Let us verify that the state $|\Omega\rangle = |0\rangle + |\Omega^{(1)}\rangle + |\Omega^{(2)}\rangle$ satisfies the subsidiary condition. It is easy to see that

$$\lambda_0^3|\Omega\rangle = 0$$

where

$$\lambda_0^3 = \frac{g}{\sqrt{2}L}\left(C_0^3+D_0^3\right)+\frac{g}{\sqrt{2}L}\sum_{m=-\infty}^{\infty}\Big((A_m^-)^\dagger A_m^- - (A_m^+)^\dagger A_m^+\Big).$$

As a matter of fact we have

$$\lambda_0^3|0\rangle = 0$$

$$\lambda_0^3|\Omega^{(1)}\rangle = -\frac{g^2}{L\sqrt{2mL}}\Bigg\{\sum_{n=\frac{1}{2}}^{\infty}\frac{C_0^3}{2k_n+m}\Big((A_n^-)^\dagger(C_n^+)^\dagger|0\rangle + (A_n^+)^\dagger(C_n^-)^\dagger|0\rangle\Big)$$
$$+\sum_{n=\frac{1}{2}}^{\infty}\frac{1}{2k_n+m}\Big((A_n^-)^\dagger(C_n^+)^\dagger|0\rangle - (A_n^+)^\dagger(C_n^-)^\dagger|0\rangle\Big)\Bigg\}$$

Using the relation $[C_0^3,(C_n^\pm)^\dagger] = \mp(C_n^\pm)^\dagger$ it is easy to see that $\lambda_0^3|\Omega^{(1)}\rangle = 0$ and that the same holds for every term in $|\Omega^{(2)}\rangle$ (p. 40). Each term in the perturbative expansion of the physical vacuum $|\Omega\rangle$ is an eigenstate of the conserved charge λ_0^3 with eigenvalue 0. As we have seen, in order to be a physical state, the vacuum must also be annihilated by the positive frequency components of λ^3. This means that it must satisfy $\lambda_N^3|\Omega\rangle = 0$ where

$$\lambda_N^3 = \frac{ik_N b_N^3}{\sqrt{L}} + \frac{g}{\sqrt{2L}}\Big((C_N^3)^\dagger + D_N^3\Big)$$
$$+\frac{g}{\sqrt{2L}}\sum_{m=-\infty}^{\infty}\Big((A_{m-N}^-)^\dagger A_m^- - (A_{m-N}^+)^\dagger A_m^+\Big)\,.$$

Clearly this condition cannot be satisfied term by term in the expansion of $|\Omega\rangle$, as is the case for λ_0^3. The action of λ_N^3 mixes up the perturbative orders and the condition cannot be satisfied exactly by our perturbative evaluation of the vacuum. We can only check that it holds for the two lowest orders in the expansion of $\lambda_N^3|\Omega\rangle$. We have

$$\lambda_N^3|0\rangle = \frac{g}{\sqrt{2L}}(C_N^3)^\dagger|0\rangle$$
$$\lambda_N^3|\Omega^{(1)}\rangle = -\frac{g}{\sqrt{2L}}(C_N^3)^\dagger|0\rangle - \frac{g^2}{L\sqrt{L}}\sum_{M=1}^{\infty}\left(\frac{{a_M^3}^\dagger}{2k_M} + \frac{i{b_M^3}^\dagger}{(2k_M)^2}\right)(C_N^3)^\dagger(C_M^3)^\dagger|0\rangle$$
$$-\frac{g^2}{L\sqrt{2mL}}\sum_{n=\frac{1}{2}}^{\infty}\frac{(C_N^3)^\dagger}{2k_n+m}\Big((C_n^-)^\dagger(A_n^+)^\dagger + (C_n^+)^\dagger(A_n^-)^\dagger\Big)|0\rangle$$
$$-\frac{g^2}{L\sqrt{2mL}}\sum_{n=\frac{1}{2}}^{\infty}\frac{1}{2k_n+m}\Big((C_n^+)^\dagger(A_{n-N}^-)^\dagger - (C_n^-)^\dagger(A_{n-N}^+)^\dagger\Big)|0\rangle\,.$$

Disregarding higher order terms we can write

$$\lambda_N^3|\Omega^{(1)}\rangle \simeq -\frac{g}{\sqrt{2L}}(C_N^3)^\dagger|0\rangle - \frac{g^2}{L\sqrt{L}}\sum_{M=1}^{\infty}\left(\frac{{a_M^3}^\dagger}{2k_M} + \frac{i{b_M^3}^\dagger}{(2k_M)^2}\right)(C_N^3)^\dagger(C_M^3)^\dagger|0\rangle$$

$$
-\frac{g^2}{L\sqrt{2mL}}\Bigg\{ \sum_{n=\frac{1}{2}}^{\infty} \frac{(C_N^3)^\dagger}{2k_n} \Big((C_n^-)^\dagger (A_n^+)^\dagger + (C_n^+)^\dagger (A_n^-)^\dagger \Big) |0\rangle
+ \sum_{n=\frac{1}{2}}^{\infty} \frac{1}{2k_n} \Big((C_n^+)^\dagger (A_{n-N}^-)^\dagger - (C_n^-)^\dagger (A_{n-N}^+)^\dagger \Big) |0\rangle \Bigg\}
$$

and keeping only the lowest order terms in $\lambda_N^3|\Omega^{(2)}\rangle$ we get

$$
\begin{aligned}
\lambda_N^3|\Omega^{(2)}\rangle \simeq & \frac{g^2}{L\sqrt{L}} \sum_{M=1}^{\infty} \left(\frac{{a_M^3}^\dagger}{2k_M} + \frac{i{b_M^3}^\dagger}{(2k_M)^2} \right) (C_N^3)^\dagger (C_M^3)^\dagger |0\rangle \\
& + \frac{g^2}{L\sqrt{2mL}} \Bigg\{ \sum_{n=\frac{1}{2}}^{\infty} \frac{k_N (C_N^3)^\dagger \Big((A_n^+)^\dagger (C_n^-)^\dagger + (A_n^-)^\dagger (C_n^+)^\dagger \Big)}{2k_n k_{n+N}} |0\rangle \\
& + \sum_{n=-\infty}^{\infty} \frac{\Big((A_n^+)^\dagger (C_n^-)^\dagger + (A_n^-)^\dagger (C_n^+)^\dagger \Big) (C_N^3)^\dagger}{2k_{n+N}} |0\rangle \Bigg\} .
\end{aligned}
$$

Using the relation $[(C_n^\pm)^\dagger, (C_N^3)^\dagger] = \pm (C_{n+N}^\pm)^\dagger$ we get

$$
\begin{aligned}
\lambda_N^3|\Omega^{(2)}\rangle \simeq & \frac{g^2}{L\sqrt{L}} \sum_{M=1}^{\infty} \left(\frac{{a_M^3}^\dagger}{2k_M} + \frac{i{b_M^3}^\dagger}{(2k_M)^2} \right) (C_N^3)^\dagger (C_M^3)^\dagger |0\rangle \\
& + \frac{g^2}{L\sqrt{2mL}} \Bigg\{ \sum_{n=\frac{1}{2}}^{\infty} \frac{1}{2k_n} (C_N^3)^\dagger \Big((A_n^+)^\dagger (C_n^-)^\dagger + (A_n^-)^\dagger (C_n^+)^\dagger \Big) |0\rangle \\
& + \sum_{n=-N+\frac{1}{2}}^{\infty} \frac{1}{2k_{n+N}} \Big(-(A_n^+)^\dagger (C_{n+N}^-)^\dagger + (A_n^-)^\dagger (C_{n+N}^+)^\dagger \Big) |0\rangle \Bigg\}
\end{aligned}
$$

and one can immediately see that

$$
\lambda_N^3 \Big(|0\rangle + |\Omega^{(1)}\rangle + |\Omega^{(2)}\rangle \Big) \simeq 0 .
$$

Note that this condition would not be satisfied even for the two lowest orders if A^3 had been quantized with a positive definite metric, while no such inconsistency appears as a consequence of our quantization of A^+ and A^-.

3.1.2 The degenerate vacua

We have seen that under the residual gauge transformation

$$U(x) = e^{iN\frac{\pi}{L}(t+\mathrm{x})\tau^3} \tag{86}$$

the gauge fields transform according to

$$\begin{aligned} A^{-\prime} &= e^{iN\frac{\pi}{L}(t+\mathrm{x})}A^- \quad , \qquad A^{+\prime} = e^{-iN\frac{\pi}{L}(t+\mathrm{x})}A^+ \quad , \qquad A^{3\prime} = A^3 - \frac{N\pi\sqrt{2}}{gL} \\ F^{-\prime} &= e^{iN\frac{\pi}{L}(t+\mathrm{x})}F^- \quad , \qquad F^{+\prime} = e^{-iN\frac{\pi}{L}(t+\mathrm{x})}F^+ \quad , \qquad F^{3\prime} = F^3 \ . \end{aligned}$$

As a consequence, under the action of the operator T^N representing this transformation, the Fourier modes into which the fields are decomposed at $t = 0$ transform as

$$T^N a_n^+ (T^N)^\dagger = a_{n-N}^+ \quad , \qquad T^N b_n^+ (T^N)^\dagger = b_{n-N}^+ \tag{87}$$

$$T^N a_n^- (T^N)^\dagger = a_{n+N}^- \quad , \qquad T^N b_n^- (T^N)^\dagger = b_{n+N}^- \tag{88}$$

$$T^N a_M^3 (T^N)^\dagger = a_M^3 \quad \text{for } M \neq 0 \tag{89}$$

$$T^N a_0^3 (T^N)^\dagger = a_0^3 - \frac{N\pi\sqrt{2}}{g\sqrt{L}} \tag{90}$$

$$T^N b_M^3 (T^N)^\dagger = b_M^3 \ . \tag{91}$$

From (87) and (88) we also get

$$T^N A_n^+ (T^N)^\dagger = A_{n-N}^+ \tag{92}$$

$$T^N A_n^- (T^N)^\dagger = A_{n+N}^- \tag{93}$$

Let us consider the transformation of the Fermi field. It is easy to see that

$$\Psi' = U\Psi U^\dagger = e^{iN\frac{\pi}{L}(t+\mathrm{x})}\psi\tau^+ + e^{-iN\frac{\pi}{L}(t+\mathrm{x})}\psi^\dagger\tau^- + \phi\tau^3 \ .$$

Therefore, at $t = 0$,

$$\psi'(0,\mathrm{x}) = e^{iN\frac{\pi}{L}\mathrm{x}}\psi(0,\mathrm{x}) \quad , \qquad \phi'(0,\mathrm{x}) = \phi(0,\mathrm{x})$$

From the bosonized form of $\psi_{R/L}$ (eqs. 58, 59), using relations (48—53) and the identity

$$e^A B = B e^A e^c \quad \text{if } [A,B] = cB, \quad \text{where } c \text{ is a c} - \text{number} ,$$

one can easily prove that

$$\sigma_L^\dagger \sigma_R e^{i\pi(C_0^3+D_0^3)} \psi(0,\mathrm{x}) \left(\sigma_L^\dagger \sigma_R e^{i\pi(C_0^3+D_0^3)}\right)^\dagger = e^{iN\frac{\pi}{L}\mathrm{x}} \psi(0,\mathrm{x}) \; .$$

We also have

$$e^{-\frac{i\pi\sqrt{2}}{g\sqrt{L}} b_0^3} a_0^3 \, e^{\frac{i\pi\sqrt{2}}{g\sqrt{L}} b_0^3} = a_0^3 - \frac{\pi\sqrt{2}}{g\sqrt{L}} \; .$$

The operator T representing the residual gauge transformation can therefore be expressed as

$$T = \sigma_L^\dagger \sigma_R e^{i\pi(C_0^3+D_0^3)} e^{-\frac{i\pi\sqrt{2}}{g\sqrt{L}} b_0^3} \, \widetilde{T} \; .$$

where $\widetilde{T}$ is the operator transforming A^+ and A^- under the residual gauge transformation. In order to determine the transformation properties of $C_n^\pm$ let us consider ψ_R'. We can write

$$\begin{aligned} \psi_R'(0,\mathrm{x}) &= \frac{1}{\sqrt{2L}} \sum_{n=\frac{1}{2}}^{\infty} \left(b_n e^{in\frac{\pi}{L}\mathrm{x}} e^{iN\frac{\pi}{L}\mathrm{x}} + d_n^\dagger e^{-in\frac{\pi}{L}\mathrm{x}} e^{iN\frac{\pi}{L}\mathrm{x}} \right) \\ &= \frac{1}{\sqrt{2L}} \left(\sum_{n=N+\frac{1}{2}}^{\infty} b_{n-N} e^{in\frac{\pi}{L}\mathrm{x}} + \sum_{n=-N+\frac{1}{2}}^{\infty} d_{n+N}^\dagger e^{-in\frac{\pi}{L}\mathrm{x}} \right) . \end{aligned}$$

For $N > 0$ ψ_R' can be written as

$$\psi_R'(0,\mathrm{x}) = \frac{1}{\sqrt{2L}} \left(\sum_{n=N+\frac{1}{2}}^{\infty} b_{n-N} e^{in\frac{\pi}{L}\mathrm{x}} + \sum_{n=\frac{1}{2}}^{N-\frac{1}{2}} d_{N-n}^\dagger e^{in\frac{\pi}{L}\mathrm{x}} + \sum_{n=\frac{1}{2}}^{\infty} d_{n+N}^\dagger e^{-in\frac{\pi}{L}\mathrm{x}} \right)$$

and for $N < 0$

$$\psi_R'(0,\mathrm{x}) = \frac{1}{\sqrt{2L}} \left(\sum_{n=\frac{1}{2}}^{\infty} b_{n-N} e^{in\frac{\pi}{L}\mathrm{x}} + \sum_{n=\frac{1}{2}}^{-N-\frac{1}{2}} b_{-n-N} e^{-in\frac{\pi}{L}\mathrm{x}} + \sum_{n=-N+\frac{1}{2}}^{\infty} d_{n+N}^\dagger e^{-in\frac{\pi}{L}\mathrm{x}} \right) .$$

We can therefore see that we must have

$$
\begin{aligned}
T^N b_n (T^N)^\dagger &= b_{n-N} \quad \text{for } N < n \\
T^N b_n (T^N)^\dagger &= d^\dagger_{N-n} \quad \text{for } N > n \\
T^N d_n (T^N)^\dagger &= d^\dagger_{n+N} \quad \text{for } N > -n \\
T^N d_n (T^N)^\dagger &= b^\dagger_{-N-n} \quad \text{for } N < -n \ .
\end{aligned}
$$

We can now determine the transformation properties of $C_n^{\pm}$.
For $n > N > 0$ we have

$$
\begin{aligned}
T^N C_n^- (T^N)^\dagger = & \sum_{m=\frac{1}{2}}^{\infty} d^\dagger_{m+N} r_{n+m} - \sum_{M=0}^{\infty} r^\dagger_M b_{M+n-N} \\
& - \sum_{m=N+\frac{1}{2}}^{n} r_{n-m} b_{m-N} - \sum_{m=\frac{1}{2}}^{N-\frac{1}{2}} r_{n-m} d^\dagger_{N-m} \\
= & \sum_{m=N+\frac{1}{2}}^{\infty} d^\dagger_m r_{n-N+m} - \sum_{M=0}^{\infty} r^\dagger_M b_{M+n-N} \\
& - \sum_{m=\frac{1}{2}}^{n-N} r_{n-N-m} b_m + \sum_{m=\frac{1}{2}}^{N-\frac{1}{2}} d^\dagger_m r_{n-N+m} \\
= & C^-_{n-N}
\end{aligned}
$$

and for $N > n > 0$

$$
\begin{aligned}
T^N C_n^- (T^N)^\dagger = & \sum_{m=\frac{1}{2}}^{\infty} d^\dagger_{m+N} r_{n+m} - \sum_{M=0}^{N-n-\frac{1}{2}} r^\dagger_M d^\dagger_{N-n-M} \\
& - \sum_{M=N-n+\frac{1}{2}}^{\infty} r^\dagger_M b_{M+n-N} - \sum_{m=\frac{1}{2}}^{n} r_{n-m} d^\dagger_{N-m} \\
= & \sum_{M=n+\frac{1}{2}}^{\infty} d^\dagger_{M-n+N} r_M + \sum_{M=0}^{n-\frac{1}{2}} d^\dagger_{M-n+N} r_M \\
& - \sum_{m=\frac{1}{2}}^{N-n} r^\dagger_{N-n-m} d^\dagger_m - \sum_{m=\frac{1}{2}}^{\infty} r^\dagger_{m+N-n} b_m \\
= & (C^+_{N-n})^\dagger \equiv C^-_{n-N}
\end{aligned}
$$

Analogously it is possible to show that for any positive or negative N

$$T^N C_n^- (T^N)^\dagger = C_{n-N}^- \tag{94}$$

$$T^N C_n^+ (T^N)^\dagger = C_{n+N}^+ \ . \tag{95}$$

It follows from (87), (88), (90) and (94–95) that the action of the transformation T^N on the Hamiltonian (60) is given by

$$T^N \hat{H} (T^N)^\dagger = \hat{H} - \frac{g^2 N\pi}{2L}(C_0^3 + D_0^3) + \frac{ig^2 N\pi}{2L} \sum_{n=\frac{1}{2}}^{\infty} \left((a_n^+)^\dagger b_n^+ + (b_n^-)^\dagger a_n^- - (a_n^-)^\dagger b_n^- - (b_n^+)^\dagger a_n^+ \right) \ .$$

$\hat{H}$ is not invariant under the action of T^N but

$$T^N \hat{H} (T^N)^\dagger = \hat{H} - \frac{gN\pi}{\sqrt{2}} \lambda_0^3$$

which means that $\hat{H}$ is invariant when restricted to the physical subspace. Note also that

$$T^N \lambda_0^3 (T^N)^\dagger = \lambda_0^3$$

so that if $\lambda_0^3 |\varphi\rangle = 0$ then also $\lambda_0^3 T^N |\varphi\rangle = 0$ and

$$\hat{H} T^N |\varphi\rangle = T^N \hat{H} (T^N)^\dagger T^N |\varphi\rangle + \frac{gN\pi}{\sqrt{2}} \lambda_0^3 T^N |\varphi\rangle = T^N \hat{H} |\varphi\rangle \ .$$

In particular this means that the states

$$|\Omega_N\rangle \equiv T^N |\Omega\rangle \ , \quad N = 0, \pm 1, \pm 2, \ldots$$

are an infinite set of degenerate vacua. These states are clearly not gauge-invariant. Physically acceptable gauge-invariant vacua can be obtained by constructing superpositions that diagonalize the operators T^N. We can create eigenstates of T by forming θ-states as in the case of the Schwinger model. If we take

$$|\theta\rangle \equiv \sum_{N=-\infty}^{\infty} e^{-iN\theta} |\Omega_N\rangle \ ,$$

we have

$$T^M|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{-iN\theta}|\Omega_{N+M}\rangle = e^{iM\theta}|\theta\rangle$$

so that $|\theta\rangle$ is invariant up to a phase factor under the action of T^M.

The theory is also invariant [6] under the transformation R such that

$$\begin{aligned} &R\psi R^{-1} = \psi^\dagger \\ &R\phi R^{-1} = -\phi \\ &RA^\pm R^{-1} = A^\mp \\ &RA^3 R^{-1} = -A^3 \end{aligned}$$

corresponding to the $SU(2)$ transformation $U{=}e^{i\pi\tau_1}$.
The action of R on the fermion Fock operators is

$$\begin{aligned} &Rb_n R^{-1} = d_n \\ &R\beta_n R^{-1} = \delta_n \\ &Rr_N R^{-1} = -r_N \\ &R\rho_N R^{-1} = -\rho_N \ . \end{aligned}$$

As a consequence we have

$$\begin{aligned} &RC_n^\pm R^{-1} = C_n^\mp \\ &RC_N^3 R^{-1} = -C_N^3 \ . \end{aligned}$$

The gauge Fock operators transform as

$$\begin{aligned} &RA_n^\pm R^{-1} = A_n^\mp \\ &RA_N^3 R^{-1} = -A_N^3 \ . \end{aligned}$$

Note that the state $R|0\rangle$ is annihilated by all the destruction operators and, since $R^2 = 1$, we must have

$$R|0\rangle = \pm|0\rangle \ .$$

Without loss of generality we may take $R|0\rangle = |0\rangle$. This relation, together with the transformation properties of the Fock operators, defines the action

of R on all states.
One can immediately see that the state $|\Omega\rangle$ is invariant under the action of R:

$$R|\Omega\rangle = |\Omega\rangle \ .$$

Let us consider now the action of R on the other vacuum states $|\Omega_N\rangle \equiv T^N|\Omega\rangle$. From the definition of the spurion operators (p. 14) it is not hard to see that

$$R\sigma_{R/L}R^{-1} = \sigma^{\dagger}_{R/L}$$

and it is straightforward to verify that

$$RTR^{-1} = -T^{\dagger}$$

and

$$RT^NR^{-1} = (-1)^N(T^{\dagger})^N \equiv (-1)^N T^{-N} \ .$$

As a consequence, R interchanges $|\Omega_N\rangle$ and $|\Omega_{-N}\rangle$

$$R|\Omega_N\rangle = (-1)^N|\Omega_{-N}\rangle \ .$$

By applying R to the θ-vacuum we obtain

$$R|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{-iN\theta} R|\Omega_N\rangle = \sum_{N=-\infty}^{\infty} e^{-iN\theta} e^{iN\pi}|\Omega_{-N}\rangle = \sum_{N=-\infty}^{\infty} e^{-iN(\pi-\theta)}|\Omega_N\rangle$$

and we see that only two values of the parameter θ , namely $\theta = \pm\frac{\pi}{2}$, give rise to states which are invariant under both the T and R residual symmetries. We therefore have two physically acceptable vacua, in agreement with refs. [1][3][5][6].

We want to use our results for the vacuum to obtain a perturbative evaluation of the gauge-invariant fermion condensate, defined as

$$\frac{\langle\theta|\mathrm{Tr}\bar{\Psi}\Psi|\theta\rangle}{\langle\theta|\theta\rangle} .$$

We have

$$\mathrm{Tr}\bar{\Psi}\Psi = i\mathrm{Tr}(\Psi_L^\dagger\Psi_R - \Psi_R^\dagger\Psi_L) = i\left(\phi_L\phi_R + \psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right)$$

and

$$\langle\theta|\mathrm{Tr}\bar{\Psi}\Psi|\theta\rangle = i\sum_{N,M=-\infty}^{\infty} e^{i(M-N)\theta}\,\langle\Omega_M|\left(\phi_L\phi_R + \psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right)|\Omega_N\rangle .$$

Being a time-independent quantity, the fermion condensate can be evaluated at $t = 0$.
Writing

$$\Omega_N = \Omega_N^{(0)} + \Omega_N^{(1)} + \Omega_N^{(2)} + \ldots$$

we have

$$\langle\theta|\mathrm{Tr}\bar{\Psi}\Psi|\theta\rangle = i\sum_{j,k}\sum_{N,M=-\infty}^{\infty} e^{i(M-N)\theta}\,\langle\Omega_M^{(j)}|\left(\phi_L\phi_R + \psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L\right)(0,\mathrm{x})\,|\Omega_N^{(k)}\rangle$$

where

$$|\Omega_N^{(i)}\rangle = T^N|\Omega^{(i)}\rangle \quad \text{and} \quad |\Omega^{(0)}\rangle \equiv |0\rangle .$$

Explicitly

$$|\Omega_N^{(0)}\rangle = T^N|0\rangle = e^{-\frac{iN\pi\sqrt{2}}{g\sqrt{L}}b_0^3}\left(\sigma_L^\dagger\sigma_R\right)^N|0\rangle \equiv e^{-\frac{iN\pi\sqrt{2}}{g\sqrt{L}}b_0^3}|N\rangle$$

where

$$|N\rangle = \beta^{\dagger}_{N-\frac{1}{2}} d^{\dagger}_{N-\frac{1}{2}} \cdots \beta^{\dagger}_{\frac{1}{2}} d^{\dagger}_{\frac{1}{2}} |0\rangle \qquad \text{for } N > 0$$
$$|N\rangle = \delta^{\dagger}_{N-\frac{1}{2}} b^{\dagger}_{N-\frac{1}{2}} \cdots \delta^{\dagger}_{\frac{1}{2}} b^{\dagger}_{\frac{1}{2}} |0\rangle \qquad \text{for } N < 0\,,$$

$$|\Omega_N^{(1)}\rangle = -\frac{g\sqrt{2}}{\sqrt{L}} \sum_{N=1}^{\infty} \left(\frac{a_N^{3\,\dagger}}{2k_N} + \frac{ib_N^{3\,\dagger}}{(2k_N)^2} \right) C_N^{3\,\dagger} |\Omega_N^{(0)}\rangle$$
$$-\frac{g}{\sqrt{mL}} \sum_{n=\frac{1}{2}}^{\infty} \frac{1}{2k_n+m} \Big((A^-_{n+N})^\dagger (C^+_{n+N})^\dagger |\Omega_N^{(0)}\rangle + (A^+_{n-N})^\dagger (C^-_{n-N})^\dagger |\Omega_N^{(0)}\rangle \Big)\,,$$

$$|\Omega_N^{(2)}\rangle = \frac{g^2}{L} \sum_{M=1}^{\infty} \sum_{J=1}^{\infty} \left(\frac{a_M^{3\,\dagger}}{2k_M} + \frac{ib_M^{3\,\dagger}}{(2k_M)^2} \right) \left(\frac{a_J^{3\,\dagger}}{2k_J} + \frac{ib_J^{3\,\dagger}}{(2k_J)^2} \right) C_M^{3\,\dagger} C_J^{3\,\dagger} |\Omega_N^{(0)}\rangle$$
$$+\frac{g^2\sqrt{2}}{\sqrt{mL}} \sum_{M=1}^{\infty} \sum_{n=\frac{1}{2}}^{\infty} \Bigg(\frac{a_M^{3\,\dagger}}{2k_M+2k_n+m}$$
$$+\frac{ib_M^{3\,\dagger}}{(2k_n+2k_M+m)^2} \Bigg) C_M^{3\,\dagger} \frac{(A^-_{n+N})^\dagger (C^+_{n+N})^\dagger + (A^+_{n-N})^\dagger (C^-_{n-N})^\dagger}{2k_n+m} |\Omega_N^{(0)}\rangle$$
$$+\frac{g^2\sqrt{2}}{\sqrt{mL}} \sum_{J=1}^{\infty} \sum_{p=-\infty}^{\infty} \Bigg\{ \frac{1}{2k_p+2k_J+m} \left(\frac{a_J^{3\,\dagger}}{2k_J} + \frac{ib_J^{3\,\dagger}}{(2k_J)^2} \right)$$
$$+\frac{ib_J^{3\,\dagger}}{2k_J(2k_p+2k_J+m)^2} \Bigg\} \Big((A^-_{p+N})^\dagger (C^+_{p+N})^\dagger + (A^+_{p-N})^\dagger (C^-_{p-N})^\dagger \Big) C_J^{3\,\dagger} |\Omega_N^{(0)}\rangle$$
$$+\frac{g^2}{mL} \Bigg\{ \sum_{n=\frac{1}{2}}^{\infty} \frac{\Big((A_0^3)^\dagger - \frac{N\pi\sqrt{m}}{g\sqrt{L}} \Big) \Big(-(A^-_{n+N})^\dagger (C^+_{n+N})^\dagger + (A^+_{n-N})^\dagger (C^-_{n-N})^\dagger \Big)}{2(k_n+m)(2k_n+m)} |\Omega_N^{(0)}\rangle$$
$$+\sum_{p=\frac{1}{2}}^{\infty} \sum_{n=\frac{1}{2}}^{\infty} \frac{(A^+_{p-N})^\dagger (A^+_{n-N})^\dagger (C^-_{p-N})^\dagger (C^-_{n-N})^\dagger}{2(2k_p+m)(2k_n+m)} |\Omega_N^{(0)}\rangle$$
$$+\sum_{p=\frac{1}{2}}^{\infty} \sum_{n=\frac{1}{2}}^{\infty} + \frac{(A^-_{p+N})^\dagger (A^-_{n+N})^\dagger (C^+_{p+N})^\dagger (C^+_{n+N})^\dagger}{2(2k_p+m)(2k_n+m)} |\Omega_N^{(0)}\rangle$$
$$+\sum_{p=-\infty}^{\infty} \sum_{n=\frac{1}{2}}^{\infty} \frac{(A^-_{p+N})^\dagger (A^+_{n-N})^\dagger (C^+_{p+N})^\dagger (C^-_{n-N})^\dagger}{2(k_n+k_p+m)(2k_n+m)} |\Omega_N^{(0)}\rangle$$
$$+\sum_{p=-\infty}^{\infty} \sum_{n=\frac{1}{2}}^{\infty} \frac{(A^+_{p-N})^\dagger (A^-_{n+N})^\dagger (C^-_{p-N})^\dagger (C^+_{n+N})^\dagger}{2(k_n+k_p+m)(2k_n+m)} |\Omega_N^{(0)}\rangle \Bigg\}.$$

We define

$$|\theta^{(i)}\rangle = \sum_{N=-\infty}^{\infty} e^{-iN\theta}|\Omega_N^{(i)}\rangle$$

so that we can write

$$|\theta\rangle = |\theta^{(0)}\rangle + |\theta^{(1)}\rangle + |\theta^{(2)}\rangle + \dots$$

and

$$\langle\theta|\mathrm{Tr}\bar{\Psi}\Psi|\theta\rangle = \sum_{i,j}\langle\theta^{(i)}|\mathrm{Tr}\bar{\Psi}\Psi|\theta^{(j)}\rangle$$

The calculation of this quantity is long and tedious, involving very lengthy expressions. Here, we shall just summerize a few of the intermediate steps and give the result. We first consider the contribution of the complex field. We find that

$$\begin{aligned}\langle\theta|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta\rangle \simeq \langle\theta^{(0)}|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta^{(0)}\rangle + \langle\theta^{(1)}|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta^{(1)}\rangle \\ +2\langle\theta^{(1)}|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta^{(2)}\rangle + \langle\theta^{(2)}|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta^{(2)}\rangle/;,\end{aligned}$$

where

$$\langle\theta^{(0)}|i\left(\psi_L^\dagger(0,\mathrm{x})\psi_R(0,\mathrm{x}) - \psi_R^\dagger(0,\mathrm{x})\psi_L(0,\mathrm{x})\right)|\theta^{(0)}\rangle = -\frac{1}{L}\sum_{N=-\infty}^{\infty}\sin\theta e^{-\frac{\pi}{2mL}} \;, \quad (96)$$

$$\begin{aligned}&\langle\theta^{(1)}|\, i\left(\psi_L^\dagger(0,\mathrm{x})\psi_R(0,\mathrm{x}) - \psi_L^\dagger(0,\mathrm{x})\psi_R(0,\mathrm{x})\right)|\theta^{(1)}\rangle = \\ &-\frac{g^2}{mL^2}e^{-\frac{\pi}{2mL}}\sum_{N=-\infty}^{\infty}\sin\theta\left\{\sum_{n=\frac{1}{2}}^{\infty}\frac{2n}{(2k_n+m)(2k_{n+1}+m)} - m\sum_{J=1}^{\infty}\frac{(J-1)}{2k_J^3}\right\}, \quad (97)\end{aligned}$$

$$\begin{aligned}&\langle\theta^{(1)}|\, i\left(\psi_L^\dagger(0,\mathrm{x})\psi_R(0,\mathrm{x}) - \psi_R^\dagger(0,\mathrm{x})\psi_L(0,\mathrm{x})\right)|\theta^{(2)}\rangle = \\ &= \frac{g^2\pi^2}{mL^4}e^{-\frac{\pi}{2mL}}\sum_{N=-\infty}^{\infty}\sin\theta\sum_{n=\frac{1}{2}}^{\infty}\frac{n}{2(2k_n+m)(2k_{n+1}+m)(k_n+m)(k_{n+1}+m)}\,, \quad (98)\end{aligned}$$

and

$$\langle\theta^{(2)}|i\psi_L^\dagger\psi_R(0,\mathrm{x})-i\psi_R^\dagger\psi_L|\theta^{(2)}\rangle=$$

$$\begin{aligned}
&=-\frac{g^4e^{-\frac{\pi}{2mL}}}{m^2L^3}\sum_{N=-\infty}^{\infty}\sin\theta\Bigg\{\sum_{I,J=1}^{\infty}\frac{8m^2}{(2k_I)^3((2k_J)^3}(IJ-I-J+1)\\
&+\sum_{J=1}^{\infty}\frac{8m^2}{(2k_J)^6}(J^2-2J)\\
&-8m\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{pJ-J+1-p+\theta(J-p-1)(J-p-1)}{(2k_p+m)(2k_{p+1}+m)(2k_J)^3}\\
&-4m\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{p}{(2k_p+m)(2k_{p+1}+m)}\Bigg(\frac{1}{(2k_p+2k_J+m)^2(2k_{p+1}+2k_J+m)}\\
&\qquad+\frac{1}{(2k_p+2k_J+m)(2k_{p+1}+2k_J+m)^2}\Bigg)\\
&+2m\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{1}{(2k_{p+1}+m)}\Bigg(\frac{2}{(2k_J)^3(2k_p+2k_J+m)}\\
&\qquad+\frac{1}{(2k_J)^2(2k_p+2k_J+m)^2}\Bigg)\\
&-4m\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\Bigg(\frac{2J}{(2k_J)^3(2k_p+2k_J+m)(2k_{p+1}+2k_J+m)}\\
&\qquad+\frac{J}{(2k_J)^2(2k_p+2k_J+m)^2(2k_{p+1}+2k_J+m)}\\
&\qquad+\frac{J}{(2k_J)^2(2k_p+2k_J+m)(2k_{p+1}+2k_J+m)^2}\Bigg)\\
&+\sum_{J=1}^{\infty}\frac{m}{(2k_{\frac{1}{2}}+m)}\Bigg(\frac{2}{(2k_J)^3(2k_{J-\frac{1}{2}}+m)}+\frac{1}{(2k_J)^2(2k_{J+\frac{1}{2}}+m)^2}\Bigg)\\
&-4m\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\Bigg(\frac{2\theta(J-p)(J-p)}{(2k_J)^3(2k_{J-p}+m)(2k_{J-p+1}+m)}\\
&\qquad+\frac{\theta(J-p)(J-p)}{(2k_J)^2(2k_{J-p}+m)^2(2k_{J-p+1}+m)}\\
&\qquad+\frac{\theta(J-p)(J-p)}{(2k_J)^2(2k_{J-p}+m)(2k_{J-p+1}+m)^2}\Bigg)\\
&+\sum_{p=\frac{1}{2}}^{\infty}\frac{\left(1-\frac{\pi^2m}{g^2L}\right)p}{2(k_p+m)(2k_p+m)(k_{p+1}+m)(2k_{p+1}+m)}
\end{aligned}$$

$$+\sum_{p,n=\frac{1}{2}}^{\infty}\frac{2np-n+(n-p)\theta(n-p-\frac{1}{2})}{(2k_p+m)(2k_n+m)(2k_{p+1}+m)(2k_{n+1}+m)}$$

$$+\sum_{n=\frac{1}{2}}^{\infty}\frac{n^2}{(2k_n+m)^2(2k_{n+1}+m)^2}$$

$$+\sum_{n=\frac{1}{2}}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{n}{2(2k_n+m)(2k_{n+1}+m)(k_{n+p+1}+m)^2}$$

$$+\sum_{p=\frac{1}{2}}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\frac{(n-p-1)\theta(n-p-\frac{1}{2})}{2(2k_n+m)(2k_{n+1}+m)(k_{n-p}+m)^2}$$

$$+\sum_{n=\frac{1}{2}}^{\infty}\frac{n-\frac{1}{2}}{2(2k_n+m)(2k_{n+1}+m)(k_{n+\frac{1}{2}}+m)^2}$$

$$+\frac{1}{16(2k_{\frac{1}{2}}+m)^2m^2}+\sum_{n=\frac{1}{2}}^{\infty}\frac{n(n+1)}{2(2k_n+m)(2k_{n+1}+m)m^2}\Bigg\} .$$

We shall also need the norm of the state. We have that

$$\langle\theta|\theta\rangle \simeq \langle\theta^{(0)}|\theta^{(0)}\rangle+\langle\theta^{(1)}|\theta^{(1)}\rangle+\langle\theta^{(2)}|\theta^{(2)}\rangle ,$$

where

$$\langle\theta^{(0)}|\theta^{(0)}\rangle=\sum_{N,M=-\infty}^{\infty}e^{i(M-N)\theta}\,\langle\Omega_M^{(0)}|\Omega_N^{(0)}\rangle=\sum_{N,M=-\infty}^{\infty}e^{i(M-N)\theta}\delta_{M,N}=\sum_{N=-\infty}^{\infty}1 \quad (99)$$

$$\langle\theta^{(1)}|\theta^{(1)}\rangle=\frac{g^2}{mL}\sum_{N=-\infty}^{\infty}\left\{-\sum_{J=1}^{\infty}\frac{mJ}{2k_J^3}+\sum_{n=\frac{1}{2}}^{\infty}\frac{2n}{(2k_n+m)^2}\right\} , \quad (100)$$

and

$$\langle\theta^{(2)}|\theta^{(2)}\rangle=\frac{g^4}{m^2L^2}\sum_{n=-\infty}^{\infty}\left\{\sum_{I,J=1}^{\infty}IJ\frac{8m^2}{(2k_I)^3((2k_J)^3}+\sum_{J=1}^{\infty}\frac{8m^2}{(2k_J)^6}J^2\right.$$

$$-8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{pJ-p+\theta(p-J)(p-J)}{(2k_p+m)^2(2k_J)^3}$$

$$-8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{p}{(2k_p+m)^2(2k_p+2k_J+m)^3}$$

$$-8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\left(\frac{J}{(2k_p+2k_J+m)^2(2k_J)^3}+\frac{J}{(2k_p+2k_J+m)^3(2k_J)^2}\right)$$

$$-8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\left(\frac{\theta(J-p)(J-p)}{(-2k_p+2k_J+m)^2(2k_J)^3}+\frac{\theta(J-p)(J-p)}{(-2k_p+2k_J+m)^3(2k_J)^2}\right)$$

$$+\sum_{p=\frac{1}{2}}^{\infty}\frac{p}{2(k_p+m)^2(2k_p+m)^2}+\sum_{n=\frac{1}{2}}^{\infty}\frac{n^2}{(2k_n+m)^4}$$

$$+\sum_{p,n=\frac{1}{2}}^{\infty}\frac{2np-n+(n-p)\theta(n-p-\frac{1}{2})}{(2k_p+m)^2(2k_n+m)^2}+\sum_{n=\frac{1}{2}}^{\infty}\frac{n^2}{2m^2(2k_n+m)^2}$$

$$+\sum_{n,p=\frac{1}{2}}^{\infty}\left[\frac{(n-p)\theta(n-p-\frac{1}{2})}{2(k_{n-p}+m)^2(2k_n+m)^2}+\frac{n}{2(k_{p+n}+m)^2(2k_n+m)^2}\right]\Bigg\}\,.$$

Using these results we can calculate $\frac{\langle\theta|\psi_L^\dagger\psi_R|\theta\rangle}{\langle\theta|\theta\rangle}$:

$$\frac{\langle\theta|i(\psi_L^\dagger\psi_R-\psi_R^\dagger\psi_L)|\theta\rangle}{\langle\theta|\theta\rangle}\simeq$$

$$\simeq\sin\theta e^{-\frac{\pi}{2mL}}\Bigg\{-\frac{1}{L}+\frac{g^2}{mL^2}\Bigg[\sum_{n=\frac{1}{2}}^{\infty}\frac{4k_n}{(2k_n+m)^2(2k_{n+1}+m)}-\sum_{J=1}^{\infty}\frac{4m}{(2k_J)^3}\Bigg]$$

$$+\frac{g^4}{m^2L^3}\Bigg[-\sum_{n=\frac{1}{2}}^{\infty}\frac{k_n}{(2k_n+m)^2(2k_{n+1}+m)}\sum_{p=\frac{1}{2}}^{\infty}\frac{4k_p}{(2k_p+m)^2(2k_{p+1}+m)}$$

$$+\sum_{n=\frac{1}{2}}^{\infty}\frac{4k_n}{(2k_n+m)^2(2k_{n+1}+m)}\sum_{J=1}^{\infty}\frac{4m}{(2k_J)^3}$$

$$+\sum_{J=1}^{\infty}\frac{4mJ}{(2k_J)^3}\sum_{n=\frac{1}{2}}^{\infty}\frac{4k_n}{(2k_n+m)^2(2k_{n+1}+m)}-\sum_{I,J=1}^{\infty}\frac{8m^2}{(2k_I)^3(2k_J)^3}+\sum_{J=1}^{\infty}\frac{16m^2J}{(2k_J)^6}$$

$$+\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{4mp}{(2k_p+m)(2k_p+2k_J+m)}\Bigg(\frac{1}{(2k_{p+1}+m)(2k_{p+1}+2k_J+m)^2}$$

$$+\frac{1}{(2k_{p+1}+m)(2k_{p+1}+2k_J+m)(2k_p+2k_J+m)}$$

$$-\frac{2}{(2k_p+m)(2k_p+2k_J+m)^2}\Bigg)$$

$$
-\sum_{J=1}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{2m}{(2k_{p+1}+m)(2k_p+2k_J+m)(2k_J)^2}\left(\frac{1}{(2k_p+2k_J+m)}+\frac{1}{k_J}\right)
$$

$$
-\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{8mk_J}{(2k_p+2k_J+m)(2k_J)^2}\left(\frac{1}{k_J(2k_p+2k_J+m)(2k_{p+1}+2k_J+m)}\right.
$$

$$
\left.+\frac{1}{(2k_p+2k_J+m)(2k_{p+1}+2k_J+m)^2}+\frac{2}{(2k_p+2k_J+m)^2(2k_{p+1}+2k_J+m)}\right)
$$

$$
+\sum_{J=1}^{\infty}\frac{m}{(2k_{\frac{1}{2}}+m)(2k_{J-\frac{1}{2}}+m)(2k_J)^2}\left(\frac{1}{(2k_{J-\frac{1}{2}}+m)}+\frac{1}{k_J}\right)
$$

$$
-\frac{\pi}{L}\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{8m\theta(J-p)(J-p)}{(-2k_p+2k_J+m)(2k_J)^2}\left(\frac{1}{(-2k_p+2k_J+m)(-2k_{p-1}+2k_J+m)^2}\right.
$$

$$
+\frac{1}{k_J(-2k_p+2k_J+m)(-2k_{p-1}+2k_J+m)}
$$

$$
\left.+\frac{2}{(-2k_p+2k_J+m)^2(-2k_{p-1}+2k_J+m)}\right)
$$

$$
-\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{16m}{(2k_J)^3}\frac{k_pJ-k_J+\theta(J-p)(k_J-k_p)}{(2k_p+m)^2(2k_{p+1}+m)}
$$

$$
+\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{8m}{(2k_J)^3}\frac{1-\theta(J-p-1)}{(2k_p+m)(2k_{p+1}+m)}
$$

$$
-\sum_{p=\frac{1}{2}}^{\infty}\frac{4m}{(2k_{p+\frac{1}{2}})^3(2k_p+m)(2k_{p+1}+m)}
$$

$$
+\sum_{p=\frac{1}{2}}^{\infty}\frac{k_p}{2(k_p+m)^2(2k_p+m)(2k_{p+1}+m)}\left(\frac{2}{(2k_p+m)}+\frac{1}{(k_{p+1}+m)}\right)
$$

$$
+\frac{3\pi^2m}{2g^2L}\sum_{p=\frac{1}{2}}^{\infty}\frac{p}{(k_p+m)(2k_p+m)(k_{p+1}+m)(2k_{p+1}+m)}
$$

$$
+\sum_{p,n=\frac{1}{2}}^{\infty}\frac{-2k_n\theta(p-n+\frac{1}{2})-2k_p\theta(n-p-\frac{1}{2})}{(2k_p+m)(2k_n+m)^2(2k_{p+1}+m)}\left(\frac{1}{(2k_p+m)}+\frac{1}{(2k_{n+1}+m)}\right)
$$

$$
+\sum_{n=\frac{1}{2}}^{\infty}\frac{nk_n}{(2k_n+m)^3(2k_{n+1}+m)}\left(\frac{1}{(2k_{n+1}+m)}+\frac{1}{(2k_n+m)}\right)
$$

$$
-\sum_{n=\frac{1}{2}}^{\infty}\frac{n}{2m(2k_n+m)^2(2k_{n+1}+m)}
$$

$$
+\sum_{n,p=\frac{1}{2}}^{\infty}\frac{(k_n-k_p)\theta(n-p-\frac{1}{2})}{(k_{n-p}+m)^2(2k_n+m)^2(2k_{n+1}+m)}
$$

$$
+\sum_{p=\frac{1}{2}}^{\infty}\sum_{n=\frac{1}{2}}^{\infty}\frac{\theta(n-p-\frac{1}{2})}{2(2k_n+m)(2k_{n+1}+m)(k_{n-p}+m)^2}
$$

$$
+\sum_{n=\frac{1}{2}}^{\infty}\frac{n-\frac{1}{2}}{2(2k_n+m)(2k_{n+1}+m)(k_{n+\frac{1}{2}}+m)^2}+\frac{1}{16(2k_{\frac{1}{2}}+m)^2m^2}
$$

$$
+\sum_{n=\frac{1}{2}}^{\infty}\sum_{p=\frac{1}{2}}^{\infty}\frac{k_n}{2(2k_n+m)(2k_{n+1}+m)(k_{p+n}+m)}\Bigg(\frac{1}{(k_{p+n}+m)(k_{p+n+1}+m)}
$$

$$
+\frac{1}{(k_{p+n+1}+m)^2}+\frac{2}{(k_{p+n}+m)(2k_n+m)}\Bigg)\Bigg]\Bigg\}
$$

While some of the sums in $\langle\theta|i(\psi_L^\dagger\psi_R-\psi_R^\dagger\psi_L)|\theta\rangle$ and $\langle\theta|\theta\rangle$ are divergent, the ratio is finite.

We also need to consider the potential contribution of the real field. We find that

$$
\frac{\langle\theta|\phi_L(0,\mathrm{x})\phi_R(0,\mathrm{x})|\theta\rangle}{\langle\theta|\theta\rangle}\simeq\langle 0|\overset{o}{\phi}_L\overset{o}{\phi}_R|0\rangle\Bigg\{1-2\frac{g^2}{mL}\sum_{n=\frac{1}{2}}^{\infty}\frac{1}{(2k_n+m)^2}
$$

$$
+\frac{g^4}{m^2L^2}\Bigg[8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{1}{(2k_p+m)^2(2k_p+2k_J+m)^3}
$$

$$
+8m\sum_{p=\frac{1}{2}}^{\infty}\sum_{J=1}^{\infty}\frac{1}{(2k_{J+p-\frac{1}{2}})^2(2k_{J-\frac{1}{2}}+m)^3}
$$

$$
-\sum_{p=\frac{1}{2}}^{\infty}\frac{1}{4(k_p+m)^2(2k_p+m)^2}-\sum_{p,n=\frac{1}{2}}^{\infty}\frac{-2+2p\delta_{n,p}}{(2k_p+m)^2(2k_n+m)^2}\Bigg]\Bigg\}
$$

and it is not hard to verify that it goes to zero in large-L limit, as one might have expected considering that this contribution is an artifact of the breaking of chiral invariance in free theory due to the zero modes.

We now want to evaluate the condensate in the large L limit. By studying the large-L behaviour one can see that, while several terms go to zero, others diverge with L. The divergent behaviour is expected since it is found also in the expansion of the factor multiplying the exponential $e^{-\frac{\pi}{2mL}}$ in the finite-L condensate for the Schwinger model. In that case, the full nonperturbative result has a finite limit [10].

Setting $g = m\sqrt{\pi}$ we can see that the condensate takes the form (recall that $\theta = \pm\frac{\pi}{2}$)

$$\frac{\langle\theta|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta\rangle}{\langle\theta|\theta\rangle} = m\sin\theta e^{-\frac{\pi}{2mL}} f(mL) \tag{101}$$

where f is a function of mL that goes to a pure number, if convergent, in the large-L limit . We therefore find that, as in the case of the Schwinger model, the condensate is proportional to the coupling constant.

Disregarding contributions that vanish as L goes to infinity we get the following estimate for the large-L behaviour of the condensate

$$\begin{aligned}\frac{\langle\theta|i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)|\theta\rangle}{\langle\theta|\theta\rangle} \simeq m\sin\theta e^{-\frac{\pi}{2mL}} \Bigg(&\frac{1}{12\pi}\sum_J \frac{1}{J^3} - \frac{7}{8\pi}\sum_J \frac{1}{J^2} \\ &-\frac{m^3L^3}{8\pi^4}\left(\sum_J \frac{1}{J^3}\right)^2 + \frac{m^3L^3}{4\pi^4}\sum_J \frac{1}{J^5}\Bigg)\end{aligned}$$

A standard technique to estimate the value of a function in the limit where the argument goes to infinity when the function is defined by a power series is that of Padé approximants[18]. The method of quadratic approximants gives the following function

$$f(x) = \frac{-1 + \sqrt{1 + 4x^3(a + (a^2 + b)x^3)}}{2x^3}$$

which has the power series expansion

$$f(x) \simeq a + bx^3 \ .$$

For our case

$$\begin{aligned} a &= \frac{1}{12\pi}\sum_J \frac{1}{J^3} - \frac{7}{8\pi}\sum_J \frac{1}{J^2} \simeq -0.426 \\ b &= -\frac{1}{8\pi^4}\left(\sum_J \frac{1}{J^3}\right)^2 + \frac{1}{4\pi^4}\sum_J \frac{1}{J^5} \simeq 0.000807 \\ x &= mL\,. \end{aligned}$$

For $0 < x < \infty$ f is between -0.426 and 0.427. It is therefore likely that the number multiplying $m \sin\theta$ in the condensate is within this range. But nothing definite can be said about the accuracy of this result. Since we can only form one approximant, we cannot test for convergence, even empirically, and there is no mathematical theorem giving a bound on the error that is made with this approximation. A similar procedure for the Schwinger model gives a correct estimate of the order of magnitude, with an asymptotic value of about 0.45, while the correct value is about 0.28. In all likelihood our number is of order 1.

4 Summary

In quantizing QCD_{1+1} with quarks in the adjoint representation with twisted boundary conditions we have encountered the unexpected difficulty that if all the components of the gauge field are quantized in indefinite metric (the quantization procedure we expected to have to use and the one that is required in the case of the Schwinger model), the residual gauge symmetry present at the classical level cannot be implemented at the quantum level. On the other hand, if we quantize all the components of the gauge field in positive metric we find that, as expected, we cannot consistently remove the unphysical states from the system. Our solution to this quandary relies explicitly on the twisted boundary conditions: we quantize the periodic gauge field in indefinite metric and the antiperiodic components of the gauge field in positive metric. It is an open question as to what procedure should be used in the continuum or in a case where each of the components of the gauge field is subject to the same periodicity conditions. The problem may have some importance since the same issue arises in the light-cone quantization of standard QCD.

With that mixed quantization scheme in place we show that a physical subspace exists which consists of color singlets . We explicitly demonstrate that the physical subspace is dynamically stable by showing that the Lagrange multiplier fields are free fields. We give the algebra of the Lagrange multiplier fields, which is likely to be of the same form as that in a continuum solution. We show that the use of the mixed quantization scheme allows the quantum implementation of the residual gauge symmetry which exists at the classical level. We then showed that there are two possible vacua, in agreement with the results of [1][3][5][6].

We worked out the Hamiltonian. If the Hamiltonian is partitioned into the kinetic energies, as the unperturbed Hamiltonian, and the interaction, as the perturbing Hamiltonian, perturbation theory cannot be applied. That is to be expected since the condensate, and the vacuum which leads to it, are nonperturbative quantities in the usual meaning of the word. But we find that if we include a small part of the interaction in the unperturbed Hamiltonian we can diagonalize this new unperturbed Hamiltonian by hand and in doing so we include all the singular structure in these analytic unperturbed eigenstates. We can now apply standard perturbation theory using the rest of the interaction as the perturbing operator. We applied that perturbation theory to work out an expansion for the vacuum through the first three orders (two orders in the perturbation). We then showed that the resulting state satisfies, through the relevant order, the subsidiary condition and is therefore a physical state. With that vacuum we then evaluated, through the same order in perturbation theory, the chiral condensate. We were able to show the dependence of the condensate on the parameters but have only an approximate value for a constant of proportionality. We used Padé approximants to find an approximate value for the constant in the limit where $L \to \infty$ (the so called, decompactification limit).

Several extensions of the work are possible. Perhaps the most interesting, and possibly important, is to study the continuum case or a case where all components of the gauge field are periodic (untwisted boundary conditions) and determine what quantization procedure will allow the implementation of the residual gauge symmetry and a consistent implementation of the dynamics. The perturbation theory could be used to find approximation to states other than the vacuum such as the one particle states (and thus determine an approximate spectrum). It would be interesting to compare such results, and also the results we have presented here, with results from other techniques if such results become available. It is possible that, if the calculations are automated on a computer, higher orders of perturbation theory could be worked out, thus allowing better estimates of the values and of the accuracy of the values. We do not think it practical to try to go to higher order by hand.

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