The fully differential decay rate of top quark at next-to-next-to leading order in QCD

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We perform a first calculation of the fully differential decay rate for top quark leptonic decay $t \to W^+(l^+\nu)b$ at next-to-next-to leading order in QCD. The method is based on the understanding of the invariant mass distribution of final-state jet in the singular limit from effective field theory. One-loop electroweak corrections as well as finite bottom quark mass and W boson width effects are also included. Our result can be used to study arbitrary infrared-safe observables of top quark decay with the highest perturbative accuracy.

Introduction. The top quark is the heaviest fermion in the Standard Model (SM), and frequently plays an important role in new physics extensions of the SM. Therefore detailed studies of its production and decay are highly desirable. Their precise measurements at the LHC will be crucial for the understanding of electroweak symmetry breaking and also search of new physics. Due to its large mass, the lifetime of top quark is much smaller than the typical time scale of hadronization. For this reason, the top quark can be treated as a free particle in good approximation, and perturbative calculation of higher order quantum corrections to its decay rate can be performed.

Within the SM, the next-to-leading order (NLO) QCD corrections to top quark decay width, Γ_t, are calculated more than 20 years ago [1]. Employing the method developed in Ref. [2], the next-to-next-to-leading order (NNLO) QCD corrections to Γ_t was calculated in Ref. [3], in the limit of $m_t \gg m_W$. Later, finite W boson mass effect in the NNLO computation is taken into account in Refs. [4] based on the calculations of top quark self-energy as expansion in m_W^2/m_t^2. While all the previous calculations at NNLO concentrate on the inclusive decay width, differential decay rate is also of substantial interest, especially when considering the measurement of top quark mass [5] and electroweak (EW) couplings [6]. In particular, it's an important ingredient in a fully differential top quark pair production [7] and decay at NNLO. To the best of our knowledge, such a calculation is still unknown in the literature, and is the subject of this Letter.

The formalism. We consider the SM top quark decay,

$$t \to W^+ + b + X, \tag{1}$$

where X represents any partons in the final state. NNLO QCD corrections to this process consists of three parts: two-loop virtual contribution ($X = 0$), ${\Gamma_t^{(2)}}_1$, one-loop real-virtual contribution ($X = 1$), ${\Gamma_t^{(2)}}_2$, and tree-level double real contribution ($X = 2$), ${\Gamma_t^{(2)}}_3$. While the amplitudes for each part are well defined, integrals over the phase space induce infrared singularities, which must be extracted to cancel those from virtual corrections in order to obtain a finite result. In particular, the double real contribution is the primary obstacle for obtaining fully differential NNLO corrections. In this Letter, we solve this problem for processes of heavy to light decay at NNLO, using a phase space slicing method inspired by a factorization formula for heavy to light current in Soft-Collinear Effective Theory (SCET) [8]. Below we describe our method.

Cluster all the partons in the final state into a single jet. Let $\tau = (p_b + p_X)^2/m_t^2$, which essential measures the invariant mass of the jet. the complete NNLO corrections to Γ_t can be divided into two parts:

$$\Gamma_t = \int_0^{\tau_0} \mathrm{d}\tau \frac{\mathrm{d}\Gamma_t}{\mathrm{d}\tau} + \int_{\tau_0}^{\infty} \mathrm{d}\tau \frac{\mathrm{d}\Gamma_t}{\mathrm{d}\tau} \equiv \Gamma_A + \Gamma_B, \tag{2}$$

where τ_0 is a dimensionless cutoff for τ. We set bottom quark mass $m_b = 0$ in the NLO and NNLO QCD calculations. Effects of finite m_b will be considered later as a correction to the LO results. In the limit of $\tau \to 0$, only soft radiations and (or) radiations collinear to the b quark are allowed. In this region, $\frac{\mathrm{d}\Gamma_t}{\mathrm{d}\tau}$ obeys a factorization formula [9]:

$$\frac{1}{\Gamma_t^{(0)}} \frac{\mathrm{d}\Gamma_t}{\mathrm{d}\tau} = \mathcal{H}\left(x \equiv \frac{m_W^2}{m_t^2}, \mu\right) \int \mathrm{d}k\, \mathrm{d}m^2 J(m^2,\mu) S(k,\mu) \times \delta\left(\tau - \frac{m^2 + 2E_J k}{m_t^2}\right) + \cdots, \tag{3}$$

where we have neglected non-singular terms in τ. $\Gamma_t^{(0)}$ is the top quark decay width at LO, μ is the renormalization scale, and E_J is the energy of the jet. $\mathcal{H}(x,\mu)$ is the hard function, which results from integrating out hard modes of QCD in matching to SCET. It has been calculated to NNLO in α_s [10]. $J(m^2,\mu)$ is the quark jet function with mass m, whose NNLO expression can be found in Ref. [11]. It can be thought of as the probability of finding a jet with invariant mass m, generated by collinear radiations. $S(k,\mu)$ is the soft function, which describes the probability of measuring the light-cone component of the momentum of soft radiations $k_s \cdot n$, where n is a unit light-cone vector along the direction of the jet, to be k. It has also been calculated to NNLO in Ref. [12].

Using the NNLO results for the hard function, jet function, and soft function, we can then calculate Γ_A at NNLO, utilizing Eq. (3), up to terms proportional to τ_0. For sufficiently small τ_0, they can be safely neglected. The most difficult part of double real contributions are included in the calculation of the jet function and soft function. Note that Γ_A is infrared finite, because the infrared divergences in the jet and soft function cancel against those from the hard function. The spin information of the b quark is lost since spin summation has been performed in the jet function. But polarization information of the top quark is retained, due to the fact that soft radiations do not change spin. In practice, instead of a convolution form, it's more convenient to write Eq. (3) in a product form:

$$\frac{1}{\Gamma_t}\frac{\mathrm{d}\Gamma_t}{\mathrm{d}\tau} = \mathcal{H}(x,\mu) \times \lim_{\eta\to 0}\widetilde{j}\left(\partial_\eta + \ln\frac{m_t^2}{\mu^2},\mu\right)\widetilde{s}\left(\partial_\eta + \ln\frac{m_t^2}{2E_J\mu},\mu\right)\frac{\tau^\eta}{\tau}\frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}, \tag{4}$$

where $\widetilde{j}$ and $\widetilde{s}$ are the Laplace trasformed jet and soft function, respectively:

$$\widetilde{j}\left(\ln\frac{\nu m_t^2}{\mu^2},\mu\right) = \int_0^\infty \mathrm{d}m^2 \,\exp\left(-\frac{\nu m^2}{e^{\gamma_E}m_t^2}\right) J(m^2,\mu),$$
$$\widetilde{s}\left(\ln\frac{\nu m_t^2}{2E_J\mu},\mu\right) = \int_0^\infty \mathrm{d}k \,\exp\left(-\frac{2\nu E_J k}{e^{\gamma_E}m_t^2}\right) S(k,\mu), \tag{5}$$

and τ^η/τ should be expanded in terms of plus distribution:

$$\frac{\tau^\eta}{\tau} = \frac{1}{\eta}\delta(\tau) + \sum_{n=0}^{\infty}\frac{\eta^n}{n!}\left[\frac{\ln^n\tau}{\tau}\right]_+ . \tag{6}$$

Substituting the NNLO expansion for hard function, jet function and soft function into Eq. (4) gives a closed form solution of $\mathrm{d}\Gamma_t/\mathrm{d}\tau$ at small τ.

Γ_B is also infrared finite. In fact it can be calculated from the NLO QCD corrections to $t\to W^+ b$ plus 1 jet, as long as $\tau_0 > 0$. In our calculation, the one-loop helicity amplitudes for this specific process is extracted from the NLO QCD corrections to single top production associated with W boson [13]. The tree level helicity amplitudes are calculated with HELAS [14]. Infrared divergences in the phase space integral of tree level matrix elements are canceled by adding appropriate dipole subtraction terms [15].

It should be pointed out that the method used here to calculate the NNLO corrections is similar to the q_T subtraction method of Catani and Grazzini [16]. In fact they both employ the universality of infrared divergences and the knowledge of resummation to facilitate the calculation.

Total width. For top quark SM decay, the total decay width in G_F parametrization scheme [17] at LO is given by

$$\Gamma_t^{(0)} = \frac{G_F m_t^3}{8\sqrt{2}\pi}\Big[1 - 3(\frac{m_W^2}{m_t^2})^2 + 2(\frac{m_W^2}{m_t^2})^3\Big],$$

assuming CKM matrix element $|V_{tb}| = 1$ and $m_b = 0$. We choose $m_W = 80.385\,\mathrm{GeV}$, $G_F = 1.16638\,\mathrm{GeV}$, and $m_t = 173.5\,\mathrm{GeV}$ [18], unless specified. Other constants used in followed calculations include m_Z, $\alpha_s(m_Z)$, and m_b, which are also chosen as in Ref. [18]. Corrections to the LO width considered here include finite b quark mass and W boson width effects, δ_f^b and δ_f^W, NLO electroweak corrections, δ_{EW}, NLO and NNLO QCD corrections, $\delta_{QCD}^{(1)}$ and $\delta_{QCD}^{(2)}$, which are defined as

$$\Gamma_t = \Gamma_t^{(0)}(1 + \delta_f^b + \delta_f^W + \delta_{EW} + \delta_{QCD}^{(1)} + \delta_{QCD}^{(2)}),$$

where Γ_t is the corrected total width. In Table I we show the LO total width together with all the corrections in percentage for different top quark mass values. The renormalization scale is set to top quark mass. Our results agree with the ones shown in previous literatures for finite width and mass effects [1], electroweak corrections [17, 19], NLO [1] and NNLO QCD corrections [3, 4], with the updated input parameters. All the corrections are stable with respect to the top quark mass.

m_t	$\Gamma_t^{(0)}$	δ_f^b	δ_f^W	δ_{EW}	$\delta_{QCD}^{(1)}$	$\delta_{QCD}^{(2)}$
172.5	1.4806	-0.26	-1.49	1.68	-8.58	-2.09
173.5	1.5109	-0.26	-1.49	1.69	-8.58	-2.09
174.5	1.5415	-0.25	-1.48	1.69	-8.58	-2.09

TABLE I. Top quark total width at LO and corrections in percentage from finite W boson width, finite b quark mass, and high orders, including NLO in EW couplings, NLO and NNLO in QCD coupling. Mass and width are shown in unit of GeV.

As mentioned earlier, the NNLO QCD corrections can be divided into three pieces, which are ${\Gamma_t^{(2)}}_1$, ${\Gamma_t^{(2)}}_2$, and ${\Gamma_t^{(2)}}_3$. Each of them depends strongly on the cutoff parameter τ_0 up to the fourth power of $\ln\tau_0$. While their sum should only have weak dependences propotional to τ_0, and approach the genuine NNLO QCD corrections when τ_0 is small enough. Thus in Fig. 1 we show the separate contributions to the NNLO corrections. When τ_0 varies from 10^{-3} to about 10^{-6}, the separate contributions can reach as large as twice of the LO width, while the sum keeps almost unchanged at the value of about 2.1% of the LO width. Stability of such a large cancellation proves the validity of our NNLO calculation. On the other hand the NLO QCD corrections have an uncertainty of about 1.6% of the LO width due to the arbitrary choice of renormalization scale as shown in Fig. 2, which comes directly from running of QCD coupling constant α_s. After adding the NNLO QCD corrections, the scale dependence is reduced to about 0.8%, which makes the predictions more reliable.

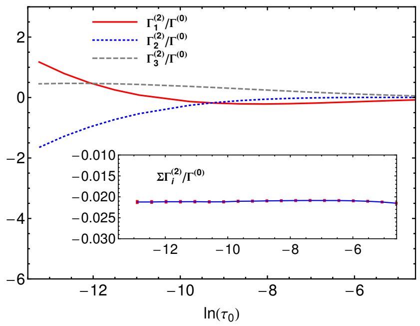

FIG. 1. Separate contributions $\Gamma_{t\ \ i}^{(2)}$ of the NNLO QCD corrections and their sum as functions of the cutoff τ_0, normalized to the LO width.

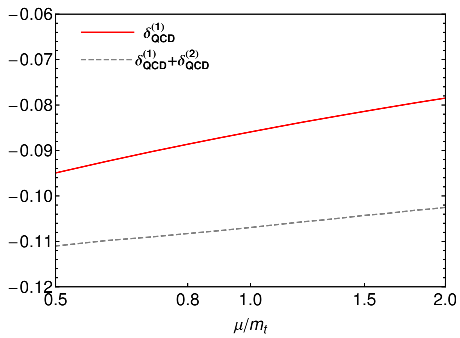

FIG. 2. Renormalization scale dependence of the NLO and NLO+NNLO QCD corrections, normalized to the LO width at central scale $\mu = m_t$.

Differential distributions. Within our framework we can calculate the fully differential decay width of top quark semileptonic decay $t \to W^+(l^+\nu)b$ up to NNLO in QCD, which is not possible for the method based on calculations of top quark self-energy. Precise predictions for differential distributions of top quark decay products are of great importance, especially for the measurement of top quark mass [5] and test of the $V - A$ structure of tWb charged current [6]. Below we will show several final state distributions for $t \to W^+(l^+\nu)b$, including all the corrections as in the total width results. We use parton level JADE algorithm [20] for jet clustering with jet resolution threshold chosen to 0.1. The shape measurements are more relevant for the experimental studies, having both small experimental and theoretical uncertaities. Thus all the distrbutions are normalized to unit area for comparisions. We show results for several cases, i.e., pure LO prediction (LO1), LO predictions plus corrections from finite m_b, W boson width and NLO EW effects (LO2), LO2 with NLO QCD corrections in addition (NLO), and further including NNLO QCD corrections (NNLO).

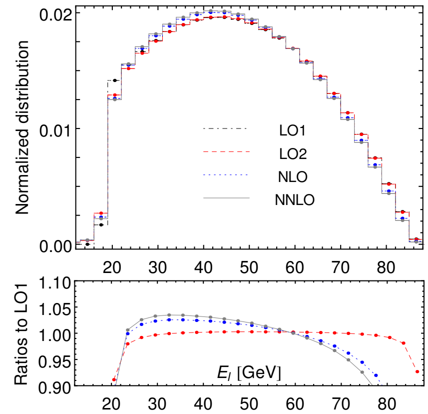

FIG. 3. Energy distribution of the charged lepton from top quark decay in top quark rest frame.

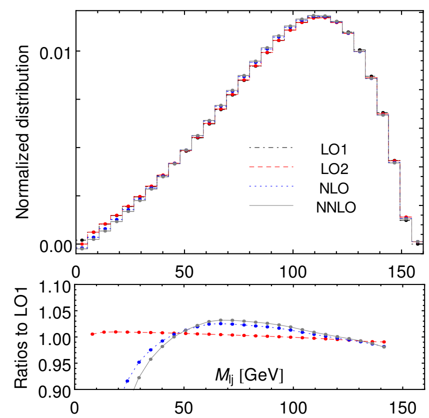

FIG. 4. Invariant mass distribution of the charged lepton and hardest jet from top quark decay in top quark rest frame.

From Fig. 3 to 6, we present the charged lepton energy distribution, invariant mass distribution of the charged lepton and the hardest jet in energy, in top quark rest frame, and two angular distribution of $\cos(\theta^*)$ and $\cos(\theta_{lj})$. All of them are normalized to unit area. θ^* are defined in W boson rest frame as the angle between charged lepton and the opposite of top quark direction, and θ_{lj} is the angle between charged lepton and hardest jet in top quark rest frame. In each figure the uper panel shows the normalized distribution while the lower panel gives their ratios with respect to the one of LO1. As we can see, the differences between LO1 and LO2 are small in general, especially for the central region of each plot. Both the NLO and NNLO QCD corrections push the energy and invariant mass distributions into the central region since the recoil constituents are then massive. The NNLO corrections here are about one-fourth of the NLO ones, similar to the results of total width. Inclusive angular distribution of $\cos(\theta^*)$ reflects the W boson helicity fractions in top quark decay, which can be also predicted up to NNLO in QCD through top quark self-energy calculations [21]. $\cos(\theta^*)$ distributon has been extensively studied at both the Tevatron and LHC for testing potential anomalous tWb couplings induced by new physics [6]. By a least χ^2 fit we get the W boson helicity fractions ratio as $\mathcal{F}_L : \mathcal{F}_+ : \mathcal{F}_- = 0.689 : 0.0017 : 0.309$ using the $\cos(\theta^*)$ distribution. The results incorporate finite b quark mass and W boson width effects, one-loop EW corrections, and QCD corrections up to NNLO, and are in very good agreement with the one shown in [21]. Our calculations are more helpful for the corresponding measurements since experimentalists can include precise corrections for the acceptance in different kinematic regions using our results. As for $\cos(\theta_{lj})$ distribution, QCD corrections are more pronounced there since changes of the energy spectrum also modify the distribution.

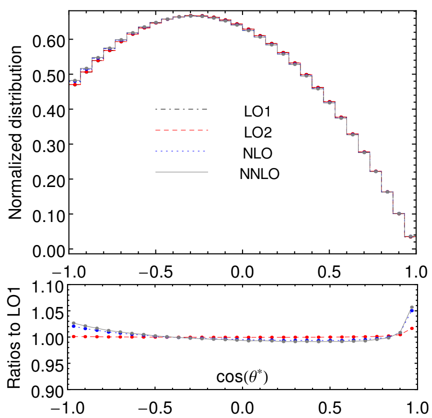

FIG. 5. Angular distribution of the charged lepton from top quark decay in W boson rest frame.

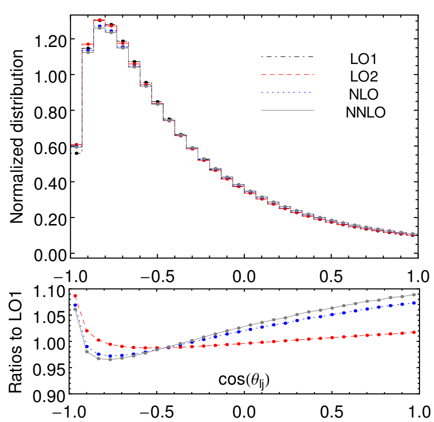

FIG. 6. Angular distribution of the charged lepton from top quark decay in top quark rest frame.

Conclusions. We presented the NNLO QCD corrections to top quark total decay width, which do not depend on expansion in W boson mass, and fully differential distributions of $t \to W^+(l^+\nu)b$ based on SCET. One-loop EW corrections as well as effects from finite b quark mass and W boson width are also included. All together they made the current most precise predictions for top quark decay, which are helpful for top quark mass measurement and testing of weak charged current structure. We have implemented the calculation into a parton level Monte Carlo program, in which arbitrary infrared-safe cut can be imposed on the final state. Our calculations are complementary to the NNLO QCD predictions for top quark pair production [7]. Moreover, the method can be further applied to the precise predictions of b quark semileptonic decay, which will be presented elsewhere.

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